

SK INVARIANTS FOR G -MANIFOLDS WITH BOUNDARY

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Let G be a finite abelian group. A G -manifold means an unoriented compact smooth manifold, which may have boundary, together with a smooth action of G . Let N_i ($i = 1, 2$) be G -manifolds with the same dimension, L a codimension zero invariant submanifold of each boundary ∂N_i and $\varphi, \psi : L \rightarrow L$ G -equivariant diffeomorphisms. Pasting along L , we have G -manifolds $M_1 = N_1 \cup_{\varphi} N_2$ and $M_2 = N_1 \cup_{\psi} N_2$. Then M_1 and M_2 are said to be obtained from each other by an equivariant cutting and pasting or a G -SK process. The abbreviation SK stands for Schneiden und Kleben in German.

Definition. Consider a map T defined for all G -manifolds which takes its values in the ring $\mathbf{Z}$ of rational integers and is additive with respect to the disjoint union of G -manifolds. We call T a G -SK invariant or simply an invariant if it is invariant under the G -SK process, i.e., $T(M_1) = T(M_2)$ for the above M_1 and M_2 . Further, such a T is said to be *multiplicative* if $T(M \times N) = T(M) \cdot T(N)$ for any G -manifolds M and N .

As an example, χ^H given by $\chi^H(M) = \chi(M^H)$ is a multiplicative invariant, where $H \leq G$, a subgroup of G , and χ is the Euler characteristic.

The purpose of this note is to characterize a form of multiplicative invariants.

By a G -slice type, we mean a pair $\sigma = [H; V]$ of $H (\leq G)$ and an H -module V , i.e., a finite-dimensional real vector space together with a natural linear action of H which satisfies that $V^G = \{0\}$. Let $St(G)$ be the set of all G -slice types. There exists a partial ordering on $St(G)$ as follows: $[H; V] \preceq [K; W]$ means that $H \leq K$ and $W = V \oplus W^H$ as H -modules. In this case, we denote $[K; W]_H = [H; V]$.

Let $SK_*^G(\partial)$ be an SK group resulting from equivariant cuttings and pastings of G -manifolds.

Proposition(cf.[1], [2]). $SK_*^G(\partial)$ is a free SK_* -module with basis $\{[G \times_H D(V)], [G \times_H D(V \times \mathbf{R})] \mid [H; V] \in St(G)\}$, where $D(V)$ denotes the associated H -disk.

An invariant T induces an additive homomorphism $SK_*^G(\partial) \rightarrow \mathbf{Z}$ and denote by $\mathcal{T}_*$ the set of all these homomorphisms. For $\sigma = [H; V]$, let χ_σ be an invariant defined by $\chi_\sigma(M) = \chi(M_\sigma)$, where M_σ is a submanifold of M consisting those points $x \in M$ whose slice types σ_x satisfy that $\sigma \preceq \sigma_x$. Further, consider an invariant θ_σ as

$$\theta_\sigma(M) := |G/H|^{-1} \left\{ \chi(M_\sigma) + \sum_{H < K \leq G} n_H(K) \left(\sum_{\sigma \prec \tau = [K; W]} \chi(M_\tau) \right) \right\},$$

where an integer $n_H(K)$ for K with $H \leq K \leq G$ is defined inductively as follows : $n_H(H) = 1$ and $n_H(K) = |K/H| - \sum_{H \leq L < K} n_H(L)$ ($|K/H|$; the order of K/H). By evaluating θ_σ on the basis elements for $SK_*(\partial)$ in Proposition, we have the following theorem.

Theorem(cf.[3]). The class $\{\theta_\sigma \mid \sigma \in St(G)\}$ provides a basis for $\mathcal{T}_*$ as a free $\mathbf{Z}$ -module.

A multiplicative invariant T is considered to be a ring homomorphism $SK_*^G(\partial) \rightarrow \mathbf{Z}$.

Definition. Such a (non-trivial) invariant T is said to be of type $\langle G/H \rangle$ if H is the minimum element with respect to the inclusion $\leq$ of subgroups in the set consisting of those subgroups K of G such that $T(G/K) \neq 0$.

In fact, it is seen from the multiplicative structure of $SK_*^G(\partial)$ that $H = \cap_\lambda K_\lambda$, where $\{K_\lambda\}$ is the set of all subgroups of G such that $T(G/K_\lambda) \neq 0$. For example, χ^H is of type $\langle G/H \rangle$.

Theorem(cf.[4]). If T is of type $\langle G \rangle$, then it is uniquely determined by the value $a = T(D^1)$ on the one-dimensional disk D^1 with the trivial action and has a form $T(M) = a^{\dim(M)} \chi(M)$ for any G -manifold M . Here, if $a = 0$, then a^0 is regarded as 1.

Let T be a multiplicative invariant of type $\langle G/H \rangle$ with $H \neq \{1\}$ in general and let $\mathcal{V}_T = \{a\} \cup \{\gamma_j\}_j$ be integers given by $a = T(D^1)$ and $\gamma_j = |G/H|^{-1} T(G \times_H D(V_j))$

on G -manifolds $G \times_H D(V_j)$, where $\{V_j\}$ is the complete set of non-trivial irreducible H -modules.

Denote by $St[H]$ the set of all G -slice types with H as an isotropy subgroup.

Main Theorem(cf.[4]). Let T be a multiplicative invariants of type $\langle G/H \rangle$ with $H \neq \{1\}$. Then it is uniquely determined by the class of integers $\mathcal{V}_T$ and has a form

$$T(M) = \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)} \gamma_\sigma \cdot \chi(M_\sigma)$$

for any G -manifold M , where $\gamma_\sigma = \prod_j \gamma_j^{a(j)}$ if $\sigma = [H; \prod_j V_j^{a(j)}] \in St[H]$. In case where a or $\gamma_j = 0$ for some j , we regard a^0 or γ_j^0 as 1 respectively.

Example.

Multiplicative invariants T of type $\langle G/H \rangle$, $H \neq \{1\}$, with $a, \gamma_j \in \{-1, 0, 1\}$:

$$(1) \underline{\gamma_j = 1 \quad (\forall j)},$$

$$T(M) = \begin{cases} \chi(M^H) & \text{if } a = 1, \\ \chi(M^{H, 0}) & \text{if } a = 0, \\ \chi(M^{H, \text{ev}}) - \chi(M^{H, \text{od}}) & \text{if } a = -1, \end{cases}$$

where $M^{H, 0}$ is the isolated points of M^H and $M^{H, \text{ev}}$ (or $M^{H, \text{od}}$) is the union of even-dimensional (or odd-dimensional) components of M^H respectively.

$$(2) \underline{\gamma_j = -1 \quad (\forall j)},$$

$$T(M) = \begin{cases} \chi(M_+^H) - \chi(M_-^H) & \text{if } a = 1, \\ \chi(M_+^{H, 0}) - \chi(M_-^{H, 0}) & \text{if } a = 0, \\ (-1)^{\dim M} \{\chi(M_{2,+}^H) - \chi(M_{2,-}^H)\} & \text{if } a = -1, \end{cases}$$

where $M_+^H = \{x \in M^H \mid l((\sigma_x)_H); \text{even}\}$, $M_-^H = \{x \in M^H \mid l((\sigma_x)_H); \text{odd}\}$ ($(\sigma_x)_H \preceq \sigma_x$, $l((\sigma_x)_H) = \sum_j a(j)$; the total length of $(\sigma_x)_H = [H; \prod_j V_j^{a(j)}]$), $M_\epsilon^{H, 0} = M_\epsilon^H \cap M^{H, 0}$ and $M_{2,+}^H = \{x \in M^H \mid l_2((\sigma_x)_H); \text{even}\}$, $M_{2,-}^H = \{x \in M^H \mid l_2((\sigma_x)_H); \text{odd}\}$ ($l_2((\sigma_x)_H) = \sum_j a(j)$ summing over all j with $\dim(V_j) = 2$; the total length of the two-dimensional irreducible H -modules of $(\sigma_x)_H$).

$$(3) \ \underline{\gamma_j = 0 \ (\forall j)},$$

$$T(M) = \begin{cases} \chi(M_{\sigma^H(\mathbf{0})}) & \text{if } a = 1, \\ 0^{\dim(M)} \chi(M^H) & \text{if } a = 0, \\ (-1)^{\dim(M)} \chi(M_{\sigma^H(\mathbf{0})}) & \text{if } a = -1, \end{cases}$$

where $M_{\sigma^H(\mathbf{0})}$ is the union of the components of M^H with $\dim(M_{\sigma^H(\mathbf{0})}) = \dim(M)$.

References

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