

The least energy positive solution for the nonlinear elliptic three-systems with attractive and repulsive interaction terms

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0. Introduction

In this report, for the bounded domain $\Omega \subset \mathbf{R}^n$ ($n \leq 3$) with smooth boundary, we consider the following nonlinear elliptic 3-system:

$$\begin{aligned} -\Delta u_i + \lambda_i u_i &= \mu_i u_i^3 + \sum_{j=1}^3 \beta_{i,j} u_i u_j^2 \quad \text{in } \Omega, \quad (i = 1, 2, 3), \\ u_i &\in H_0^1(\Omega) \quad (i = 1, 2, 3). \end{aligned} \quad (*)$$

where $\lambda_i, \mu_i > 0$ ($i = 1, 2, 3$) and $\beta_{i,j} = \beta_{j,i}$ ($1 \leq i < j \leq 3$). We consider about a least energy positive solution of (*) for the case $\beta_{1,2} > 0$ and $\beta_{1,3}, \beta_{2,3} \leq 0$. Here we call a solution $\vec{u} = (u_1, u_2, u_3) \in H^1(\mathbf{R}^n)^3$ is a least energy positive solution of (*) if and only if $\vec{u}$ achieves $\inf\{I(\vec{u}) \mid I'(\vec{u}) = 0, u_i > 0 \ (i = 1, 2, 3)\}$ where $I(\vec{u})$ is a functional corresponding to (*). There are many papers for the existence of non-trivial solutions of k -system ($k \geq 3$). (cf. [LWe1], [LW1], [LW2], [SW2], [S], [TT], [TTVW], [TV]...) To author's knowledge, almost existence results for (*) were given under the conditions that interaction terms $\beta_{i,j}$ are negative or not large positive.

In this report, we introduce results of our two papers [SW3]–[SW4] but we omit those proofs. For the proofs, see [SW3]–[SW4]. Roughly speaking our results, when $\beta_{1,3}, \beta_{2,3} \leq 0$ and $\beta_{1,2} > 0$ is sufficiently large, we observe the existence of least energy positive solution of (*) in Section 1 and the multiple existence of positive solution of (*) in Section 2.

1. The Existence of least energy solutions

In this section, when $\beta_{1,3}, \beta_{2,3} \leq 0$ and $\beta_{1,2} > 0$ is sufficiently large, we observe the existence of least energy positive solution of (*). Moreover, we observe that, even if Ω is ball, that solution is not radial symmetric. This is a different property from the single homogeneous equations. It is well-known in [GNN] that if Ω is ball and $f(u)$ is of class C^1 , then any positive solutions in C^2 of $-\Delta u = f(u)$ in Ω $u = 0$ on $\partial\Omega$ are radial symmetric. Also, in [LWe1], for a k -system on $\Omega = \mathbf{R}^n$, Lin and Wei showed that, if all interaction terms are positive, then, the least energy positive solutions must be radially symmetric by the Schwartz symmetrization.

Since we treat $\beta_{1,2}$ as a parameter which plays an important role, for simplicities, we often write $\beta \equiv \beta_{1,2}$. We also use the following notations:

$$\begin{aligned} \|u\|_{L^p(\Omega)}^p &= \int_{\Omega} |u|^p \quad \text{for } u \in L^p(\Omega) \quad (1 \leq p \leq \infty), \\ \|u\|_{\lambda, \Omega}^2 &= \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 \quad \text{for } u \in H_0^1(\Omega). \end{aligned}$$

The first theorem is about the existence of a least energy positive solution of (*).

Theorem 1.1. *We suppose that $\beta \equiv \beta_{1,2} > 0$ and $\beta_{1,3}, \beta_{2,3} \leq 0$. Then, there exists a $\beta_* > 0$ such that, for any $\beta > \beta_*$, (*) has a least energy positive solution $\vec{u}_\beta = (u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$. Moreover, there exist a sequence $\beta_m \rightarrow \infty$ and $U_i \in H_0^1(\Omega)$ ($i = 1, 2, 3$) such that*

$$(\sqrt{\beta_m} u_{1,\beta_m}, \sqrt{\beta_m} u_{2,\beta_m}, u_{3,\beta_m}) \rightarrow (U_1, U_2, U_3) \quad \text{strongly in } H_0^1(\Omega)^3.$$

Here U_3 is a positive least energy solution of

$$\begin{aligned} -\Delta u_3 + \lambda_3 u_3 &= \mu_3 u_3^3 \quad \text{in } \Omega, \\ u_3 &\in H_0^1(\Omega), \end{aligned} \tag{1.1}$$

and (U_1, U_2) is a positive least energy solution of

$$\begin{aligned} -\Delta u_1 + (\lambda_1 - \beta_{1,3} U_3^2) u_1 &= u_1 u_2^2 \quad \text{in } \Omega, \\ -\Delta u_2 + (\lambda_2 - \beta_{2,3} U_3^2) u_2 &= u_1^2 u_2 \quad \text{in } \Omega, \\ u_1, u_2 &\in H_0^1(\Omega). \end{aligned} \tag{1.2}$$

In particular (U_1, U_2, U_3) is a minimizer of the following minimizing problem:

$$e = \inf_{u_3 \in K_3} \inf_{(u_1, u_2) \in N_{u_3}} \left(\|u_1\|_{\lambda_1, \Omega}^2 + \|u_2\|_{\lambda_2, \Omega}^2 - \beta_{1,3} \|u_1 u_3\|_{L^2(\Omega)}^2 - \beta_{2,3} \|u_2 u_3\|_{L^2(\Omega)}^2 \right), \tag{1.3}$$

where

$$K_3 = \{u \in H_0^1(\Omega) \mid u \text{ is a least energy solution of (1.1)}\},$$

$$N_{u_3} = \left\{ (u_1, u_2) \in H_0^1(\Omega)^2 \mid \begin{array}{l} |||u_1|||_{\lambda_1, \Omega}^2 + |||u_2|||_{\lambda_2, \Omega}^2 - \beta_{1,3} |||u_1 u_3|||_{L^2(\Omega)}^2 \\ - \beta_{2,3} |||u_2 u_3|||_{L^2(\Omega)}^2 = 2 |||u_1 u_2|||_{L^2(\Omega)}^2 \neq 0 \end{array} \right\}.$$

Remark 1.2. The infimum e is also written as $e = \inf_{u_3 \in K_3} \bar{e}(u_3)$ where

$$\bar{e}(u_3) = \inf_{(u_1, u_2) \in N_{u_3}} \left(|||u_1|||_{\lambda_1, \Omega}^2 + |||u_2|||_{\lambda_2, \Omega}^2 - \beta_{1,3} |||u_1 u_3|||_{L^2(\Omega)}^2 - \beta_{2,3} |||u_2 u_3|||_{L^2(\Omega)}^2 \right).$$

For any given $u_3 \in H_0^1(\Omega)$, $\bar{e}(u_3)$ is achieved by a minimizer $(u_1, u_2) \in N_{u_3}$ which is a non-trivial least energy solution of

$$\begin{aligned} -\Delta u_1 + (\lambda_1 - \beta_{1,3} u_3^2) u_1 &= u_1 u_2^2 \quad \text{in } \Omega, \\ -\Delta u_2 + (\lambda_2 - \beta_{2,3} u_3^2) u_2 &= u_1^2 u_2 \quad \text{in } \Omega, \\ u_1, u_2 &\in H_0^1(\Omega). \end{aligned} \tag{1.3}$$

In fact, in [SW1], we showed the existence of a minimizer for $\bar{e}(u_3)$ when $V_i(x) \equiv \lambda_i - \beta_{i,3} u_3^2$ ($i = 1, 2$) are positive constants. When $V_i(x)$ are non-negative functions, we can show the existence of a minimizer by the same way.

Remark 1.3. The solution $\vec{u}_\beta$ of Theorem 1.1 was given as a minimizer of the following minimizing problem:

$$c_\beta = \inf_{\vec{u} \in M_\beta} (|||u_1|||_{\lambda_1, \Omega}^2 + |||u_2|||_{\lambda_2, \Omega}^2 + |||u_3|||_{\lambda_3, \Omega}^2),$$

$$M_\beta = \left\{ \vec{u} \in H_0^1(\Omega)^3 \mid \begin{array}{l} f_1(\vec{u}) + f_2(\vec{u}) = 0, \quad (u_1, u_2) \neq (0, 0), \\ f_3(\vec{u}) = 0, \quad u_3 \neq 0. \end{array} \right\},$$

where $f_i(\vec{u}) = |||u_i|||_{\lambda_i, \Omega}^2 - \mu_i |||u_i|||_{L^4(\Omega)}^4 - \sum_{j \neq i} \beta_{i,j} |||u_i u_j|||_{L^2(\Omega)}^2$. For details, see our paper [SW3].

Next, we will observe the non-radial symmetry for a least energy positive solution of (*) even if Ω is ball. When Ω is ball, it is well-known that least energy solution of (1.1) is a unique positive radial symmetric solution satisfying $U_3'(r) < 0, r = |x|$. (The uniqueness was proved in [K], The radial symmetry was proved in [GNN].) Let $\vec{u}_\beta = (u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$ be a least energy solution of (*). From Theorem 1.1, $u_{3,\beta}$ converges to the unique positive radial symmetric solution U_3 and a subsequence of $(\sqrt{\beta} u_{1,\beta}, \sqrt{\beta} u_{2,\beta})$ converges to a least energy solution (U_1, U_2) of (1.2) whose potential functions $\lambda_i - \beta_{i,3} U_3(x)$ ($i = 1, 2$) have minimum on the boundary $\partial\Omega$. Then, we can show that (U_1, U_2) has a concentrating point near $\partial\Omega$. That is, (U_1, U_2) is not radial symmetric.

In fact, in this report, we will observe such concentrating phenomenon for the following system including (1.2):

$$\begin{aligned} -\epsilon^2 \Delta u_1 + V_1(x)u_1 &= u_1 u_2^2 \quad \text{in } \Omega, \\ -\epsilon^2 \Delta u_2 + V_2(x)u_2 &= u_1^2 u_2 \quad \text{in } \Omega, \\ u_1, u_2 &\in H_0^1(\Omega). \end{aligned} \tag{1.4}_\epsilon$$

where $\epsilon > 0$ is a parameter, $V_i(x) \in C(\overline{\Omega}, \mathbf{R})$ ($i = 1, 2$) are positive functions. Let $(\vec{u}_\epsilon)_{\epsilon > 0}$ be a family of positive least energy solutions of $(1.4)_\epsilon$. To state the concentrating point of $(\vec{u}_\epsilon)_{\epsilon > 0}$, we need the following $b(\lambda_1, \lambda_2)$:

$$\begin{aligned} b(\lambda_1, \lambda_2) &= \inf_{(u_1, u_2) \in N_{\lambda_1, \lambda_2}} (|||u_1|||_{\lambda_1, \mathbf{R}^n}^2 + |||u_2|||_{\lambda_2, \mathbf{R}^n}^2), \\ N_{\lambda_1, \lambda_2} &= \{(u_1, u_2) \in H^1(\mathbf{R}^n)^2 \mid |||u_1|||_{\lambda_1, \mathbf{R}^n}^2 + |||u_2|||_{\lambda_2, \mathbf{R}^n}^2 = 2||u_1 u_2||_{L^2(\mathbf{R}^n)}^2 \neq 0\}. \end{aligned} \tag{1.5}$$

We easily see that $b(\lambda_1, \lambda_2)$ is achieved for some $\vec{u} = (u_1, u_2) \in H^1(\mathbf{R}^n)^2$ and $\vec{u}$ is a least energy solution of

$$\begin{aligned} -\Delta u_1 + \lambda_1 u_1 &= u_1 u_2^2 \quad \text{in } \mathbf{R}^n, \\ -\Delta u_2 + \lambda_2 u_2 &= u_1^2 u_2 \quad \text{in } \mathbf{R}^n, \\ u_1, u_2 &\in H^1(\mathbf{R}^n). \end{aligned} \tag{1.6}_{(\lambda_1, \lambda_2)}$$

Also, for $b(\lambda_1, \lambda_2)$, we have the following:

Lemma 1.4.

- (i) $b(\lambda_1, \lambda_2) : (0, \infty)^2 \rightarrow \mathbf{R}$ is a continuous function.
- (ii) $b(\lambda_1, \lambda_2)$ is increasing with respect to λ_i ($i = 1, 2$).
- (iii) $b(\eta\lambda_1, \eta\lambda_2) = \eta^2 b(\lambda_1, \lambda_2)$ for all $\eta, \lambda_1, \lambda_2 > 0$.

We regard $u \in H_0^1(\Omega)$ as $u \in H^1(\mathbf{R}^n)$ by setting $u = 0$ on $\mathbf{R}^n \setminus \Omega$. Now we have the following theorem.

Theorem 1.5. *There exist sequences $\epsilon_m \rightarrow 0$, $x_m \rightarrow x_0$ in $\overline{\Omega}$ and $\vec{u}_0 = (u_{1,0}, u_{2,0}) \in H^1(\mathbf{R}^n)^2$ which is a positive least energy solution of $(1.6)_{V_1(x_0), V_2(x_0)}$ such that*

$$\begin{aligned} u_{i, \epsilon_m}(\epsilon_m x - x_m) &\rightarrow u_{i,0}(x) \quad \text{strongly in } H^1(\mathbf{R}^n) \quad (i = 1, 2), \\ b(V_1(x_m), V_2(x_m)) &\rightarrow b(V_1(x_0), V_2(x_0)) = \underline{b}. \end{aligned}$$

Here $\underline{b} = \min_{x \in \overline{\Omega}} b(V_1(x), V_2(x))$.

Remark 1.6.

- (i) For an unique positive radial symmetric solution U_3 of (1.1), setting $V_i(x) = \lambda_i - \beta_{i,3}U_3(x)$ ($i = 1, 2$), from (ii) of Lemma 1.4, $b(V_1(x), V_2(x))$ has minimum on the boundary $\partial\Omega$. Thus, the least energy solution (U_1, U_2) of (1.2) with suitable coefficients, has a concentrating point near $\partial\Omega$.
- (ii) When a positive interaction term β closes to 0, Lin-Wei [LWe2] and Ikoma-Tanaka [IT] studied a singular perturbation problem for

$$\begin{aligned} -\epsilon^2 \Delta u_1 + V_1(x)u_1 &= \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbf{R}^n, \\ -\epsilon^2 \Delta u_2 + V_2(x)u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2 \quad \text{in } \mathbf{R}^n. \end{aligned}$$

In this case, there are possibilities that least energy positive solution $u_{1,\epsilon}$ and $u_{2,\epsilon}$ have different concentrating points. But, in our case, $u_{1,\epsilon}$ and $u_{2,\epsilon}$ always must concentrate a same point.

2. The Existence of G -symmetric least energy solutions

In this section, when Ω is a ball $B = \{x \in \mathbf{R}^N \mid |x| = 1\}$, we observe the multiple existence of positive solutions of (*). When Ω is a ball, a positive solution U_3 of (1.1) is unique and radially symmetrically. Thus, for group actions $G \subset O(n)$ ($O(n)$ is the orthogonal group for $n = 2, 3$), by solving minimizing problems on G -symmetric function's set $H_0^G(B) = \{u \in H_0^1(B) \mid u(gx) = u(x) \text{ for all } g \in G\}$, we can expect multiple existence of positive solutions of (*). In fact, we can show the following theorem by a similar way of Theorem 1.1.

Theorem 2.1. Assume $\Omega = B$, $\beta \equiv \beta_{1,2} > 0$ and $\beta_{1,3}, \beta_{2,3} \leq 0$. Let $G \subset O(n)$ be a group action. Then, there exists $\beta^G > 0$ such that, for any $\beta > \beta^G$, (*) has a G -symmetric positive solution $\vec{u}_\beta^G(x) = (u_{1,\beta}^G(x), u_{2,\beta}^G(x), u_{3,\beta}^G(x))$. Moreover, there exist a sequence $\beta_m \rightarrow \infty$ and $U_i \in H_0^1(B)$ ($i = 1, 2, 3$) such that

$$(\sqrt{\beta_m} u_{1,\beta_m}^G, \sqrt{\beta_m} u_{2,\beta_m}^G, u_{3,\beta_m}^G) \rightarrow (U_1^G, U_2^G, U_3) \quad \text{strongly in } H_0^1(B)^3.$$

Here U_3 is a unique positive radial least energy solution of (1.1) with $\Omega = B$ and (U_1^G, U_2^G) is a positive least energy G -symmetric solution of (1.2) with $\Omega = B$.

Theorem 2.1 suggests the multiplicity of positive solutions of (*). However, in order to get a multiple existence for (*), we need to show the $\vec{u}^G \neq \vec{u}^{G'}$ for group actions $G \neq G'$. To observe this, we will discuss an asymptotically behavior of limit equation (1.2) by regarding some coefficients as parameters. That is, for group actions $G \neq G'$,

we show that (U_1^G, U_2^G) and $(U_1^{G'}, U_2^{G'})$ has different asymptotically behaviors when some parameters go to limits. More precisely, we will show (U_1^G, U_2^G) has k^G -peaks near the boundary ∂B where k^G is a number of the minimum orbit for G . This argument is similar to the arguments for the multiplicity of positive solutions of $-\Delta u + u = u^p$ on annulus domain.

We will observe such asymptotically results for more general equations which including (1.2). For radial positive functions $V_i(x) \in C(B, \mathbf{R})$ ($i = 1, 2$), we consider the following system

$$\begin{aligned} -\Delta u_1 + \eta V_1(x)u_1 &= u_1 u_2^2 \text{ in } B, & u_1 &\in H_0^1(B), \\ -\Delta u_2 + \eta V_2(x)u_2 &= u_1^2 u_2 \text{ in } B, & u_2 &\in H_0^1(B). \end{aligned} \quad (2.1)$$

For a group action $G \subset O(n)$ ($n = 2, 3$), to get G -symmetric solutions of (2.1), we solve the following minimizing problem on $H_0^G(B)$:

$$\begin{aligned} \hat{b}_\eta^G &= \inf_{(u_1, u_2) \in \hat{N}_\eta^G} (|||u_1|||_{\eta V_1, B}^2 + |||u_2|||_{\eta V_2, B}^2), \\ \hat{N}_\eta^G &= \left\{ (u_1, u_2) \in H_0^G(B)^2 \mid |||u_1|||_{\eta V_1, B}^2 + |||u_2|||_{\eta V_2, B}^2 = 2||u_1 u_2||_{L^2(B)}^2 \neq 0 \right\}. \end{aligned} \quad (2.2)$$

By standard ways, we see that $\hat{b}_\eta^G$ is achieved for some $(u_{1,\eta}^G, u_{2,\eta}^G) \in \hat{N}_\eta^G$ which is a G -symmetric positive solution of (2.1). To discuss an asymptotically behavior of $(u_{1,\eta}^G, u_{2,\eta}^G)$ as $\eta \rightarrow \infty$, the function $b(\lambda_1, \lambda_2)$ which was defined in (1.5) also plays important roles. By using the Schwarz symmetrization, we see that least energy solutions of (1.6) $_{(\lambda_1, \lambda_2)}$ is radial symmetry. For $G \subset O(n)$, let $G[x] = \{gx \mid g \in G\}$ be an orbit of $x \in \mathbf{R}^n \setminus \{0\}$ and $k^G = \min\{\#G[x] \mid x \in \mathbf{R}^n \setminus \{0\}\}$ be a element number of the minimum orbit. Now, we obtain the following theorem which is essential in our arguments.

Theorem 2.2. Assume that $V_i(x) \in C(B, \mathbf{R})$ ($i = 1, 2$) are positive radial functions and a finite group $G \subset O(n)$ satisfies $k^G < \frac{b(V_1(0), V_2(0))}{\underline{b}}$ where $\underline{b} = \min_{x \in B} b(V_1(x), V_2(x))$. Then we have

$$\eta^{\frac{n}{2}-2} \hat{b}_\eta^G \rightarrow k^G \underline{b} \quad \text{as } \eta \rightarrow \infty.$$

Moreover, for a family of G -symmetric positive solutions $(u_{1,\eta}^G, u_{2,\eta}^G)$ of (2.1) which achieves the minimizing problem (2.2), there exist a subsequence $\eta_m \rightarrow \infty$ and a sequence $x_m \rightarrow x_0$ in B with $\#G[x_m] = \#G[x_0] = k^G$ and $b(V_1(x_0), V_2(x_0)) = \underline{b}$ such that

$$\left\| \left\| u_{i,\eta_m}^G - \sum_{z \in G[x_m]} \sqrt{\eta_m} w_{i,0}(\sqrt{\eta_m}(\cdot - z)) \right\|_{\eta_m V_i, B}^2 \right\| = o(\eta_m^{1-\frac{n}{2}}) \quad (i = 1, 2).$$

Here $(w_{1,0}, w_{2,0})$ is a positive least energy solution of (1.5) with $(\lambda_1, \lambda_2) = (V_1(x_0), V_2(x_0))$ and $o(\eta_m^{1-\frac{n}{2}})\eta_m^{\frac{n}{2}-1} \rightarrow 0$ as $\eta_m \rightarrow \infty$. That is, for large η_m , $(u_{1,\eta_m}^G, u_{2,\eta_m}^G)$ is close to k^G -peak functions.

Remark 2.3.

- (i) For the case $k^G \geq \frac{b(V_1(0), V_2(0))}{\underline{b}}$, G -symmetric positive solutions $(u_{1,\eta}^G, u_{2,\eta}^G)$ achieving (2.2) may be radial symmetric. Thus we can't look for any more positive solutions by only Theorem 2.2.
- (ii) Theorem 2.1 and 2.2 still hold for G -invariant domain Ω . The other corollaries and theorems below follow from Theorem 2.1 and Theorem 2.2.

From Theorem 2.2, we have the following

Corollary 2.4. Suppose that the same assumptions as Theorem 2.2 hold. Let K_η^G be a set of least energy G -symmetric solution of (2.1), K be a set of least energy solution of (1.6) with $(\lambda_1, \lambda_2) = (V_1(x_0), V_2(x_0))$ and $X = \{x \in B \mid \#G[x] = k^G, b(V_1(x), V_2(x)) = \underline{b}\}$ and $X_\rho = \{x \in B \mid \#G[x] = k^G, \text{dist}(x, X) < \rho\}$. Then, for any $\rho > 0$, we have

$$\sup_{(u_1, u_2) \in K_\eta^G} \inf_{(w_1, w_2) \in K, x \in X_\rho} \sum_{i=1,2} \left\| \left\| u_i - \sum_{z \in G[x]} \sqrt{\eta} w_i(\sqrt{\eta}(\cdot - z)) \right\|_{\eta V_i, B} \right\|^2 = o(\eta^{1-\frac{n}{2}}) \quad (2.3)$$

Here $o(\eta^{1-\frac{n}{2}})\eta^{\frac{n}{2}-1} \rightarrow 0$ as $\eta \rightarrow \infty$.

Proof. Suppose that Corollary 2.4 does not hold. Then there exists $c_0 > 0$, $\eta_m \rightarrow \infty$ and $(u_{1,\eta_m}, u_{2,\eta_m}) \in K_{\eta_m}^G$ such that

$$\inf_{(w_1, w_2) \in K, x \in X_\rho} \sum_{i=1,2} \left\| \left\| u_{i,\eta_m} - \sum_{z \in G[x]} \sqrt{\eta_m} w_i(\sqrt{\eta_m}(\cdot - z)) \right\|_{\eta_m V_i, B} \right\|^2 \geq c_0 \eta^{1-\frac{n}{2}}$$

But, this contradicts to Theorem 2.2. ■

When $n = 2$, for any $k \in \mathbf{N}$, the cyclic group $\mathbf{Z}_k \subset O(2)$ satisfies $k^{\mathbf{Z}_k} = k$. Thus, from Corollary 2.4, we easily find the following multiple existence of positive solutions.

Corollary 2.5. Suppose that $n = 2$ and $V_i(x) \in C(B, \mathbf{R})$ ($i = 1, 2$) are positive radial functions and $k < \frac{b(V_1(0), V_2(0))}{\underline{b}}$ where $\underline{b} = \min_{x \in B} b(V_1(x), V_2(x))$. Then, there exists $\eta_k > 0$ such that, for any $\eta > \eta_k$, (2.1) has a positive solution close to ℓ -peak function in

the sense of (2.3) with $G = \mathbf{Z}_\ell$ and (2.1) has a radial positive solution. That is, (2.1) has at least $k + 1$ positive solutions.

Proof. From Corollary 2.4, for any $\ell < \frac{b(V_1(0), V_2(0))}{\underline{b}}$, there exists $\eta^{\mathbf{Z}_\ell} > 0$ such that, for any $\eta > \eta^{\mathbf{Z}_\ell}$, (2.1) has a positive solution close to ℓ -peak solutions in the sense of (2.3). On the other hand, (2.1) always has a radial positive solution. Thus, for $\eta \geq \eta_k \equiv \max\{\eta^{\mathbf{Z}_1}, \dots, \eta^{\mathbf{Z}_k}\}$, (2.1) has at least $k + 1$ positive solutions. ■

When $n = 3$, the subgroups $G = \mathbf{Z}_2, P_4, P_8, P_{12} \subset O(3)$ satisfy $k^G = 2, 4, 8$, or 12 , respectively. Here P_q is the q -regular polyhedron group. Thus, from Corollary 2.4, we also find the following corollary.

Corollary 2.6. Suppose that $n = 3$ and $V_i(x) \in C(B, \mathbf{R})$ ($i = 1, 2$) are positive radial functions and 2 (or 4, 8, 12, respectively) $< \frac{b(V_1(0), V_2(0))}{\underline{b}}$ where $\underline{b} = \min_{x \in B} b(V_1(x), V_2(x))$. Then, there exists $\eta_0 > 0$ such that, for any $\eta > \eta_0$, (2.1) has a positive solution close to 2 (or 4, 8, 12, respectively)-peak functions in the sense of (2.3) with $G = \mathbf{Z}_2$ (or P_4, P_8, P_{12} , respectively) and (2.1) has a radial positive solution.

Here, we return to the our original equation (*) and limit equation (1.2). For $\eta > 0$, $\lambda'_i > 0$ and $\beta'_{i,3} < 0$ ($i = 1, 2$), we set

$$\lambda_i = \eta \lambda'_i, \quad \beta_{i,3} = \eta \beta'_{i,3} \quad (i = 1, 2).$$

For (1.2), since $U_3(x)$ is radial and decreasing with respect to $r = |x|$, from (b1)–(b2), $b(\lambda_1 - \beta_{1,3}U_3(x)^2, \lambda_2 - \beta_{2,3}U_3(x)^2) : (0, \infty)^2 \rightarrow \mathbf{R}$ has maximum at $x = 0$ and minimum on ∂B . We remark that, from (b3), $b(\lambda_1 - \beta_{1,3}U_3(x)^2, \lambda_2 - \beta_{2,3}U_3(x)^2) = \eta^2 b(\lambda'_1 - \beta'_{1,3}U_3(x)^2, \lambda'_2 - \beta'_{2,3}U_3(x)^2)$. From Corollary 2.5 and Corollary 2.6, we have the following multiple existence for (*).

Theorem 2.7. Assume that $\Omega = B$. For $\eta > 0$, we assume that $\beta \equiv \beta_{1,2} > 0$, $\lambda_i = \eta \lambda'_i > 0$, $\beta_{i,3} = \eta \beta'_{i,3} < 0$ ($i = 1, 2$). Let k be the maximum integer satisfying $k < \frac{b(\lambda'_1 - \beta'_{1,3}U_3(0)^2, \lambda'_2 - \beta'_{2,3}U_3(0)^2)}{b(\lambda'_1, \lambda'_2)}$.

(i) When $n = 2$, for any $\epsilon, \rho > 0$, there exists $\eta_k > 0$ such that, for any $\eta > \eta_k$, there exists $\beta_k(\eta) > 0$ such that, for any $\beta > \beta_k(\eta)$ and $\ell = 1, \dots, k$, we have

$$\sup_{\vec{u} \in L_\beta^{\mathbf{Z}_\ell}} \inf_{(w_1, w_2) \in K, x \in X_\rho} \sum_{i=1,2} \left\| \sqrt{\beta} u_i - \sum_{z \in \mathbf{Z}_\ell[x]} \sqrt{\eta} w_i(\sqrt{\eta}(\cdot - z)) \right\|_{V_i, B}^2 \leq \eta^{1-\frac{n}{2}} \epsilon.$$

Here $L_\beta^{\mathbf{Z}_\ell}$ is a set of least energy $\mathbf{Z}_\ell$ -symmetric solutions of (*) and K is a set of least energy solutions of (1.6) and $X_\rho = \{1 - \rho < |x| \leq 1\}$. In particular, (*) has

at least $k + 1$ positive solutions $\vec{u}^\ell = (u_1^\ell, u_2^\ell, u_3^\ell)$ of $(*)$ ($1 \leq \ell \leq k + 1$). Here $\vec{u}^\ell$ is $\mathbf{Z}_\ell$ -symmetry and (u_1^ℓ, u_2^ℓ) is close to ℓ -peak functions which peak's locations are near ∂B ($1 \leq \ell \leq k$) and $\vec{u}^{k+1}$ is radial symmetry.

- (ii) When $n = 3$, if 2 (or 4, 8, 12, respectively) $\leq k$ holds, for any $\epsilon, \rho > 0$, there exists $\eta_k > 0$ such that, for any $\eta > \eta_k$, there exists $\beta_k(\eta) > 0$ such that, for any $\beta > \beta_k(\eta)$ and $G = \mathbf{Z}_2$ (or P_4, P_8, P_{12} , respectively), we have

$$\sup_{\vec{u} \in L_\beta^G} \inf_{(w_1, w_2) \in K, x \in X_\rho} \sum_{i=1,2} \left\| \sqrt{\beta} u_i - \sum_{z \in G[x]} \sqrt{\eta} w_i(\sqrt{\eta}(\cdot - z)) \right\|_{V_i, B}^2 \leq \eta^{1-\frac{n}{2}} \epsilon.$$

Here L_β^G is a set of least energy G -symmetric solutions of $(*)$ and K is a set of least energy solutions of (1.6) and $X_\rho = \{1 - \rho < |x| \leq 1\}$.

Proof. From Corollary 2.5 and Corollary 2.6, there exists $\eta_k > 0$ such that, for all $\eta > \eta_k$,

$$\sup_{(U_1, U_2) \in K_\eta^G} \inf_{(w_1, w_2) \in K, x \in X_\rho} \sum_{i=1,2} \left\| U_i - \sum_{z \in G[x]} \sqrt{\eta} w_i(\sqrt{\eta}(\cdot - z)) \right\|_{\eta V_i, B}^2 \leq \eta^{1-\frac{n}{2}} \epsilon. \quad (2.4)$$

Here K_η^G is a set of least energy G -symmetric solution of (2.1). Next, from Theorem 2.2, there exists $\beta(\eta) > 0$ such that, for all $\beta > \beta(\eta)$, we have

$$\sup_{\vec{u} \in L_\eta^G} \inf_{(U_1, U_2) \in K_\eta^G} \sum_{i=1,2} \left\| \sqrt{\beta} u_i - U_i \right\|_{\eta V_i, B}^2 \leq \eta^{1-\frac{n}{2}} \epsilon. \quad (2.5)$$

From (2.4)–(2.5), we obtain Theorem 2.7. ■

Moreover, since $U_3(0) \rightarrow \infty$ as $\lambda_3 \rightarrow \infty$, from (b3), we observe that

$$\frac{b(\lambda'_1 - \beta'_{1,3} U_3(0)^2, \lambda'_2 - \beta'_{2,3} U_3(0)^2)}{b(\lambda'_1, \lambda'_2)} \rightarrow \infty \quad \text{as } \lambda_3 \rightarrow \infty. \quad (2.6)$$

Therefore, from Theorem 2.7 and (2.6), we easily lead the following theorem.

Theorem 2.8. Assume that $\Omega = B$ and $n = 2$. Then, for any $k \in \mathbf{N}$, there exists $\lambda_k > 0$, such that, for any $\lambda_3 > \lambda_k$, there exists $\eta_k(\lambda_3) > 0$, such that, for any $\eta > \eta_k(\lambda_3)$, there exists $\beta_k(\eta, \lambda_3) > 0$, such that, for any $\beta \equiv \beta_{1,2} > \beta_k(\eta, \lambda_3)$, $(*)$ has at least $k + 1$ positive solutions $\vec{u}^\ell = (u_1^\ell, u_2^\ell, u_3^\ell)$ ($1 \leq \ell \leq k + 1$). Here $\vec{u}^\ell$ is $\mathbf{Z}_\ell$ -symmetry and (u_1^ℓ, u_2^ℓ) is close to ℓ -peak functions which peak's locations are near ∂B ($1 \leq \ell \leq k$) and $\vec{u}^{k+1}$ is radial symmetry.

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