

SYMMETRY AND STABILITY IN DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Many mathematical models for physical, biological or sociological phenomena exhibit various kinds of symmetry, such as symmetry with respect to reflection, rotation, translation, dilation, gauge transformation, and so on. Given an equation with certain symmetry, a natural question that arises is whether or not the symmetry of the equation is inherited by its solutions.

Needless to say, the answer is generally “No”. There are abundant examples of symmetry breaking that occur in a variety of problems, such as in morphogenesis, fluid flows, crystal growth, or even in patterns of bacterial colonies. For example, mathematical models for population distribution in a spatially uniform environment have translational symmetry, but it often happens that intriguing geometric spatial patterns emerge from such an environment, thus breaking the translational symmetry. Without exaggeration one can say that the striking complexity and variety of our world are a result of innumerable sequence of symmetry-breaking procedures.

On the other hand, there are also many situations in which symmetry is well preserved. Otherwise our world would be too disorderly and chaotic, and no advanced structure could survive very long. Both aspects — symmetry breaking and symmetry preserving — are important for the nature to function properly.

In this article we give rather a general mathematical framework for studying symmetry preservation. Mathematically, symmetry is expressed by a group of transformations, such as the group of rotations and translations. Given such a group action, say G , an equation is said to have “ G -symmetry” if the equation remains unchanged under this group action.

For example, suppose that a given equation $F(u) = 0$ has the left-right mirror symmetry. This means that the exchange of left and right does not affect the equation. In other words, if we observe a mirror image of what is happening under the operation of F , nothing looks different from the way F operates in the original world. More precisely, if we denote the left-right reflection by ρ , then $\rho F(u)$ (the mirror image of F operating on u) is the same as $F(\rho u)$ (F operating on the mirror image of u). Thus our question is formulated as follows:

Suppose that a group G acts on a space X and that a mapping $F: X \rightarrow X$ is G -equivariant, that is, $F \circ g = g \circ F$ for every $g \in G$. Then can we say that solutions of the equation $F(u) = 0$ are G -invariant?

As we mentioned earlier, the answer is generally negative unless we impose additional conditions on the equation or on the solutions. We will henceforth restrict our

attention to solutions that are “stable” in a certain sense and discuss the relation between stability and symmetry, or stability and some kind of monotonicity.

In the area of nonlinear diffusion equations or heat equations, early studies in this direction can be found in Casten-Holland [3] and Matano [11]. Among many other things, they showed that if a bounded domain Ω is rotationally symmetric then any stable equilibrium solution of a semilinear diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad x \in \Omega, t > 0$$

inherits the same symmetry. Later it was discovered that the same result holds in a much more general framework, namely that of “strongly order-preserving systems”. This is a class of dynamical systems for which the comparison principle holds in a certain strong sense, whose concept was introduced in [6], [12] (see also [19], which gives a comprehensive survey on early developments of this theory). Mierczyński-Poláčik [15] (for the time-continuous case) and Takáč [21] (for the time-discrete case) considered strongly order-preserving dynamical systems with a symmetry property associated with a compact connected group G and showed that any stable orbit has a G -invariant ω -limit set. This, in particular, implies that any stable equilibrium point or stable periodic point is G -invariant.

The aim of this article is first to establish a theory analogous to [15] and [21] for a wider class of systems. We will do this in the first part of the present article. To be more precise, we will relax the requirement that the dynamical system be strongly order-preserving. This will allow us to deal with degenerate diffusion equations and equations on an unbounded domain. Secondly, we will relax the requirement that the acting group G be compact. This will allow us to discuss symmetry or monotonicity properties with respect to translation; the results will then be applied to the stability analysis of travelling waves of reaction-diffusion equations and to equations of curvature-dependent motion of surfaces. Much of the material here is a review of our earlier work [17].

In the second part of the present article, we will establish another useful general theorem, which we call the “convergence theorem”. This theorem roughly states that stability implies asymptotic stability. Combining the convergence theorem and monotonicity theorem, one can derive various useful results concerning the stability and monotonicity properties of travelling waves and periodic travelling waves for certain classes of nonlinear diffusion equations with bistable nonlinearity. Some of those results are already known for specific problems (for instance, [2], [4], [20], [22], [23]), but our aim is to treat all those results from a unified point of view. Much of the material here is a review of our earlier work [18] and some recent results.

This article is organized as follows. In Section 2, we present our main theorems: the *monotonicity theorem* and the *convergence theorem*. In Sections 3 and 4, we prove these theorems. Section 5 deals with applications of the monotonicity theorem. Among other things we prove the instability of closed orbits. We also apply the theorem to show rotational symmetry of solutions of elliptic equations and the monotonicity of travelling waves. In Section 6, we apply the convergence theorem to the stability analysis of travelling waves and periodic travelling waves. We will also study periodic growth patterns of certain equations and show that the growth pattern is unique and stable.

2. NOTATION AND MAIN RESULTS

2.1. Time-discrete systems. Let X be an ordered metric space. In other words, X is a metric space on which a closed partial order relation is defined. We will denote by d and $\preceq$ the metric and the order relation in X . Here, we say that a partial order relation in X is *closed* if $u_n \preceq v_n$ ($n = 1, 2, 3, \dots$) implies $\lim_{n \rightarrow \infty} u_n \preceq \lim_{n \rightarrow \infty} v_n$ provided that both limits exist. We also assume that, for any $u, v \in X$, the greatest lower bound of $\{u, v\}$ — denoted by $u \wedge v$ — exists and that $(u, v) \mapsto u \wedge v$ is a continuous mapping from $X \times X$ into X . We write $u \prec v$ if $u \preceq v$ and $u \neq v$. For a subset $Y \subset X$, the expression $u \preceq Y$ (resp. $u \prec Y$, $u \succeq Y$, $u \succ Y$) means $u \preceq v$ (resp. $u \prec v$, $u \succeq v$, $u \succ v$) for all points $v \in Y$.

Let F be a mapping from a subset $D(F) \subset X$ into X with the following properties (F1), (F2), (F3):

- (F1) F is order-preserving (i.e., $u \preceq v$ implies $F(u) \preceq F(v)$ for all $u, v \in D(F)$);
- (F2) F is continuous;
- (F3) any bounded orbit $\{F^k(u)\}_{k=0,1,2,\dots}$ is relatively compact.

In this paper F^n will denote the identity mapping in the case $n = 0$ and the composition mapping $\underbrace{F \circ F \circ \dots \circ F}_{n \text{ times}}$ in the case $n \in \mathbb{N}$, and

$$D(F^n) = \{u \in X \mid F^k(u) \in D(F) \text{ for } k = 1, 2, \dots, n-1\},$$

$$D(F^\infty) = \bigcap_{n=1}^{\infty} D(F^n).$$

The set

$$\omega(u) = \bigcap_{n=1}^{\infty} \overline{\{F^k(u) \mid k \geq n\}}$$

is called the *omega limit set* of u , where $\overline{K}$ denotes the closure of a set K . As is well-known, under condition (F3) $\omega(u)$ is a nonempty compact set provided that the orbit $\{F^k(u)\}_{k=0,1,2,\dots}$ is bounded. Furthermore, by (F2) it is F -invariant, that is, $F(\omega(u)) = \omega(u)$.

Let G be a metrizable topological group acting on X . We say G acts on X if there exists a continuous mapping $\gamma: G \times X \rightarrow X$ such that $g \mapsto \gamma(g, \cdot)$ is a group homomorphism of G into $\text{Hom}(X)$, the group of homeomorphisms of X onto itself. For brevity, we write $\gamma(g, u) = gu$ and identify the element $g \in G$ with its action $\gamma(g, \cdot)$. We assume that

- (G1) γ is order-preserving (that is, $u \preceq v$ implies $gu \preceq gv$ for any $g \in G$);
- (G2) γ commutes with F (that is, $gF(u) = F(gu)$ for any $u \in D(F)$, $g \in G$);
- (G3) G is connected.

We say that an element $u \in X$ is *symmetric* if it is G -invariant, that is, $gu = u$ for all $g \in G$. The set $Gu = \{gu \mid g \in G\}$ is called a *group orbit*. We will denote by e the unit element of G .

An element $u \in X$ is called a *fixed point* of F if $F(u) = u$. In what follows $\bar{u}$ will denote a fixed point of F such that the group orbit $G\bar{u}$ is locally precompact. In our previous paper [17], which studies symmetry and monotonicity properties of fixed points, we have imposed the following condition on $\bar{u}$:

(E) for any fixed point u with $u \prec \bar{u}$ and with $d(u, \bar{u})$ sufficiently small, there exists some neighborhood $B(e) \subset G$ of e such that $u \prec g\bar{u}$ for any $g \in B(e)$.

In the present paper we will impose a slightly stronger version of this condition to prove the convergence theorem :

(E_ω) for any point u with $\omega(u) \prec h\bar{u}$ (resp. $\omega(u) \succ h\bar{u}$) for some $h \in G$ and $d(u, \bar{u})$ sufficiently small, there exists some neighborhood $B(e) \subset G$ of e such that $\omega(u) \prec gh\bar{u}$ (resp. $\omega(u) \succ gh\bar{u}$) for any $g \in B(e)$.

Clearly condition (E_ω) implies condition (E) since $\omega(u) = \{u\}$ if u is a fixed point. In various applications which we will discuss in subsequent sections, both conditions (E) and (E_ω) can be verified by using the maximum principle.

Remark 2.1. In the case where the mapping F is strongly order-preserving, (E_ω) and hence (E) are automatically fulfilled. Here a mapping F is called *strongly order-preserving* if $u \prec v$ implies $F(\tilde{u}) \prec F(\tilde{v})$ for any $\tilde{u}, \tilde{v}$ that are sufficiently close to u, v , respectively ([12], [19]). To derive (E_ω), note that the strongly order-preserving property and $\omega(u) \prec h\bar{u}$ imply $F(F^k(u)) \prec F(gh\bar{u}) = gh\bar{u}$ for sufficiently large k and any $g \in G$ sufficiently close to e . It follows that $F^{k+1}(u) \prec gh\bar{u}$ for all large k , hence $\omega(u) \preceq gh\bar{u}$. Considering that $\omega(u)$ is compact and that $h\bar{u} \notin \omega(u)$, we see that $\omega(u) \prec gh\bar{u}$ if g is sufficiently close to e .

Definition 2.2. A fixed point $\bar{u} \in X$ of F is called *stable* if, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$d(v, \bar{u}) < \delta \implies v \in D(F^\infty), \quad d(F^n(v), \bar{u}) < \varepsilon \text{ for any } n = 1, 2, 3, \dots$$

It is called *G-stable* if, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$d(v, \bar{u}) < \delta \implies v \in D(F^\infty), \quad d(F^n(v), G\bar{u}) < \varepsilon \text{ for any } n = 1, 2, 3, \dots$$

Needless to say, stability implies *G-stability*. It follows from (G2) that if $\bar{u}$ is a stable fixed point of F then so are all points in $G\bar{u}$.

In our previous paper ([17]) we have obtained the following result :

Theorem 2.3. (monotonicity theorem [17, Theorem B]) *Let $\bar{u}$ be a G-stable fixed point satisfying condition (E). Then either of the following alternatives holds :*

- (a) $G\bar{u} = \{\bar{u}\}$, that is, $\bar{u}$ is symmetric.
- (b) $G\bar{u} \simeq \mathbb{R}$, or, more precisely, $G\bar{u}$ is a totally ordered set that is homeomorphic and order-isomorphic to $\mathbb{R}$.

If G is a compact group, such as the group of rotations, then $G\bar{u}$ is a compact set, therefore the case (b) in the above theorem never occurs. Thus we obtain the following corollary, which recovers the results of [15] and [21] in the case of stationary problems:

Corollary. *Let $\bar{u}$ be a stable (or G-stable) fixed point, and assume that G is a compact group. Then $\bar{u}$ is G-invariant, that is, $\bar{u}$ is symmetric.*

As we have seen in [17], this result implies, among other things, that any orbitally stable travelling waves or periodic travelling waves are monotone in x and/or t (see Section 5 of the present paper).

In this paper we present another general result which is exceedingly useful in many applications:

Theorem 2.4. (convergence theorem [18, Theorem 2.4]) *Let $\bar{u}$ be a stable fixed point satisfying condition (E_ω) and $G\bar{u} \neq \{\bar{u}\}$. Then there exists some $\delta > 0$ such that if $u \in X$ satisfies $d(u, \bar{u}) < \delta$ then $\omega(u) = \{g\bar{u}\}$ for some $g \in G$. In other words, $\lim_{n \rightarrow \infty} F^n(u) = g\bar{u}$.*

Remark 2.5. As will be clear from the proof of Theorems 2.3 and 2.4, the group G need not act on the whole space X ; it only needs to act on the set of fixed points of F provided that all points in $G\bar{u}$ are known to be stable fixed points. This will allow us much flexibility in the choice of group G .

Remark 2.6. Theorem 2.3 remains true if we replace condition (F3) by:

(F4) for any bounded monotone decreasing orbit $\{F^k(u)\}_{k=0,1,2,\dots}$ there exists some fixed point v of F and a universal constant $C > 0$ such that

$$v \preceq F^k(u) \quad \text{for any } k = 0, 1, 2, \dots, \quad d(v, u) \leq C \limsup_{k \rightarrow \infty} d(F^k(u), u).$$

Condition (F4) (or $(\Phi 4)$ which will be defined later) is fulfilled if a bounded decreasing orbit is known to converge in an appropriate weak sense.

2.2. Time-continuous systems. With minor modifications, Theorems 2.3 and 2.4 carry over to time-continuous systems. To be more precise, let $\{\Phi_t\}_{t \in [0, \infty)}$ be a family of mappings Φ_t from a subset $D(\Phi_t) \subset X$ to X that satisfies the following semigroup property:

$$\begin{aligned} D(\Phi_t) \text{ is monotone non-increasing in } t, \text{ and } D(\Phi_0) &= X, \\ \Phi_0(u) &= u \quad \text{for all } u \in X, \\ \Phi_{t_1} \circ \Phi_{t_2} &= \Phi_{t_1+t_2} \quad \text{for any } t_1, t_2 \in [0, \infty). \end{aligned}$$

We assume that

- ($\Phi 1$) Φ_t is order-preserving for each $t \in [0, \infty)$;
- ($\Phi 2$) $\Phi_t(u)$ is continuous in u for each $t \in [0, \infty)$;
- ($\Phi 3$) any bounded orbit $\{\Phi_t(u)\}_{t \in [0, \infty)}$ is relatively compact,

and that the group G satisfies (G1), (G3) and

- (G2') γ commutes with Φ_t for each $t \in [0, \infty)$ (that is, $g\Phi_t(u) = \Phi_t(gu)$ for each $g \in G, u \in D(\Phi_t), t \in [0, \infty)$).

The set

$$\omega(u) = \bigcap_{s \in (0, \infty)} \overline{\{\Phi_t(u) \mid t \in [s, \infty)\}}$$

is called the *omega limit set* of u . Under conditions ($\Phi 2$), ($\Phi 3$), an omega limit set $\omega(u)$ is a nonempty compact set that is Φ_t -invariant for all $t > 0$, provided that the orbit $\{\Phi_t(u)\}_{t \in [0, \infty)}$ is bounded.

A point $u \in X$ is called an *equilibrium point* if it satisfies $\Phi_t(u) = u$ for all $t \in [0, \infty)$. In the rest of this section $\bar{u}$ will denote an equilibrium point such that $G\bar{u}$ is locally precompact. We impose either of the following conditions on $\bar{u}$:

- (E') for any equilibrium point u with $u \prec \bar{u}$ and with $d(u, \bar{u})$ sufficiently small, there exists some neighborhood $B(e) \subset G$ of e such that $u \prec g\bar{u}$ for any $g \in B(e)$.
 (E' _{ω}) for any point u with $\omega(u) \prec h\bar{u}$ (resp. $\omega(u) \succ h\bar{u}$) for some $h \in G$ and $d(u, \bar{u})$ sufficiently small, there exists some neighborhood $B(e) \subset G$ of e such that $\omega(u) \prec gh\bar{u}$ (resp. $\omega(u) \succ gh\bar{u}$) for any $g \in B(e)$.

Clearly (E' _{ω}) implies (E') since $\omega(u) = \{u\}$ if u is an equilibrium point.

Remark 2.7. A semigroup $\{\Phi_t\}_{t \in [0, \infty)}$ is called *strongly order-preserving* if the map Φ_t is strongly order-preserving (see Remark 2.1) for every $t > 0$. It is easily seen that if $\{\Phi_t\}_{t \in [0, \infty)}$ is strongly order-preserving then any equilibrium point $\bar{u}$ satisfies (E') and (E' _{ω}). The converse is not true.

As in Definition 2.2, an equilibrium point $u \in X$ of $\{\Phi_t\}_{t \in [0, \infty)}$ is called *stable* if, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$d(v, u) < \delta \implies v \in D(\Phi_\infty), \quad d(\Phi_t(v), u) < \varepsilon \text{ for any } t \in [0, \infty),$$

where $D(\Phi_\infty) = \bigcap_{t \in [0, \infty)} D(\Phi_t)$. It is called *G-stable* if, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$d(v, u) < \delta \implies v \in D(\Phi_\infty), \quad d(\Phi_t(v), Gu) < \varepsilon \text{ for any } t \in [0, \infty).$$

The following are time-continuous versions of Theorems 2.3 and 2.4:

Theorem 2.8. (monotonicity theorem [17, Theorem B']) *Let $\bar{u}$ be a G-stable equilibrium point satisfying (E'). Then either of the following alternatives holds:*

- (a) $G\bar{u} = \{\bar{u}\}$, that is, $\bar{u}$ is symmetric.
- (b) $G\bar{u} \simeq \mathbb{R}$, or, more precisely, there exists an order-preserving homeomorphism from $G\bar{u}$ onto $\mathbb{R}$.

Corollary. *Let $\bar{u}$ be a stable (or G-stable) fixed point, and assume that G is a compact group. Then $\bar{u}$ is G-invariant, that is, $\bar{u}$ is symmetric.*

Theorem 2.9. (convergence theorem) [18, Theorem 2.10] *Let $\bar{u}$ be a stable equilibrium point satisfying condition (E' _{ω}) and $G\bar{u} \neq \{\bar{u}\}$. Then there exists some $\delta > 0$ such that if $u \in X$ satisfies $d(u, \bar{u}) < \delta$ then $\omega(u) = \{g\bar{u}\}$ for some $g \in G$. In other words, $\lim_{t \rightarrow \infty} \Phi_t(u) = g\bar{u}$.*

The same remarks as Remarks 2.5 also applies to the time-continuous systems. As we noted in Remark 2.6, Theorem 2.9 holds if we replace (Φ3) by

- (Φ4) for any bounded monotone decreasing orbit $\{\Phi_t(u)\}_{t \in [0, \infty)}$ there exists some equilibrium point v and a universal constant $C > 0$ such that

$$v \preceq \Phi_t(u) \quad \text{for any } t \in [0, \infty), \quad d(u, v) \leq C \limsup_{t \rightarrow \infty} d(u, \Phi_t(u)).$$

3. PROOF OF THE MONOTONICITY THEOREM

In this section we prove the monotonicity theorems. Since the time-continuous case (Theorem 2.9) can be treated with minor modification, we will only prove Theorem 2.3.

We begin with the following proposition:

Proposition 3.1. *One of the following holds:*

- (a) $G\bar{u} = \{\bar{u}\}$;
- (b) $G\bar{u}$ is a totally ordered set and has no maximum nor minimum;
- (c) $G\bar{u} \neq \{\bar{u}\}$, and no pair of points $w_1, w_2 \in G\bar{u}$ satisfy $w_1 \prec w_2$ or $w_1 \succ w_2$.

In the case (c), any fixed point v with $v \prec \bar{u}$ satisfies $Gv \prec \bar{u}$.

To prove the above proposition, we need some lemmas.

Lemma 3.2. *Define*

$$\begin{aligned} G_0 &= \{g \in G \mid g\bar{u} = \bar{u}\}, \\ G_{\pm} &= \{g \in G \mid g\bar{u} \prec \bar{u} \text{ or } g\bar{u} \succ \bar{u}\}, \\ G_* &= \{g \in G \mid g\bar{u} \not\leq \bar{u} \text{ and } g\bar{u} \not\geq \bar{u}\}. \end{aligned}$$

Then the subset G_0 is closed, $G_{\pm}$ and G_ are open.*

Proof. From the definition it is easily seen that G_0 is a closed subset and G_* an open subset of G . Moreover condition (E) implies that $G_{\pm}$ is also open. $\square$

Lemma 3.3. *Let $G_0, G_{\pm}$ and G_* be as in Lemma 4.1. Then one of the following holds:*

- (a) $G = G_0$;
- (b) $G_{\pm} \neq \emptyset$ and $G = G_0 \cup G_{\pm}$ with $G_0 = \partial G_{\pm}$;
- (c) $G_* \neq \emptyset$ and $G = G_0 \cup G_*$ with $G_0 = \partial G_*$.

In the last case, any fixed point v with $v \prec \bar{u}$ satisfies $Gv \prec \bar{u}$.

Proof. $G_0, G_{\pm}, G_*$ are mutually disjoint and

$$G = G_0 \cup G_{\pm} \cup G_*.$$

We first assume that $G_* \neq \emptyset$. Then by the connectedness of G we have

$$\partial G_* \neq \emptyset.$$

Since both $G_{\pm}$ and G_* are open, we have $\partial G_* \subset G_0$. This means that there exists an element $h_0 \in \partial G_* \cap G_0$. Now let g_0 be any element of G_0 and $B(g_0)$ be any neighborhood of g_0 . Then, since $h_0 g_0^{-1} B(g_0) = \{h_0 g_0^{-1} g \mid g \in B(g_0)\}$ is a neighborhood of h_0 , we have

$$h_0 g_0^{-1} B(g_0) \cap G_* \neq \emptyset.$$

It follows from this and $g_0 h_0^{-1} \in G_0$ that

$$B(g_0) \cap G_* \neq \emptyset.$$

This shows that $\partial G_* = G_0$. Now let v be any fixed point of F satisfying $v \prec \bar{u}$. By condition (E), it holds that $v \prec g_* \bar{u}$ for some $g_* \in G_*$. Define

$$A = \{g \in G \mid gv \preceq g_* \bar{u}, \quad gv \preceq \bar{u}\}, \quad A_0 = \{h \in G \mid v \preceq h \bar{u}\}.$$

Since $h \mapsto h\bar{u}: G \rightarrow X$ is continuous, A_0 is a closed subset of G . In view of this and the identity $A = A_0^{-1} \cap g_* A_0^{-1}$, we see that A is closed. (Here A_0^{-1} stands for the set $\{g^{-1} \mid g \in A_0\}$.) On the other hand, since $g_* \bar{u}$ and $\bar{u}$ are order-unrelated, neither of the equality signs in the condition $gv \preceq g_* \bar{u}$, $gv \preceq \bar{u}$ can hold. Therefore

$$A = \{g \in G \mid gv \prec g_* \bar{u}, \quad gv \prec \bar{u}\}.$$

It follows from this and (E) that A is also open. Thus by the connectedness of G we have $A = G$, hence

$$Gv \prec \bar{u}.$$

This proves the last statement of the lemma. We next show that $G_{\pm} = \emptyset$. Suppose that $G_* \neq \emptyset$ and that there exists an element $g \in G_{\pm}$. By replacing g by g^{-1} if necessary, we may assume that $g\bar{u} \prec \bar{u}$. Applying the above result to $v = g\bar{u}$, we see that $Gv \prec \bar{u}$ holds. But this is impossible since

$$Gv = Gg\bar{u} = G\bar{u} \ni \bar{u}.$$

This contradiction shows $G_{\pm} = \emptyset$, verifying case (c).

Next we assume that $G_{\pm} \neq \emptyset$. Then it follows from statement (c) that $G = G_0 \cup G_{\pm}$. The assertion $G_0 = \partial G_{\pm}$ can be shown in the same manner as in (c). The lemma is proved. $\square$

Lemma 3.4. *The maximum of $G\bar{u}$ exists if and only if $G\bar{u} = \{\bar{u}\}$. The same is true for the minimum.*

Proof. Suppose that $g_0\bar{u}$ is the maximum of $G\bar{u}$. Then

$$g\bar{u} \preceq g_0\bar{u} \quad \text{for any } g \in G.$$

In particular, $g_0^2\bar{u} \preceq g_0\bar{u}$, hence

$$g_0\bar{u} = g_0^{-1}(g_0^2\bar{u}) \preceq g_0^{-1}(g_0\bar{u}) = \bar{u} \preceq g_0\bar{u}.$$

This shows that $g_0\bar{u} = \bar{u}$, therefore $\bar{u}$ is the maximum of $G\bar{u}$. Consequently

$$g^{-1}\bar{u} \preceq \bar{u} \quad \text{for any } g \in G,$$

hence

$$\bar{u} = g(g^{-1}\bar{u}) \preceq g\bar{u} \preceq \bar{u}.$$

This implies that $G\bar{u} = \{\bar{u}\}$. The same argument applies if $G\bar{u}$ has the minimum. The lemma is proved. $\square$

Proof of Proposition 3.1. Let $G_0, G_{\pm}, G_*$ be as in Lemma 4.1. Suppose that there exist $g_1, g_2 \in G$ such that

$$g_1\bar{u} \succ g_2\bar{u}.$$

Then $\bar{u} \succ g_1^{-1}g_2\bar{u}$, hence $g_1^{-1}g_2 \in G_{\pm}$. Therefore the existence of a strictly ordered pair of points $w_1 \succ w_2$ in $G\bar{u}$ is equivalent to the condition $G_{\pm} \neq \emptyset$. In view of this and Lemma 4.2, we find that $G_* \neq \emptyset$ implies case (c) in Proposition 3.1. The last statement of the proposition also follows from Lemma 4.2. On the other hand, if $G_* = \emptyset$, then $G\bar{u}$ is clearly a totally ordered set. The alternatives (a), (b) now follows immediately from Lemma 3.4. $\square$

To prove Theorem 2.3, we further need the following lemma:

Lemma 3.5. *Let $u \in D(F^\infty)$ satisfy $F(u) \preceq u$, and assume that the sequence $\{F^n(u)\}_{n=0,1,2,\dots}$ is bounded in X . Then $F^n(u)$ converges to some point $v \in X$ as $n \rightarrow \infty$. If $v \in D(F)$, then v is a fixed point of F .*

Proof. Since assumption (F1) and $F(u) \preceq u$ imply

$$u \succeq F(u) \succeq F^2(u) \succeq F^3(u) \succeq \dots,$$

it follows from (F3) that the sequence $\{F^n(u)\}_{n=0,1,2,\dots}$ converges as $n \rightarrow \infty$ to a point, say v .

Next assume that $v \in D(F)$. Then, by (F2) we have

$$F(v) = F(\lim_{n \rightarrow \infty} F^n(u)) = \lim_{n \rightarrow \infty} F^{n+1}(u) = v.$$

Hence v is a fixed point of F . □

Proof of Theorem 2.3

Step 1 We first prove that the case (c) in Proposition 3.1 does not hold under the stronger assumption that $\bar{u}$ is stable instead of G -stable. Supposing that case (c) in Proposition 3.1 holds, we will derive a contradiction. Since G is connected, the set $G\bar{u} \subset X$ is connected. From this fact and $G\bar{u} \ni \bar{u}$, there exists a sequence $\{g_m\bar{u}\}_{m=1,2,3,\dots} \subset G\bar{u}$ converging to $\bar{u}$ and satisfying $g_m\bar{u} \not\preceq \bar{u}$, $g_m\bar{u} \not\succeq \bar{u}$ for all $m \in \mathbb{N}$. The inequalities

$$g_m\bar{u} \wedge \bar{u} \prec g_m\bar{u}, \quad g_m\bar{u} \wedge \bar{u} \prec \bar{u}$$

and assumption (F1) yield

$$F(g_m\bar{u} \wedge \bar{u}) \preceq F(g_m\bar{u}) \wedge F(\bar{u}) = g_m\bar{u} \wedge \bar{u} \prec \bar{u}.$$

Because of the stability of $\bar{u}$, we can choose $\{g_m\bar{u}\}_{m=1,2,3,\dots}$ such that the closure of the sequence $\{F^n(g_m\bar{u} \wedge \bar{u})\}_{n=1,2,3,\dots}$ is contained in $D(F)$ and is bounded for each $m \in \mathbb{N}$. Then it follows from Lemma 3.5 that $\{F^n(g_m\bar{u} \wedge \bar{u})\}_{n=1,2,3,\dots}$ converges to some fixed point of F , which we will denote by v_m . By the last statement of Proposition 3.1, for any $g \in G$ and $m \in \mathbb{N}$,

$$(3.1) \quad gv_m \prec \bar{u}$$

holds. Since $\bar{u}$ is stable and $g_m\bar{u} \wedge \bar{u}$ converges to $\bar{u}$ as $m \rightarrow \infty$, its ω -limit point v_m converges to $\bar{u}$ as $m \rightarrow \infty$. Letting $m \rightarrow \infty$ in (3.1), we obtain

$$g\bar{u} \preceq \bar{u} \quad \text{for all } g \in G,$$

which contradicts our assumption that (c) holds. Thus either (a) or (b) in Proposition 3.1 holds.

Step 2 Next we show the same result as above under the assumption that $\bar{u}$ is simply G -stable. Supposing case (c) in Proposition 3.1, we will derive a contradiction. Let $\{g_m\bar{u}\}_{m=1,2,3,\dots}$ be as in Step 1. Put $u_m = g_m\bar{u} \wedge \bar{u}$. If

$$\liminf_{m \rightarrow \infty} \sup_k d(F^k(u_m), \bar{u}) = 0,$$

then repeating the same argument as Step 1, we obtain a contradiction. Thus we only need to consider the case where there exists an $\varepsilon_0 > 0$ such that

$$\sup_k d(F^k(u_m), \bar{u}) > \varepsilon_0 \quad \text{for } m = 1, 2, 3, \dots.$$

Since the mapping F is continuous, if we choose a $\delta_0 \in (0, \varepsilon_0)$ sufficiently small then

$$d(w, \bar{u}) < \delta_0 \quad \text{implies} \quad d(F(w), \bar{u}) < \varepsilon_0.$$

By taking a subsequence if necessary we may assume without loss of generality that $d(u_m, \bar{u}) < \delta_0$. For each m we set

$$k(m) = \min\{k \in \mathbb{N} \mid d(F^k(u_m), \bar{u}) > \delta_0\},$$

$$w_m = F^{k(m)}(u_m).$$

Then

$$(3.2) \quad w_m \prec \bar{u}, \quad \delta_0 < d(w_m, \bar{u}) < \varepsilon_0.$$

Since $\bar{u}$ is G -stable, $d(w_m, G\bar{u}) \rightarrow 0$ as $m \rightarrow \infty$. Hence there exists some $h_m \in G$ such that

$$(3.3) \quad d(w_m, h_m \bar{u}) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

It follows from (3.2) and (3.3) that $\{h_m \bar{u}\}_{m=1,2,3,\dots}$ is bounded. By the local precompactness of $G\bar{u}$, there exists a subsequence $\{h_{m_j} \bar{u}\}_{j=1,2,3,\dots}$ that converges to some point z_{ε_0} . From this and (3.3), we see that $\{w_{m_j}\}_{j=1,2,3,\dots}$ also converges to z_{ε_0} . Letting $m_j \rightarrow \infty$ in (3.2), we get

$$(3.4) \quad z_{\varepsilon_0} \prec \bar{u}, \quad \delta_0 < d(z_{\varepsilon_0}, \bar{u}) < \varepsilon_0.$$

Furthermore, since each $h_{m_j} \bar{u}$ is a fixed point of F and since F is continuous, the limit z_{ε_0} is also a fixed point. Hence by the last statement of Proposition 3.1, it holds that

$$Gz_{\varepsilon_0} \prec \bar{u}.$$

Combining this with (3.4) and letting $\varepsilon_0 \rightarrow 0$, we get $G\bar{u} \preceq \bar{u}$, and equivalently $G\bar{u} \succeq \bar{u}$. Thus $G\bar{u} = \bar{u}$, yielding a contradiction. Therefore either (a) or (b) in Proposition 3.1 must hold.

Step 3 The conclusion (a) of this theorem follows from (a) in Proposition 3.1. The conclusion (b) follows from (b) in Proposition 3.1 and Proposition Y2 in [17], which we state below without proof. The proof of the theorem is completed. $\square$

Lemma 3.6. ([17, Prop. Y2]) *Let Y be a totally ordered connected subset of X and suppose that Y is locally precompact (that is, $\bar{Y}$ is locally compact) and that Y has neither the maximum nor the minimum; more precisely suppose that for any $x \in Y$ there exist points $y, z \in Y$ satisfying $y \prec x \prec z$. Then Y is homeomorphic and order-isomorphic to $\mathbb{R}$.*

4. PROOF OF THE CONVERGENCE THEOREM

In this section we prove Theorem 2.4. As the proof of Theorem 2.10 is almost identical to that of Theorem 2.4, we omit its proof. In what follows $\bar{u}$ will denote a fixed point of F satisfying (E_ω) .

Lemma 4.1. *Let $\bar{u}$ be stable. If $\omega(u) \cap G\bar{u} \neq \emptyset$ then $\omega(u) = \{g\bar{u}\}$ for some $g \in G$.*

Proof. Since $\bar{u}$ is a stable fixed point, so is every point in $G\bar{u}$. It is also easy to see that if $\omega(u)$ contains a stable fixed point, say x , then $\omega(u) = \{x\}$. The conclusion of the lemma now follows immediately. $\square$

Lemma 4.2. *Under the condition of Theorem 2.4 there exists some neighborhood U of $\bar{u}$ such that, if $u \in U$ satisfies $\omega(u) \preceq g_1\bar{u}$ or $\omega(u) \succeq g_1\bar{u}$ for some $g_1 \in G$, then $\omega(u) = \{g_2\bar{u}\}$ for some $g_2 \in G$.*

Proof. Let V be a neighborhood of $\bar{u}$ such that condition (E_ω) holds for all $u \in V$. Suppose that a point $u \in V$ satisfies $\omega(u) \preceq g_1\bar{u}$ for some $g_1 \in G$ and

$$(4.1) \quad \omega(u) \neq \{g\bar{u}\} \quad \text{for any } g \in G.$$

Then by Lemma 4.1 we have

$$(4.2) \quad \omega(u) \cap G\bar{u} = \emptyset.$$

Define $A = \{g \in G \mid \omega(u) \preceq g\bar{u}\}$. Clearly A is a closed subset of G and is nonempty since $g_1 \in A$. Furthermore, (4.2) implies $A = \{g \in G \mid \omega(u) \prec g\bar{u}\}$. Hence from condition (E_ω) we see that A is also open. Since G is connected, we have $A = G$, that is, $\omega(u) \preceq g\bar{u}$ for any $g \in G$. Similarly, if a point $u \in V$ satisfies $\omega(u) \succeq g_1\bar{u}$ for some $g_1 \in G$ together with (4.1), then $\omega(u) \succeq g\bar{u}$ for any $g \in G$.

Now suppose the conclusion of the lemma does not hold. Then, in view of the above argument, there exists some sequence $\{u_m\} \subset V$ converging to $\bar{u}$ such that

$$\omega(u_m) \preceq g\bar{u} \quad \text{for any } g \in G \quad \text{or} \quad \omega(u_m) \succeq g\bar{u} \quad \text{for any } g \in G$$

for $m = 1, 2, 3, \dots$. Without loss of generality we may assume that the former holds for all m . Since $\bar{u}$ is stable, $\omega(u_m) \rightarrow \{\bar{u}\}$ as $m \rightarrow \infty$ in the Hausdorff metric. Thus, letting $m \rightarrow \infty$ in the above inequality yields

$$\bar{u} \preceq g\bar{u}, \quad g \in G.$$

Replacing g with g^{-1} and applying g on both sides, we get

$$g\bar{u} \preceq \bar{u},$$

hence $g\bar{u} = \bar{u}$ for all $g \in G$. This, however, contradicts the assumption that $G\bar{u} \neq \{\bar{u}\}$. The proof is complete. $\square$

Proof of Theorem 2.4. Let U be as in Lemma 4.2 and take a neighborhood W of $\bar{u}$ such that $W \subset U$ and that $u \wedge \bar{u} \in U$ for all $u \in W$. Clearly

$$(4.3) \quad u \wedge \bar{u} \preceq u \quad \text{and} \quad u \wedge \bar{u} \preceq \bar{u}.$$

Since the latter inequality implies $\omega(u \wedge \bar{u}) \preceq \bar{u}$, it follows from Lemma 4.2 that $\omega(u \wedge \bar{u}) = \{g_*\bar{u}\}$ for some $g_* \in G$. Therefore, by the former inequality of (4.3), we get $g_*\bar{u} \preceq \omega(u)$. Applying Lemma 4.2 again, we see that $\omega(u) = \{g\bar{u}\}$ for some $g \in G$. $\square$

5. APPLICATIONS OF THE MONOTONICITY THEOREM

5.1. Instability of closed orbits. In this subsection we prove that closed orbits (periodic motion) of order-preserving systems are always unstable. This result is first proved by Hirsch [6] by using his celebrated "almost everywhere quasi-convergence theorem". Our proof here is different, and is simpler.

Let $\{\Phi_t\}_{t \in [0, \infty)}$ be a semigroup of mappings as in Section 3. We assume that $\Phi_t(u)$ is continuous in t as well as in u . Such a semigroup of mappings is called a *local semiflow* on X . It is called a *semiflow* if we further have $D(\Phi_t) = X$ for every $t \geq 0$. By definition, any local semiflow satisfies $(\Phi 2)$.

An orbit $O^+(u) = \{\Phi_t(u) \mid t \in [0, \infty)\}$ is called a *periodic orbit* if there exists a $\tau > 0$ such that $\Phi_\tau(u) = u$. In this case the point u is called a *periodic point* or, more precisely, a τ -*periodic point*. Note that the quantity τ need not be the minimal period in this definition. A periodic orbit $O^+(u)$ is called a *closed orbit* if u is not an equilibrium point.

Definition 5.1. A closed orbit $O^+(u) = \{\Phi_t(u) \mid t \in [0, \infty)\}$ is called *orbitally stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(v, O^+(u)) < \delta \implies d(\Phi_t(v), O^+(u)) < \varepsilon \text{ for any } t \in [0, \infty).$$

It is called *stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(v, u) < \delta \implies d(\Phi_t(v), \Phi_t(u)) < \varepsilon \text{ for any } t \in [0, \infty).$$

Clearly stability implies orbital stability.

We consider local semiflows satisfying the following conditions:

(P) for any $\tau > 0$ and any τ -periodic points $u, v \in X$ satisfying $u \prec v$, there exists a $\delta > 0$ such that

$$\Phi_t(u) \prec \Phi_s(v) \quad \text{for any } t, s \in [0, \delta].$$

We are now ready to state our main result of this section:

Theorem 5.2. Let $\{\Phi_t\}_{t \in [0, \infty)}$ be a local semiflow satisfying conditions $(\Phi 1)$, $(\Phi 3)$ in Section 3 and condition **(P)** above. Then any closed orbit is orbitally unstable (hence unstable).

Proof. Let $O^+(\bar{u})$ be an orbitally stable closed orbit with period τ . Denote by $\mathcal{P}_\tau$ the set of all τ -periodic points of the semigroup $\{\Phi_t\}_{t \in [0, \infty)}$ and let $F = \Phi_\tau$. Then $\{F^n\}_{n=0,1,2,\dots}$ defines a discrete semigroup on X , and $\mathcal{P}_\tau$ coincides with the set of fixed points of F . It is easily seen that conditions (F1), (F2), (F3) in Section 2 are all fulfilled. Furthermore, since each $u \in \mathcal{P}_\tau$ is a periodic point of $\{\Phi_t\}_{t \in [0, \infty)}$, $\Phi_t(u)$ can be defined for all $t \in \mathbb{R}$ and we clearly have $\Phi_t(\mathcal{P}_\tau) = \mathcal{P}_\tau$ for any $t \in \mathbb{R}$. Thus $\{\Phi_t\}_{t \in [0, \infty)}$ is extended to a one-parameter group acting on $\mathcal{P}_\tau$. Denote this group by G . Then conditions (G1), (G2), (G3) can easily be checked. Conditions **(P)** and $(\Phi 1)$ imply condition (E') . Furthermore $\bar{u}$ is a G -stable fixed point of F such that $G\bar{u} = O^+(\bar{u})$ is a compact subset of X . Applying Theorem B and Remarks 2.5, we see that either of the following holds:

$$(a) \quad G\bar{u} = \{\bar{u}\}; \quad (b) \quad G\bar{u} \simeq \mathbb{R}.$$

Since $G\bar{u}$ is compact, case (b) is excluded. This means that $\bar{u}$ is an equilibrium point of the semigroup $\{\Phi_t\}_{t \in [0, \infty)}$, contradicting the assumption that $O^+(\bar{u})$ is a closed orbit. The theorem is proved. $\square$

Example. The above Theorem applies, for example, to semilinear parabolic equations of the form

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + f(x, u, \nabla u), & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where Ω is a domain in $\mathbb{R}^N$. This result has been known if Ω is a bounded domain, but our theorem also covers the case where Ω is unbounded, provided that $\partial_u f(x, 0, 0) \leq -\eta$ ($x \in \Omega$) for some $\eta > 0$.

5.2. Various other applications. There are many other applications of the monotonicity theorem. Let us list a few.

Rotational symmetry in PDE: We can apply the monotonicity theorem to show the rotational symmetry of stable equilibrium solutions of an initial boundary value problem for a nonlinear parabolic equation of the form

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad x \in \Omega, t > 0,$$

where $\Omega \subset \mathbb{R}^N$ is a rotationally symmetric domain that is not necessarily bounded. This generalizes the result of Casten–Holland [3] and Matano [11] considerably. In this problem, we choose G to be the group of rotations.

Monotonicity of travelling waves: we apply our theory to so-called travelling waves for an equation of the form

$$(5.1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f\left(u, \frac{\partial u}{\partial x}\right). \quad x \in \mathbb{R}, t > 0,$$

A nonconstant solution $\tilde{u}(x, t)$ is called a travelling wave if it is written in the form

$$\tilde{u}(x, t) = v(x - ct)$$

for some constant $c \in \mathbb{R}$, which represents the speed of the travelling wave. The function $v(z)$ is called the *profile* of the travelling wave and satisfies the equation

$$v'' + cv' + f(v, v') = 0.$$

Here we deal with travelling waves whose limiting values

$$\lim_{z \rightarrow \pm\infty} v(z) = u^\pm$$

are both stable zeros of $f(u, 0)$. In order to apply our theory to study (5.1), we rewrite the equation in the moving coordinates $z = x - ct$, to obtain

$$(5.2) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + c \frac{\partial u}{\partial z} + f\left(u, \frac{\partial u}{\partial z}\right), \quad z \in \mathbb{R}, t > 0.$$

There is one-to-one correspondence between the equilibrium solutions of (5.2) and the travelling waves of (5.1) with speed c . Let G be the group of translations on $\mathbb{R}$:

$$(5.3) \quad G = \{\sigma_\ell \mid \ell \in \mathbb{R}\} \simeq \mathbb{R}, \quad \text{where} \quad \sigma_\ell: u(z) \mapsto u(z - \ell).$$

Thus, given any equilibrium solution $v(z)$ of (5.2), its G -orbit is expressed as

$$Gv = \{\sigma_\ell v \mid \ell \in \mathbb{R}\}.$$

Then Theorem 2.9 implies that if Gv is stable, then it is a totally ordered set. This means that, for any $\ell \in \mathbb{R}$, we have

$$\text{either } v(x - \ell) \leq v(x) \quad (x \in \mathbb{R}) \quad \text{or} \quad v(x - \ell) \geq v(x) \quad (x \in \mathbb{R}).$$

In other words, $v(z)$ is a monotone function. Consequently, any stable (or orbitally stable) travelling wave is monotone both in x and t .

The same argument applies to a system of equations of the form

$$(5.4) \quad \begin{cases} \frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + f_1(u_1, \dots, u_m), \\ \vdots \\ \frac{\partial u_m}{\partial t} = d_m \frac{\partial^2 u_m}{\partial x^2} + f_m(u_1, \dots, u_m), \end{cases} \quad x \in \mathbb{R}, t > 0,$$

where constants $d_1, \dots, d_m$ are positive and functions $f_1, \dots, f_m$ satisfy certain conditions so that the system is of the cooperation type or of the competition type. (We assume $m = 2$ in the latter case.)

With minor modifications, our results extend to travelling waves for equations in higher space dimensions such as

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(x_1, \dots, x_{N-1}, u), \\ \frac{\partial u}{\partial n} = 0 \end{cases} \quad \begin{matrix} x \in \Omega, t > 0, \\ x \in \partial\Omega, t > 0, \end{matrix}$$

where Ω is a cylindrical domain of the form $\Omega = D \times \mathbb{R}$ with D being a bounded $(N - 1)$ -dimensional domain. A solution $u(x, t)$ is called a travelling wave if it is written in the form

$$u(x, t) = v(x_1, \dots, x_{N-1}, x_N - ct).$$

We consider travelling waves whose limiting profiles

$$\lim_{z_N \rightarrow \pm\infty} v(z_1, \dots, z_{N-1}, z_N) = u^\pm(z_1, \dots, z_{N-1})$$

are stable in a certain sense. We can show that any stable (or orbitally stable) travelling wave is monotone in the axial direction. Moreover these travelling waves inherit the symmetry properties of D provided that its symmetry group is connected. For the monotonicity of the travelling wave, we choose G to be the group of translations along the x_N -axis. For the latter result, we choose G to be the symmetry group for the cross section D .

The above results can be extended to so-called periodic travelling waves, which we will discuss in the next section.

6. APPLICATIONS OF THE CONVERGENCE THEOREM

6.1. Asymptotic stability of travelling waves. In Section 5 we have applied the monotonicity theorem to show that stable travelling waves are monotone either in x or in t . In this subsection we prove the converse of this result in a certain sense. We give only an outline. More details can be found in [18]

Let us first consider an equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, t > 0$$

or a system of equations of the cooperation type or of the competition type in the form (5.4), or an equation with time-delay of the form

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t), u(x, t-1)), \quad x \in \mathbb{R}, t > 0.$$

It follows from the monotonicity theorem that any stable travelling wave is monotone. Conversely, it is known that monotone travelling waves for such equations are stable. This can be shown by constructing a suitable pair of super- and subsolutions; see [17] for details. Using this fact and the convergence theorem, one can show that monotone travelling waves are stable with asymptotic phase. This means that any solution whose initial data is somewhat close to the travelling wave will eventually converge to the travelling wave (or its phase-shift) as $t \rightarrow \infty$. More precisely, if $v(x - ct)$ denotes the travelling wave, then

$$\sup_{x \in \mathbb{R}} |u(x, t) - v(x - ct - \alpha)| \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where α is some real number representing a phase shift. In the above problems, we choose G to be the group of translations along $\mathbb{R}$.

6.2. Generalized travelling waves. The above results for travelling waves can be extended to the class of *generalized travelling waves* (or periodic travelling waves) in temporally or spatially inhomogeneous media. More specifically, let us consider an initial value problem for the equation

$$(6.1) \quad \frac{\partial u}{\partial t} = a(t) \frac{\partial^2 u}{\partial x^2} + b(t, u) \frac{\partial u}{\partial x} + f(t, u), \quad x \in \mathbb{R}, t > 0,$$

and one for the equation

$$(6.2) \quad \frac{\partial u}{\partial t} = \alpha(x) \frac{\partial^2 u}{\partial x^2} + \beta(x, u) \frac{\partial u}{\partial x} + g(x, u), \quad x \in \mathbb{R}, t > 0,$$

where functions a, b, f are T -periodic with respect to t while α, β, g are L -periodic with respect to x . A nonconstant solution $u(x, t)$ for (6.1) is called a periodic travelling wave if there exists a $\lambda \in \mathbb{R}$ such that

$$u(x, t + T) = u(x - \lambda, t), \quad x \in \mathbb{R}, t \in \mathbb{R},$$

and one for (6.2) is called a periodic travelling wave if

$$u(x, t + \tau) = u(x - L, t), \quad x \in \mathbb{R}, t \in \mathbb{R}$$

for some $\tau \neq 0$. The ratio $c := \lambda/T$ or $c := L/\tau$ is called the *average speed* or the *effective speed* of the periodic travelling wave.

Under suitable conditions, we can show that

- (i) *any stable periodic travelling wave for (6.1) is either monotone increasing in x or monotone decreasing in x ;*
- (ii) *any periodic travelling wave for (6.1) that is monotone in x is stable with asymptotic phase.*

Similarly we have:

- (i') *any stable periodic travelling wave for (6.2) is monotone in t ;*
- (ii') *any periodic travelling wave for (6.2) that is monotone in t is stable with asymptotic phase.*

In the above problems, the travelling waves do not keep a constant profile nor a constant speed. They fluctuate periodically in time. We handle these problems in a discrete-time setting. More precisely, in problem (6.2), we define

$$F(u) := \sigma_{-L} \circ \Phi_\tau(u),$$

where σ is as in (5.3) and Φ_t ($t \geq 0$) is the evolution operator for (6.2). Then $\tilde{u}(x, t)$ is a periodic travelling wave in the above sense if and only if it is a fixed point of F . We then choose G to be the group of time shifts.

In problem (6.1), we fix $t_0 \in \mathbb{R}$ arbitrarily and introduce an evolution operator Ψ_t ($t \geq 0$) by $\Psi_t : u(x, t_0) \mapsto u(x, t_0 + t)$. Then we define

$$F(u) := \sigma_{-\lambda} \circ \Psi_T(u).$$

and choose G to be the group of spatial translations on $\mathbb{R}$.

6.3. Travelling waves for surface motion. Our theory applies also to an evolution equation of $N-1$ dimensional surfaces $\{S(t)\}_{t \geq 0}$ contained in a domain $\Omega \subset \mathbb{R}^N$ and intersecting with $\partial\Omega$ perpendicularly or at a prescribed angle. We can prove uniqueness, monotonicity and asymptotic stability of travelling waves and periodic travelling waves.

Let us explain the outline of our results using some simple examples. First, let Ω be a two-dimensional cylindrical domain of the form $\Omega = \{(x_1, x_2) \mid |x_1| < h, x_2 \in \mathbb{R}\}$ for some $h > 0$ and consider the equation

$$(6.3) \quad V = -\kappa + A(x_1) \quad \text{on } \Gamma(t),$$

where V and κ are, respectively, the normal velocity and the curvature of the time-dependent curve $\Gamma(t)$, and $A(x_1)$ is a smooth function defined on $[-h, h]$. Then our results imply that any smooth travelling wave whose endpoints meet both sides of $\partial\Omega$ with a given contact angle is unique up to translation and asymptotically stable in a certain sense. Furthermore, this travelling wave is monotone in the x_2 -direction, that is, it is expressed in the form of a graph $x_2 = \psi(x_1) + ct$ for some function ψ defined on $[-h, h]$. Our results also apply to the equation

$$(6.4) \quad V = -\kappa + A(x_1, t) \quad \text{on } \Gamma(t),$$

where $A(x_1, t)$ is T -periodic in t . A solution $\{\Gamma(t)\}_{t \geq 0}$ of (6.4) is called a periodic travelling wave if there exists some λ such that $\Gamma(t+T) = \Gamma(t) + \lambda e_2$ for $t \in \mathbb{R}$, where $e_2 = {}^t(0, 1) \in \mathbb{R}^2$. It follows from our general results that a smooth periodic travelling wave of (6.4) is unique up to translation and asymptotically stable in a certain sense. Moreover it is monotone in the x_2 -direction, that is, it is expressed as

a graph $x_2 = \psi(x_1, t)$, where ψ is a function on $[-h, h] \times \mathbb{R}$ satisfying $\psi(x_1, t + T) = \psi(x_1, t) + \lambda$.

Another interesting example is the case where Ω is a periodically undulating cylindrical domain of the form (with $h > 0$ some smooth L -periodic function)

$$\Omega = \{(x_1, x_2) \mid |x_1| < h(x_2), x_2 \in \mathbb{R}\},$$

and the equation is of the form (with $A(x_2)$ a smooth L -periodic function)

$$(6.5) \quad V = -\kappa + A(x_2) \quad \text{on } \Gamma(t).$$

It follows from the convergence theorem (Theorem 2.4) that a smooth periodic travelling wave of (6.5) is unique up to translation and monotone in t . Using this observation, one of the authors is studying homogenization limit of this problem when the period of undulation depends on a small parameter ε as follows: $h_\varepsilon(x_2) := 1 + \varepsilon g(x_2/\varepsilon)$. Details are discussed in the papers [9] and [10].

7. PERIODIC GROWTH PATTERNS

In this section we study a system which gives rise to a periodic growth pattern. We then apply our convergence theorem to show that the growth pattern is unique and stable.

First consider a simple ODE:

$$\frac{du}{dt} = f(u).$$

Here $u = u(t)$ can be interpreted, for example, as the total asset of some individual. In that case, du/dt is the gain per unit time. Or u can be the mass of certain material, such as crystals. In this case, the above equation describe some kind of crystal growth.

If $f(u) \equiv c$ (constant), then u grows linearly at a constant speed: $u(t) = ct + u(0)$. Suppose that the gain is a function of the current asset, and that f (gain) depends on u (asset) periodically:

$$f(u + L) \equiv f(u) \quad \text{for some } L > 0.$$

In this case, we have

$$\begin{aligned} f(u) > 0 \quad (u \in \mathbb{R}) &\implies u(t) \rightarrow \infty \quad (t \rightarrow \infty), \\ f(u) < 0 \quad \text{for some } u &\implies \text{the growth is blocked.} \end{aligned}$$

In the former situation, it is easily seen that the speed of growth fluctuate periodically in time, which we may call *periodic growth*.

Next consider the case where there are many individuals $x_1, \dots, x_m$ and that each individual is subject to its own growth law:

$$(7.1) \quad \frac{du_i}{dt} = f_i(u_i) \quad (i = 1, \dots, m).$$

Here $u_i(t)$ denotes the asset of the i -th individual x_i . As before, we assume

$$(7.2) \quad f_i(u + L) \equiv f_i(u_i) \quad (i = 1, 2, \dots, m).$$

If x_i is in a favorable condition such that $f_i(u) > 0$ for every $u \in \mathbb{R}$, then u_i grows periodically, as we have seen before for the case $m = 1$. On the other hand, if

$$(7.3) \quad f_i(\alpha_i) < 0 \quad \text{for some } \alpha_i,$$

then the growth is blocked.

Now suppose that everybody is in an unfavorable environment, so that (7.3) holds for every $i = 1, 2, \dots, m$. If everybody is acting completely independently, we have the (uncoupled) system (7.1), and nobody can grow. However, quite interestingly, if the individuals are cooperating in some way, and if $\alpha_1, \alpha_2, \dots, \alpha_m$ are not identical (which means the bad period differs from individual to individual), then there are some chances that the total asset can grow without being blocked. To be more precise, consider the following system:

$$(7.4) \quad \frac{du_i}{dt} = f_i(u_i) + g_i(u_1, \dots, u_m) \quad (i = 1, \dots, m).$$

Here g_i represents the effect of cooperation, thus it satisfies

$$\frac{\partial g_i}{\partial u_j} \geq 0 \quad (i \neq j).$$

An example is the linear cooperation

$$g_i = \beta(u_{i+1} - u_i) + \gamma(u_{i-1} - u_i) \quad (u_0 = u_m),$$

which implies an averaging effect among adjacent individuals. We can construct an example of the cooperation (7.4) satisfying (7.3) and yet its solution satisfies

$$(7.5) \quad u_i(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty \quad (i = 1, 2, \dots, m).$$

By using our convergence theorem, we can show the following:

Theorem 7.1. *Let (7.4) be a cooperation system satisfying the condition (7.2) and suppose that there exists at least one initial data for which the solution satisfies (7.5). Then there exists a periodically growing solution $\bar{u}_i(t)$ ($i = 1, \dots, m$) satisfying*

$$\bar{u}_i(t + T) = \bar{u}_i + L \quad (i = 1, \dots, m).$$

Moreover, such a periodically growing solution is unique up to time shift, and is stable with asymptotic phase.

In order to derive Theorem 7.1 from Theorem 2.4, we set

$$F(u) := \Phi_T(u) - L,$$

where Φ_t ($t \geq 0$) is the evolution operator for (7.1), and choose G to be the group of time shifts, as we have done for equation (6.2).

We can extend the above model to the case where the individuals are distributed continuously. The cooperation effect (some sort of averaging) can be expressed by diffusion. In such a case, the model equation can be written as

$$(7.6) \quad \frac{\partial u}{\partial t} = \Delta u + f(x, u) \quad (x \in \Omega, t > 0).$$

The nonlinearity f is assumed to satisfy

$$f(x, u + L) \equiv f(x, u).$$

Then one can prove a result completely analogous to Theorem 7.1. This equation appears also as a model for crystal growth. For further details, see the recent work of Nakamura and Ogiwara [16], in which the existence, uniqueness and stability of periodic growth pattern are established for (7.6). In some cases, their growth pattern exhibits spiraling behavior.

Appendix

In the case where the system is strongly order-preserving, the convergence theorem (Theorems 2.4 and 2.10) can be derived from the following more general convergence theorem. See [1] for a related result.

Theorem A.1. *Let X be an ordered metric space such that any order interval $[x, y]$ is bounded. Let $F : X \rightarrow X$ be a strongly order-preserving compact map. Assume that there exists a set of fixed points M that is totally ordered and connected. Let $v_1 < v_2$ be any points of M . Then for any $w \in [v_1, v_2]$, the orbit $F^n(w)$ converges to some point in $[v_1, v_2] \cap M$ as $n \rightarrow \infty$.*

(Proof) For each $x \in [v_1, v_2]$ we define

$$A(x) := \{y \in M \cap [v_1, v_2] \mid y \geq x\}.$$

Then $A(x) \neq \emptyset$ since $v_2 \in A(x)$. Clearly $A(x)$ is a bounded closed set. Since $F(A(x)) = A(x)$ and since F is a compact map, $A(x)$ is a compact set. Moreover $A(x)$ is totally ordered, since M is totally ordered. Consequently $A(x)$ has the minimal element, which we denote by $\mu(x)$. For each n we have

$$w_n \leq \mu(w_n) \quad \text{where} \quad w_n := F^n(w).$$

Applying F to the above inequality yields $w_{n+1} \leq F(\mu(w_n)) = \mu(w_n)$, which implies $\mu(w_{n+1}) \leq \mu(w_n)$. Consequently $\mu(w) \geq \mu(w_1) \geq \mu(w_2) \geq \dots$. By the compactness of the map F , this monotone sequence is relatively compact, hence the limit

$$\mu_\infty(w) := \lim_{n \rightarrow \infty} \mu(w_n).$$

exists. Now let v be any ω -limit point of the orbit $\{w_n\}_{n=1}^\infty$. Then $w_{n_j} \rightarrow v$ as $j \rightarrow \infty$ for some sequence $n_1 < n_2 < n_3 < \dots \rightarrow \infty$, hence

$$w_{n_j} \leq \mu(w_{n_j}) \quad \text{for } j = 1, 2, 3, \dots$$

Letting $j \rightarrow \infty$ we obtain $v \leq \mu_\infty(w)$. We will show that $v = \mu_\infty(w)$. Suppose the contrary and assume $v < \mu_\infty(w)$. Then by the convergence $w_{n_j} \rightarrow v$ and by the strongly orderpreserving property of F , we have

$$w_{n_j+1} = F(w_{n_j}) < F(v) = v$$

for sufficiently large j and for any $z \in M \cap [v_1, v_2]$ that is sufficiently close to $\mu_\infty(w)$. If $\mu_\infty(w) = v_1$, then we have $v_1 \leq v \leq \mu_\infty(w)$, contradicting our assumption $v < \mu_\infty(w)$. Thus $\mu_\infty(w) > v_1$. Since M is totally ordered and connected, we can choose z to be sufficiently close to $\mu_\infty(w)$ and satisfy $z < \mu_\infty$. It follows that

$$w_{n_j+1} < z < \mu_\infty(w) \leq \mu(w_{n_j+1}),$$

which contradicts the minimality of $\mu(w_{n_j+1})$ in the set $A(w_{n_j+1})$. Therefore $v = \mu_\infty(w)$. Consequently, the ω -limit set of w coincides with $\mu_\infty(w)$, which implies the convergence $w_n \rightarrow \mu_\infty(w)$ as $n \rightarrow \infty$. The theorem is proved. $\square$

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