

ヒルベルト C^* -加群上の Selberg 不等式について

Selberg type inequalities on Hilbert C^* -modules

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1. INTRODUCTION

This paper is based on [15].

We briefly review the Selberg inequality and its generalization in a Hilbert space.

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. The Selberg inequality [2, 17] states that if $y_1, y_2, \dots, y_n$ and x are nonzero vectors in H , then

$$(1) \quad \sum_{i=1}^n \frac{|\langle y_i, x \rangle|^2}{\sum_{j=1}^n |\langle y_j, y_i \rangle|} \leq \|x\|^2.$$

Moreover, Furuta [10] posed conditions enjoying the equality: The equality in (1) holds if and only if $x = \sum_{i=1}^n a_i y_i$ for some scalars $a_1, a_2, \dots, a_n \in \mathbb{C}$ such that for arbitrary $i \neq j$

$$(2) \quad \langle y_i, y_j \rangle = 0 \quad \text{or} \quad |a_i| = |a_j| \quad \text{with} \quad \langle a_i y_i, a_j y_j \rangle \geq 0,$$

also see [7]. Note that the Selberg inequality is simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality. As a matter of fact, if $n = 1$ and $y = y_1$, then we have the Cauchy-Schwarz inequality $|\langle y, x \rangle| \leq \|y\| \|x\|$. If $\{y_i\}$ is an orthonormal system, then we have the Bessel inequality $\sum_{i=1}^n |\langle y_i, x \rangle|^2 \leq \|x\|^2$.

Fujii and Nakamoto [9] showed a refinement of the Selberg inequality (1): If $\langle y, y_i \rangle = 0$ for given nonzero vectors $y_1, \dots, y_n \in H$, then

$$(3) \quad |\langle x, y \rangle|^2 + \sum_{i=1}^n \frac{|\langle x, y_i \rangle|^2}{\sum_{j=1}^n |\langle y_j, y_i \rangle|} \|y\|^2 \leq \|x\|^2 \|y\|^2$$

holds for all $x \in H$. Also, Bombieri [1] showed the following generalization of the Bessel inequality: If $x, y_1, \dots, y_n$ are nonzero vectors in H , then

$$(4) \quad \sum_{i=1}^n |\langle x, y_i \rangle|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \sum_{j=1}^n |\langle y_j, y_i \rangle|.$$

Moreover, Mitrinović, Pecarić and Fink [17, Theorem 5 in pp394] mentioned the following inequality equivalent to Bombieri's type (4): If $x, y_1, \dots, y_n$ are nonzero vectors in H and $a_1, \dots, a_n \in \mathbb{C}$, then

$$(5) \quad \left| \sum_{i=1}^n a_i \langle x, y_i \rangle \right|^2 \leq \|x\|^2 \sum_{i=1}^n |a_i|^2 \sum_{j=1}^n |\langle y_j, y_i \rangle|.$$

In this paper, from a viewpoint of the operator theory, we propose a Selberg type inequality in a Hilbert C^* -module, which is simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality in a Hilbert C^* -module. As applications, we show Hilbert C^* -module versions of Fujii-Nakamoto type (3), Bombieri type (4) and Mitrinović, Pecarić and Fink type (5).

2. PRELIMINARIES

Let $\mathcal{A}$ be a unital C^* -algebra with the unit element e . An element $a \in \mathcal{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For $a \in \mathcal{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathcal{A}$, the operator geometric mean of a and b is defined by

$$a \sharp b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \right)^{\frac{1}{2}} a^{\frac{1}{2}}$$

for invertible a . If a and b are non invertible, then $a \sharp b$ belongs to the double commutant $\mathcal{A}''$ of $\mathcal{A}$ in general. In fact, since $a \sharp b$ satisfies the upper semicontinuity, it follows that $a \sharp b = \lim_{\epsilon \rightarrow +0} (a + \epsilon e) \sharp (b + \epsilon e)$ in the strong operator topology. If $\mathcal{A}$ is monotone complete in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have $a \sharp b \in \mathcal{A}$, see [12]. The operator geometric mean has the symmetric property: $a \sharp b = b \sharp a$. In the case that a and b commute, we have $a \sharp b = \sqrt{ab}$. For more details on the operator geometric mean, see [11, 8].

A complex linear space $\mathcal{X}$ is said to be an inner product $\mathcal{A}$ -module (or a pre-Hilbert $\mathcal{A}$ -module) if $\mathcal{X}$ is a right $\mathcal{A}$ -module together with a C^* -valued map $(x, y) \mapsto \langle x, y \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}$ such that

- (i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle \quad (x, y, z \in \mathcal{X}, \alpha, \beta \in \mathbb{C})$,
- (ii) $\langle x, ya \rangle = \langle x, y \rangle a \quad (x, y \in \mathcal{X}, a \in \mathcal{A})$,
- (iii) $\langle y, x \rangle = \langle x, y \rangle^* \quad (x, y \in \mathcal{X})$,
- (iv) $\langle x, x \rangle \geq 0 \quad (x \in \mathcal{X})$ and if $\langle x, x \rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathcal{A}$ and $\mathcal{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y \rangle = a^* \langle x, y \rangle$ for all $x, y \in \mathcal{X}, a \in \mathcal{A}$. If $\mathcal{X}$ satisfies all conditions for an inner-product $\mathcal{A}$ -module except for the second part of (iv), then we call $\mathcal{X}$ a semi-inner product $\mathcal{A}$ -module.

In this case, we write $\|x\| := \sqrt{\|\langle x, x \rangle\|}$, where the latter norm denotes the C^* -norm of $\mathcal{A}$. If an inner-product $\mathcal{A}$ -module $\mathcal{X}$ is complete with respect to its norm, then $\mathcal{X}$ is called a *Hilbert C^* -module*. In [6], from a viewpoint of operator theory, we presented the following Cauchy-Schwarz inequality in the framework of a semi-inner product C^* -module over a unital C^* -algebra: If $x, y \in \mathcal{X}$ such that the inner product $\langle x, y \rangle$ has a polar decomposition $\langle x, y \rangle = u|\langle x, y \rangle|$ with a partial isometry $u \in \mathcal{A}$, then

$$(6) \quad |\langle x, y \rangle| \leq u^* \langle x, x \rangle u \sharp \langle y, y \rangle.$$

Under the assumption that $\mathcal{X}$ is an inner product $\mathcal{A}$ -module and $\langle y, y \rangle$ is invertible, the equality in (6) holds if and only if $xu = yb$ for some $b \in \mathcal{A}$. As applications of the Cauchy-Schwarz inequality (6), we cite [5, 18].

An element x of a Hilbert C^* -module $\mathcal{X}$ is called nonsingular if the element $\langle x, x \rangle \in \mathcal{A}$ is invertible. The set $\{x_i\} \subset \mathcal{X}$ is called orthonormal if $\langle x_i, x_j \rangle = \delta_{ij}e$. For more details on Hilbert C^* -modules, see [16].

3. MAIN THEOREM

Fiest of all, we show the following Selberg type inequality in a Hilbert C^* -module.

Theorem 1. Let $\mathcal{X}$ be an inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \dots, y_n$ are nonsingular, then

$$(7) \quad \sum_{i=1}^n \langle x, y_i \rangle \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right)^{-1} \langle y_i, x \rangle \leq \langle x, x \rangle.$$

The equality in (7) holds if and only if $x = \sum_{i=1}^n y_i a_i$ for some $a_i \in \mathcal{A}$ and $i = 1, \dots, n$ such that for arbitrary $i \neq j$ $\langle y_i, y_j \rangle = 0$ or $|\langle y_j, y_i \rangle| a_i = \langle y_i, y_j \rangle a_j$.

Theorem 1 is simultaneous extensions of the Bessel inequality [4] and the Cauchy-Schwarz inequality [6] in a Hilbert C^* -module. As a matter of fact, if $\{y_1, \dots, y_n\}$ is orthonormal in Theorem 1, then we have the Bessel inequality:

$$\sum_{i=1}^n |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle$$

holds for all $x \in \mathcal{X}$. If $n = 1$ and $y = y_1$ in Theorem 1 and $\langle x, y \rangle$ has a polar decomposition $\langle x, y \rangle = u |\langle x, y \rangle|$ with a partial isometry $u \in \mathcal{A}$, then we have $u |\langle x, y \rangle| \langle y, y \rangle^{-1} |\langle y, x \rangle| u^* \leq \langle x, x \rangle$ and hence

$$|\langle x, y \rangle| = |\langle x, y \rangle| \langle y, y \rangle^{-1} |\langle y, x \rangle| \# \langle y, y \rangle \leq u^* \langle x, x \rangle u \# \langle y, y \rangle.$$

This implies the Cauchy-Schwarz inequality (6).

To prove Theorem 1, we need the following two lemmas:

Lemma 2. If $a \in \mathcal{A}$, then the operator matrix on $\mathcal{A} \oplus \mathcal{A}$

$$A = \begin{pmatrix} |a^*| & -a \\ -a^* & |a| \end{pmatrix}$$

is positive, and $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in N(A)$ if and only if $|a^*| \xi = a \eta$, where $N(A)$ is the kernel of A .

Lemma 3. For any $y_1, y_2, \dots, y_n \in \mathcal{X}$

$$\begin{pmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ & \ddots & \\ \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle \end{pmatrix} \leq \begin{pmatrix} \sum_{j=1}^n |\langle y_j, y_1 \rangle| & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^n |\langle y_j, y_n \rangle| \end{pmatrix}.$$

Proof of Theorem 1 For each $i = 1, \dots, n$, put $c_i = \sum_{j=1}^n |\langle y_j, y_i \rangle|$. Since y_i is nonsingular, it follows that c_i is invertible in $\mathcal{A}$. It follows from Lemma 3 that

$$\begin{aligned}
 & \sum_{i=1}^n \langle x, y_i \rangle c_i^{-1} \langle y_i, y_j \rangle c_j^{-1} \langle y_j, x \rangle \\
 &= (\langle x, y_1 \rangle c_1^{-1} \cdots \langle x, y_n \rangle c_n^{-1}) \begin{pmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ & \ddots & \\ \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle \end{pmatrix} \begin{pmatrix} c_1^{-1} \langle y_1, x \rangle \\ \vdots \\ c_n^{-1} \langle y_n, x \rangle \end{pmatrix} \\
 &\leq (\langle x, y_1 \rangle c_1^{-1} \cdots \langle x, y_n \rangle c_n^{-1}) \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix} \begin{pmatrix} c_1^{-1} \langle y_1, x \rangle \\ \vdots \\ c_n^{-1} \langle y_n, x \rangle \end{pmatrix} \\
 &= \sum_{i=1}^n \langle x, y_i \rangle c_i^{-1} \langle y_i, x \rangle
 \end{aligned}$$

and this implies

$$\begin{aligned}
 0 &\leq \langle x - \sum_{i=1}^n y_i c_i^{-1} \langle y_i, x \rangle, x - \sum_{i=1}^n y_i c_i^{-1} \langle y_i, x \rangle \rangle \\
 &= \langle x, x \rangle - 2 \sum_{i=1}^n \langle x, y_i \rangle c_i^{-1} \langle y_i, x \rangle + \sum_{i=1}^n \langle x, y_i \rangle c_i^{-1} \langle y_i, y_j \rangle c_j^{-1} \langle y_j, x \rangle \\
 &\leq \langle x, x \rangle - \sum_{i=1}^n \langle x, y_i \rangle c_i^{-1} \langle y_i, x \rangle.
 \end{aligned}$$

Hence we have the desired inequality (7).

The equality in (7) holds if and only if the following (8) and (9) are satisfied:

$$(8) \quad x = \sum_{i=1}^n y_i c_i^{-1} \langle y_i, x \rangle$$

and for arbitrary $i \neq j$

$$(9) \quad (\langle x, y_i \rangle c_i^{-1} \langle x, y_j \rangle c_j^{-1}) \begin{pmatrix} |\langle y_j, y_i \rangle| & -\langle y_i, y_j \rangle \\ -\langle y_j, y_i \rangle & |\langle y_i, y_j \rangle| \end{pmatrix} \begin{pmatrix} c_i^{-1} \langle y_i, x \rangle \\ c_j^{-1} \langle y_j, x \rangle \end{pmatrix} = 0.$$

Put $A = \begin{pmatrix} |\langle y_j, y_i \rangle| & -\langle y_i, y_j \rangle \\ -\langle y_j, y_i \rangle & |\langle y_i, y_j \rangle| \end{pmatrix}$ and it follows that the condition (9) holds if and only if

$$A^{1/2} \begin{pmatrix} c_i^{-1} \langle y_i, x \rangle \\ c_j^{-1} \langle y_j, x \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff A \begin{pmatrix} c_i^{-1} \langle y_i, x \rangle \\ c_j^{-1} \langle y_j, x \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence it follows from Lemma 2 that the condition (9) is equivalent to the following (10) and (11): For arbitrary $i \neq j$

$$(10) \quad \langle y_i, y_j \rangle = 0$$

or

$$(11) \quad |\langle y_j, y_i \rangle| c_i^{-1} \langle y_i, x \rangle = \langle y_i, y_j \rangle c_j^{-1} \langle y_j, x \rangle.$$

Conversely, suppose that $x = \sum_{i=1}^n y_i a_i$ for some $a_i \in \mathcal{A}$ and for $i \neq j$ $\langle y_i, y_j \rangle = 0$ or $|\langle y_j, y_i \rangle| a_i = \langle y_i, y_j \rangle a_j$. Then

$$\begin{aligned}
 & \sum_{i=1}^n \langle x, y_i \rangle \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right)^{-1} \langle y_i, x \rangle = \sum_{i=1}^n \langle x, y_i \rangle \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right)^{-1} \sum_{j=1}^n \langle y_i, y_j \rangle a_j \\
 &= \sum_{i=1}^n \langle x, y_i \rangle \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right)^{-1} \sum_{j=1}^n |\langle y_j, y_i \rangle| a_i \\
 &= \sum_{i=1}^n \langle x, y_i \rangle \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right)^{-1} \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right) a_i \\
 &= \sum_{i=1}^n \langle x, y_i \rangle a_i \\
 &= \langle x, x \rangle.
 \end{aligned}$$

Whence the proof is complete. $\square$

Remark 4. (1) In the case that $\mathcal{X}$ is a Hilbert space, the equality condition $|\langle y_j, y_i \rangle| a_i = \langle y_i, y_j \rangle a_j$ in Theorem 1 implies the condition (2) in Introduction. In fact, for some scalars $a_i, a_j \in \mathbb{C}$, it follows that $\langle a_i y_i, a_j y_j \rangle = a_i^* \langle y_i, y_j \rangle a_j = a_i^* |\langle y_j, y_i \rangle| a_i \geq 0$, and $|\langle y_j, y_i \rangle| = |\langle y_j, y_i \rangle^*|$ implies $|a_i| = |a_j|$.

(2) In the Hilbert space setting, K. Kubo and F. Kubo [14] showed another proof of Selberg's inequality (1) using Geršgorin's location of eigenvalues [13, Theorem 6.1.1] and a diagonal domination theorem of positive semidefinite matrix.

4. APPLICATIONS

In [4], Dragomir, Khosravi and Moslehian showed a version of the Bessel inequality and some generalizations of this inequality in the framework of Hilbert C^* -modules. Moreover, in [3], Bounader and Chahbi showed a type and refinement of Selberg inequality in Hilbert C^* -modules. In this section, by using Theorem 1, we consider several Hilbert C^* -module versions of the Selberg inequality and the Bessel inequality.

Bounader and Chahbi in [3, Theorem 3.1] showed that if $\mathcal{X}$ is an inner product C^* -module and $y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$, and $x \in \mathcal{X}$, then

$$(12) \quad \sum_{i=1}^n \frac{|\langle y_i, x \rangle|^2}{\sum_{j=1}^n \|\langle y_j, y_i \rangle\|} \leq \langle x, x \rangle.$$

By Theorem 1, we have the following corollary, which is an improvement of (12):

Corollary 5. *Let $\mathcal{X}$ be an inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \dots, y_n$ are nonsingular, then*

$$\sum_{i=1}^n \frac{|\langle y_i, x \rangle|^2}{\sum_{j=1}^n \|\langle y_j, y_i \rangle\|} \leq \langle x, x \rangle.$$

Moreover, Bounader and Chahbi showed a Hilbert C^* -module version of Fujii-Nakamoto type (3), which is a refinement of (12): If y and $y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $\langle y, y_i \rangle = 0$ for $i = 1, \dots, n$, and $x \in \mathcal{X}$, then

$$(13) \quad |\langle y, x \rangle|^2 + \sum_{i=1}^n \frac{|\langle y_i, x \rangle|^2}{\sum_{j=1}^n \|\langle y_i, y_j \rangle\|} \|\langle y, y \rangle\| \leq \|\langle y, y \rangle\| \langle x, x \rangle.$$

We show a Hilbert C^* -module version of a refinement of the Selberg inequality due to Fujii and Nakamoto, which is another version of (13):

Theorem 6. *Let $\mathcal{X}$ be an inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. If $x, y, y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \dots, y_n$ are nonsingular, $\langle y, y_i \rangle = 0$ for $i = 1, \dots, n$ and $\langle x, y \rangle = u|\langle x, y \rangle|$ is a polar decomposition in $\mathcal{A}$, i.e., $u \in \mathcal{A}$ is a partial isometry, then*

$$|\langle y, x \rangle| \leq u^* \langle y, y \rangle u \# \left(\langle x, x \rangle - \sum_{i=1}^n \langle x, y_i \rangle \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right)^{-1} \langle y_i, x \rangle \right) \\ (\leq u^* \langle y, y \rangle u \# \langle x, x \rangle).$$

In [3, Corollary 3.5], Bounader and Chahbi showed a Hilbert C^* -module version of Bombieri type (4): If $y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ and $x \in \mathcal{X}$, then

$$(14) \quad \sum_{i=1}^n |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle \max_{1 \leq i \leq n} \sum_{j=1}^n \|\langle y_i, y_j \rangle\|.$$

We show a Hilbert C^* -module version of Bombieri type, which is an improvement of (14):

Theorem 7. *Let $\mathcal{X}$ be an inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \dots, y_n$ are nonsingular, then*

$$\sum_{i=1}^n |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle \max_{1 \leq i \leq n} \left\| \sum_{j=1}^n |\langle y_j, y_i \rangle| \right\|.$$

As a corollary, we have the following Boas-Bellman type inequality [3, Corollary 3.6]:

Corollary 8. *Let $\mathcal{X}$ be an inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \dots, y_n$ are nonsingular, then*

$$\sum_{i=1}^n |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle \left(\max_{1 \leq i \leq n} \|\langle y_i, y_i \rangle\| + (n-1) \max_{j \neq i} \|\langle y_j, y_i \rangle\| \right).$$

Finally, we show a Mitrinović-Pečarić-Fink type inequality [17, Theorem 5 in pp394] in Hilbert C^* -modules, which is another version of [4, Theorem 3.8]:

Theorem 9. *Let $\mathcal{X}$ be an inner product C^* -module over a unital C^* -algebra $\mathcal{A}$. If $x, y_1, \dots, y_n$ are nonzero vectors in $\mathcal{X}$ and $a_1, \dots, a_n \in \mathcal{A}$ such that $y_1, \dots, y_n$ are nonsingular and $\langle x, \sum_{i=1}^n y_i a_i \rangle = u|\langle x, \sum_{i=1}^n y_i a_i \rangle|$ is a polar decomposition in $\mathcal{A}$, i.e., $u \in \mathcal{A}$ is a partial isometry, then*

$$\left| \sum_{i=1}^n \langle x, y_i \rangle a_i \right| \leq u^* \langle x, x \rangle u \# \left(\sum_{i=1}^n a_i^* \left(\sum_{j=1}^n |\langle y_j, y_i \rangle| \right) a_i \right).$$

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