

Learning Complex Numbers assisted by Mathematica

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1 INTRODUCTION

In this article, I would like to report my experience of learning complex numbers assisted by Mathematica ver.7 following the text [1] from a beginner's viewpoint. Where I wrote Mathematica code, I often described it in traditional form instead of source code when it was adequate to show the logic clearly.

2 THE ALGEBRA OF COMPLEX NUMBERS

2.1 Arithmetic Operations, Square Roots, Conjugation and Absolute Value.

I show the arithmetic operations with complex numbers using Mathematica symbolic calculation.

```
In[1] := (α + iβ) + (γ + iδ) // ComplexExpand
Out[1] = α + γ + i(β + δ)
In[2] := (α + iβ)(γ + iδ) // ComplexExpand
Out[2] = αγ - βδ + i(βγ + αδ)
In[3] := (α + iβ) / (γ + iδ) // ComplexExpand
Out[3] =  $\frac{\alpha\gamma}{\gamma^2 + \delta^2} + \frac{\beta\delta}{\gamma^2 + \delta^2} + i(\frac{\beta\gamma}{\gamma^2 + \delta^2} - \frac{\alpha\delta}{\gamma^2 + \delta^2})$ 
In[4] := 1 / (α + iβ) // ComplexExpand
Out[4] =  $\frac{\alpha}{\alpha^2 + \beta^2} - \frac{i\beta}{\alpha^2 + \beta^2}$ 
```

The numbers $0 + 0i$ and $1 + 0i$ are neutral elements under addition and multiplication respectively. The complex number system $\mathbf{C}$ is a field with these operations.

I defined an own function $\text{sqrt}[z_]$ for calculating the square root of a complex number z with the equations (1).

For $\beta \neq 0$

$$\sqrt{\alpha + i\beta} = \pm \left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i \frac{\beta}{|\beta|} \sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \right) \quad (1)$$

For $\beta = 0$ the values are $\pm\sqrt{\alpha}$ if $\alpha \geq 0$, $\pm i\sqrt{-\alpha}$ if $\alpha < 0$.

The code of `sqrt[z_]` is shown in In[6]. I show an example to calculate the square root of a complex number using the function `sqrt[z_]`.

```
In[5] := sqrt[1/2 - i*sqrt(3)/2]
Out[5] = {-i/2 + sqrt(3)/2, i/2 - sqrt(3)/2}
```

Using the built-in symbol, `Sqrt[1/2 - i*sqrt(3)/2]` gave the output "`sqrt[1/2 - i*sqrt(3)/2]`."

[EXERCISE] Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0. \quad (2)$$

I show Mathematica codes to answer and check the result.

```
In[6] := Clear["Global`*"];
sqrt[z_] := Module[{alpha = Re[z], beta = Im[z]},
  If[beta != 0, {sqrt[(alpha + sqrt(alpha^2 + beta^2))/2] + i*beta/|beta|*sqrt[-(alpha + sqrt(alpha^2 + beta^2))/2],
    i*beta/|beta|*sqrt[(alpha + sqrt(alpha^2 + beta^2))/2]},
  If[alpha >= 0, {sqrt[alpha], -sqrt[alpha]},
    {i*sqrt[-alpha], -i*sqrt[-alpha]}]];
b = alpha + i*beta; c = gamma + i*delta;
expr[z_, b_, c_] := z^2 + b*z + c;
d[b_, c_] := b^2 - 4*c//ComplexExpand;
sol[b_, c_] := -b/2 + sqrt[d[b, c]];
In[7] := sol[2i, -2]//ComplexExpand;
Out[7] = {1 - i, -1 - i}
In[8] := SeedRandom[]; {alpha, beta, gamma, delta} = RandomInteger[{0, 10}, 4];
z = sol[b, c];
Row[{"expr = ", expr[#, b, c]}]&/@{z[[1]], z[[2]]}//Simplify
Out[8] = {expr = 0, expr = 0}
```

In In[6], I defined an own function `sqrt[z_]`. Then I let b be $\alpha + i\beta$ and c be $\gamma + i\delta$. I also defined the equation `expr[z, b, c]` as (2). In the definition of `sol[b, c]`, I gave the solution, where the second term had two symmetric values.

In[7] and Out[7] are numerical calculation for check. I defined an equation substituting $2i$ for b and -2 for c . Out[7] showed the set of solutions for the equation $z^2 + 2iz - 2 = 0$.

In In[8], I substituted for b and c with randomly given integers α, β, γ and δ . In Out[8], it was confirmed that the two solutions of `sol` satisfied the equation with random coefficients.

I defined own functions for symbolic calculations with complex numbers as follows: `conj[z]` for the complex conjugation of z , `abs[z]` which calculates the absolute value of z , `re[z]` which gives the real part of z , and `im[z]` which gives the imaginary part of z .

```

In[9] := Clear["Global`*"];
complex[z_] := ComplexExpand[z];
conj[z_] := complex[z]/.{Complex[p_, q_] -> Complex[p, -q]};
abs[z_] := Sqrt[ComplexExpand[complex[z] * conj[z]]];
re[z_] := Simplify[(complex[z] + conj[z])/2];
im[z_] := Simplify[(complex[z] - conj[z])/(2i)];
In[10] := a =  $\alpha + i\beta$ ;
In[11] := {complex[a], conj[a], abs[a], re[a], im[a]}
Out[11] = { $\alpha + i\beta$ ,  $\alpha - i\beta$ ,  $\sqrt{\alpha^2 + \beta^2}$ ,  $\alpha$ ,  $\beta$ }

```

In In[10], I defined a complex number $a = \alpha + i\beta$ and checked the outcomes in In[11] and Out[11].

In In[12], I defined own function *replace*[z] for transforming $\alpha^2 + \beta^2$ to $|z|^2$. With *replace*[z], I checked that the calculations of $abs[a + b]^2$, $abs[a - b]^2$ and $abs[a + b]^2 + abs[a - b]^2$ were carried out correctly.

```

In[12] := a =  $\alpha + i\beta$ ; b =  $\gamma + i\delta$ ;
replace[z_] := re[z]^2 + im[z]^2 -> |z|^2;
Column[{ abs[a + b]^2 /. replace[a] /. replace[b],
          abs[a - b]^2 /. replace[a] /. replace[b],
          Simplify[abs[a + b]^2 + abs[a - b]^2] /. replace[a] /.
replace[b]}]
Out[12] = 2 $\alpha\gamma + 2\beta\delta + |\alpha + i\beta|^2 + |\gamma + i\delta|^2$ 
          -2 $\alpha\gamma - 2\beta\delta + |\alpha + i\beta|^2 + |\gamma + i\delta|^2$ 
          2(| $\alpha + i\beta|^2 + |\gamma + i\delta|^2$ )

```

3 THE GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

Using *Manipulate* or *Dynamic*, we can draw the pictures and move any variable. After the pictures have been drawn, we can observe them and change the view angles. Various types of coloring, lighting and styling are possible. All the pictures I present here were drawn using Mathematica. Mathematica helped me to understand the situations and experiment on the ideas.

In order to draw graphs in complex planes, I defined own functions as follows:

```

cc[z_] := {re[z], im[z]};
plr[r_,  $\theta$ _] := r(cos  $\theta + i$  sin  $\theta$ );

```

The function *cc*[z_] transforms z to its coordinate and *plr*[r_, θ _] makes polar form of a complex number.

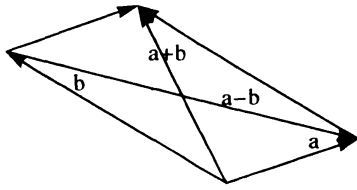


FIG. 1: Vector addition.

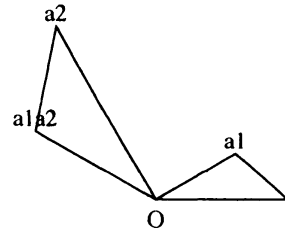


FIG. 2: Vector multiplication.

3.1 Geometric Addition and Multiplication.

A complex number $z = x + iy$ corresponds to a point in a complex plane with the coordinates $(\Re z, \Im z)$. The addition of complex numbers can be visualized as vector additions. Introducing polar coordinates, for two complex numbers $a_1 = r_1(\cos \phi_1 + i \sin \phi_1)$ and $a_2 = r_2(\cos \phi_2 + i \sin \phi_2)$, their symbolic multiplication can be calculated with own functions as shown in In[13] and Out[13]. FIG.1 and FIG.2 were drawn using own functions.

```
In[13] := plr[r-, ϕ-] := r(cos ϕ + i sin ϕ);
        a1 = plr[r1, ϕ1]; a2 = plr[r2, ϕ2];
        Simplify[a1 * a2 / ComplexExpand]
Out[13] = r1 r2 (cos(ϕ1 + ϕ2) + i sin(ϕ1 + ϕ2))
```

[EXERCISE] Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

Let $\{c_k, d_k\}$ be the two other vertices. Because $\arg \frac{i(a-b)}{a-b} = \frac{\pi}{2}$, $c_1 = a + i(a-b)$, $d_1 = b + i(a-b)$, $c_2 = a - i(a-b)$, $d_2 = b - i(a-b)$. Because $\frac{a+b}{2}$ is the midpoint of $a-b$, $c_3 = \frac{a+b+i(a-b)}{2}$, $d_3 = \frac{a+b-i(a-b)}{2}$. Hence $\{c_1, d_1\}$, $\{c_2, d_2\}$ and $\{c_3, d_3\}$ are the two other vertices in all possible cases.

To check, the cross ratio

$$\frac{a-c}{b-c} : \frac{a-d}{b-d}$$

was calculated. It was 0.5 for $\{c_1, d_1\}$ and $\{c_2, d_2\}$. i.e. inseparable. It was -1 for $\{c_3, d_3\}$ i.e. separative.(FIG.3).

3.2 The Binomial Equation.

Let $a = r(\cos \phi + i \sin \phi) \neq 0$. For the equation $z^n = a$, the complete solution is given by

$$z = \sqrt[n]{r} \left[\cos\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) + i \sin\left(\frac{\phi}{n} + k\frac{2\pi}{n}\right) \right], k = 0, 1, \dots, n-1. \quad (3)$$

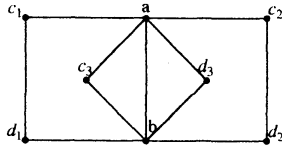
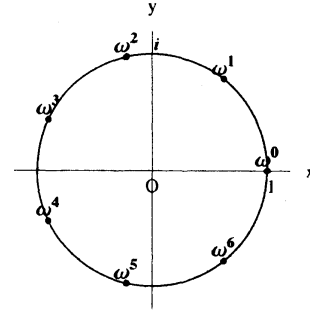


FIG. 3: 2.1.exercise. All squares.

FIG. 4: 7th roots of unity.

In the case $z^n = 1$, setting ω by (4), $1, \omega, \omega^2, \dots, \omega^{n-1}$ are n th roots of unity.(FIG.4).

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \quad (4)$$

[EXERCISE]

1. Express $\cos 3\phi$, $\cos 4\phi$, and $\sin 5\phi$ in terms of $\cos \phi$ and $\sin \phi$.

The built-in symbol *TrigExpand* gives such a transformation.

```
In[14] := TrigExpand[Cos[3φ]]
Out[14] = Cos[φ]3 - 3Cos[φ]Sin[φ]2
In[15] := TrigExpand[Cos[4φ]]
Out[15] = Cos[φ]4 - 6Cos[φ]2Sin[φ]2 + Sin[φ]4
In[16] := TrigExpand[Sin[5φ]]
Out[16] = 5Cos[φ]4Sin[φ] - 10Cos[φ]2Sin[φ]3 + Sin[φ]5
```

2. If ω is given by (4), prove that

$$1 + \omega^h + \omega^{2h} + \dots + \omega^{(n-1)h} = 0. \quad (5)$$

I show the process to answer the exercise with Mathematica in the case $h = 3$ and symbolic calculation for n .

```
In[17] := Clear["Global`*"]; h = 3;
Sum[(ωh)k, {k, 0, (n - 1)}]
Out[17] =  $\frac{-1 + (\omega^3)^n}{-1 + \omega^3}$ 
In[18] := % /. ω -> Cos[2π/n] + i Sin[2π/n]
Out[18] =  $\frac{-1 + ((\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n})^3)^n}{-1 + (\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n})^3}$ 

In[19] := % /. (Cos[q_-] + i Sin[q_-])p_- -> (Cos[q/r] + i Sin[q/r])
Out[19] = 0
```

3.3 Analytic Geometry.

Problems of finding intersections between lines and circles, parallel or orthogonal lines, tangents, and the like usually become simple when expressed in complex form.

[EXERCISE] Show that all circles that pass through a and $\frac{1}{\bar{a}}$ intersect the circle $|z| = 1$ at right angles. (FIG.5, FIG.6)

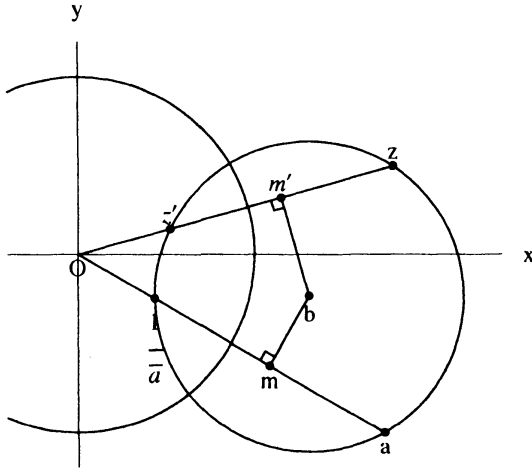


FIG. 5: 2.3. exercise.

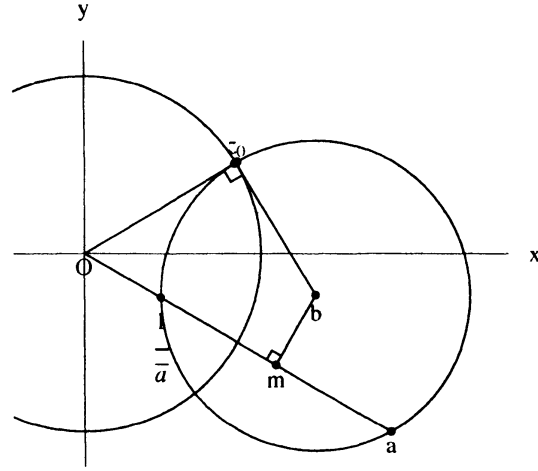


FIG. 6: 2.3. exercise.

Let the circle $|z| = 1$ be C , a circle which passes through a and $\frac{1}{\bar{a}}$ be B and its center be a point b . $\arg a = \arg \frac{1}{\bar{a}}$. $|a||\frac{1}{\bar{a}}| = 1$. Hence B and C intersect. Let the mid-point of a and $\frac{1}{\bar{a}}$ be m . That is $m = (a + \frac{1}{\bar{a}})/2$. Taking any real number t , $b = i t(a - \frac{1}{\bar{a}}) + m$. For any point z on B , there exists a correspondent point z' on S such that $\arg z' = \arg z$ and $|z||z'| = 1$ from the power theorem. Let the mid-point of z and z' be m' . Now take a point z_0 on the intersection of B and C . Then $|z_0| = 1$. Hence $z_0 = z'_0 = m'$. Therefore the tangent to B and the tangent to C intersect on z_0 .

3.4 The Spherical Representation.

The equation of the unit sphere S in three-dimensional space is $x_1^2 + x_2^2 + x_3^2 = 1$. Every point $P(x_1, x_2, x_3)$ on S , except $N(0, 0, 1)$, corresponds one-to-one to a complex number z as follows: (FIG.7, FIG.8)

$$z = \frac{x_1 + ix_2}{1 - x_3}, \quad (6)$$

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2}, \quad x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \quad (7)$$

By letting the point at infinity ∞ correspond to $N(0, 0, 1)$, we can regard the sphere as a representation of the extended complex plane. Writing $z = x + iy$ we can verify that

$$x : y : -1 = x_1 : x_2 : x_3 - 1, \quad (8)$$

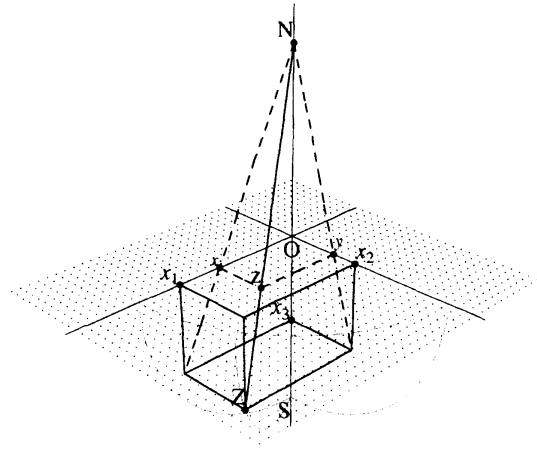
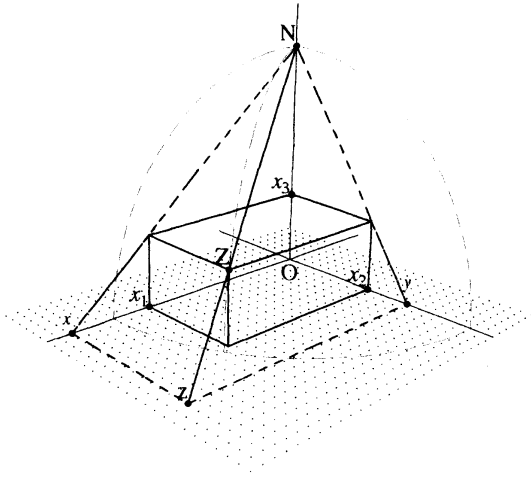


FIG. 7: Stereographic projection. ($x_3 > 0$) FIG. 8: Stereographic projection. ($x_3 < 0$)

This means that the point $z(x, y, 0)$, $Z(x_1, x_2, x_3)$ and $N(0, 0, 1)$ are in a straight line. The correspondence is a central projection from the center $N(0, 0, 1)$. This is called a stereographic projection.

Any circle on the sphere corresponds to a circle in the z -plane. When a circle on the sphere goes through N , the corresponding image in the z -plane is a straight line. (FIG.9, FIG.10)

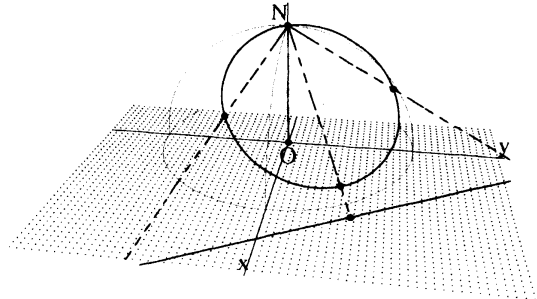
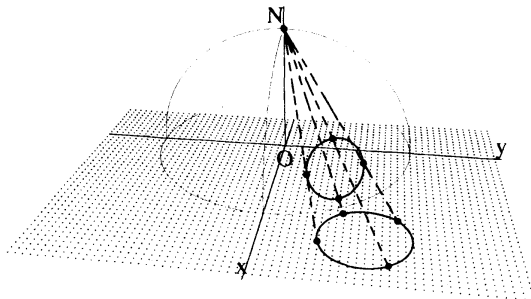


FIG. 9: A circle to a circle.

FIG. 10: A circle to a straight line.

Let $Z = (x_1, x_2, x_3)$ and $Z' = (x'_1, x'_2, x'_3)$ be stereographic projections of z and z' , and let $d(z, z')$ be the distance between Z and Z' ,

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}. \quad (9)$$

[EXERCISES]

1. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$.

Let $p(x_1, x_2, x_3)$ and $p'(x'_1, x'_2, x'_3)$ be opposite points on the Riemann sphere and let them correspond to z and z' . As p and p' are in the position of point symmetry about

the center of the sphere, $(x'_1, x'_2, x'_3) = -(x_1, x_2, x_3)$. From (6)

$$z = \frac{x_1 + ix_2}{1 - x_3}, \quad z' = \frac{-x_1 - ix_2}{1 + x_3}$$

By calculation $z\bar{z}' = -1$. On the other hand, if $z\bar{z}' = -1$, then $\bar{z}' = -\frac{1}{z}$, $z' = -\frac{1}{\bar{z}}$, $|z'|^2 = \frac{1}{|z|^2}$. From (7), $(x'_1, x'_2, x'_3) = -(x_1, x_2, x_3)$. i.e. p and p' are the opposite points on the Riemann sphere.

2. A cube has its vertices on the sphere S and its edges parallel to the coordinate axes. Find the stereographic projections of the vertices.

Let a vertex (a, a, a) , $(a > 0)$, be v and another vertex $(a, a, -a)$ be v' . Let the stereographic projections of v and v' be z and z' . Then $3a^2 = 1$, $a = \frac{1}{\sqrt{3}}$. From (6) $z = \frac{1+i}{\sqrt{3}-1}$, $z' = \frac{1+i}{\sqrt{3}+1}$. $z\bar{z}' = 1$. Hence $\bar{z}' = z^{-1}$. Therefore, the stereographic projections outside of the unit circle run counterclockwise $\{z = \frac{1+i}{\sqrt{3}-1}, -\bar{z}, -z, \bar{z}\}$, and inside the unit circle they run counterclockwise $\{\bar{z}^{-1}, -z^{-1}, -\bar{z}^{-1}, z^{-1}\}$. (FIG.11, FIG.12)

(CHECK) For each pair of $\{z, z''\}$ which corresponds to the diametrically opposite

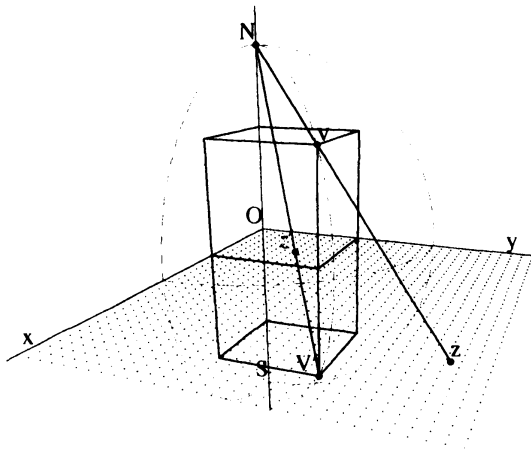


FIG. 11: exercise 2.

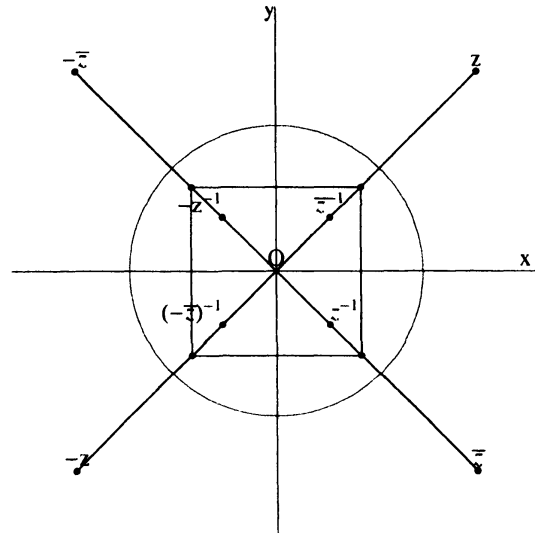


FIG. 12: exercise 2. Plan View.

points on the Riemann sphere, $z\bar{z}'' = -1$. This coincides with the fact mentioned in Exercise 1.

3. The same problem for a regular tetrahedron in a general position.(FIG.13, FIG.14)

Let the vertices of the tetrahedron be V_0, V_1, V_2, V_3 . Let the central angle of each edge of the tetrahedron be θ . Let the length of each edge of the tetrahedron be l . First, let $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $V_1 = \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix}$, then $V_2 = \begin{pmatrix} \cos \frac{2}{3}\pi \sin \theta \\ \sin \frac{2}{3}\pi \sin \theta \\ \cos \theta \end{pmatrix}$ and $V_3 = \begin{pmatrix} \cos \frac{4}{3}\pi \sin \theta \\ \sin \frac{4}{3}\pi \sin \theta \\ \cos \theta \end{pmatrix}$. When H is the center of gravity of $\triangle V_1 V_2 V_3$, $V_0 H = \frac{4}{3}$. $l = V_0 V_1 = \sqrt{(\frac{4}{3})^2 + \sin^2 \theta}$. On

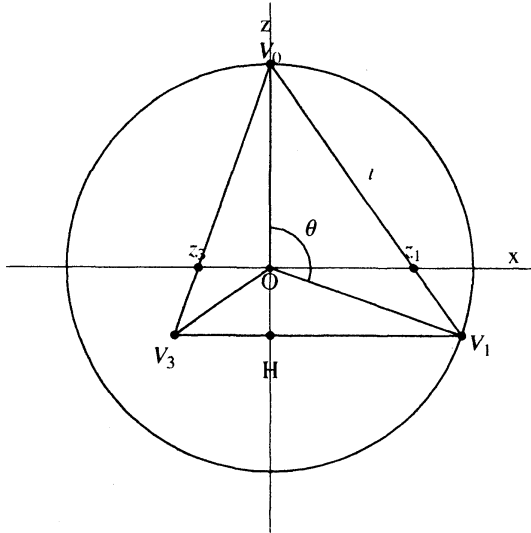


FIG. 13: exercise 3. Front Elevation.

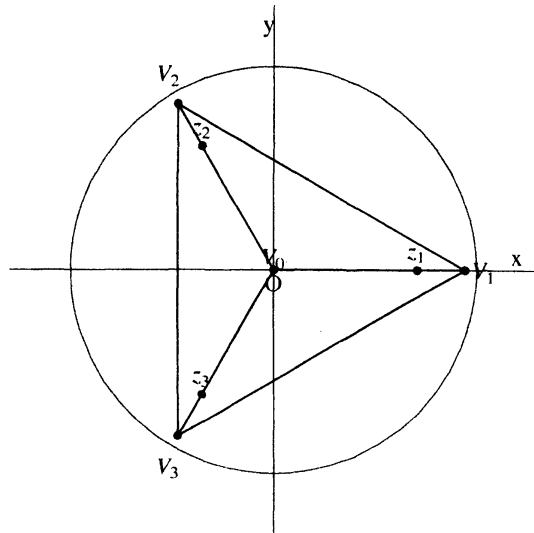


FIG. 14: exercise 3. Plan View.

the other hand, $l = V_2V_3 = \sqrt{3} \sin \theta$. By calculation, $\sin \theta = \frac{2\sqrt{2}}{3}$. Hence

$$V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, V_1 = \begin{pmatrix} \frac{2\sqrt{2}}{3} \\ 0 \\ -\frac{1}{3} \end{pmatrix}, V_2 = \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ \sqrt{\frac{2}{3}} \\ -\frac{1}{3} \end{pmatrix}, V_3 = \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ -\sqrt{\frac{2}{3}} \\ -\frac{1}{3} \end{pmatrix}$$

From (6),

$$z_0 = \infty, z_1 = \frac{1}{\sqrt{2}}, z_2 = -\frac{1}{2\sqrt{2}} + i\frac{\sqrt{3}}{2\sqrt{2}}, z_3 = -\frac{1}{2\sqrt{2}} - i\frac{\sqrt{3}}{2\sqrt{2}}$$

Let $M(\kappa, \mu, \lambda)$ be a matrix which yields κ rotation around the x-axis, μ rotation around the rotated z-axis, and λ rotation around the rotated y-axis.

$$M(\kappa, \mu, \lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \kappa & -\sin \kappa \\ 0 & \sin \kappa & \cos \kappa \end{pmatrix} \begin{pmatrix} \cos \mu & -\sin \mu & 0 \\ \sin \mu & \cos \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \lambda & 0 & -\sin \lambda \\ 0 & 1 & 0 \\ \sin \lambda & 0 & \cos \lambda \end{pmatrix}$$

Then $\{V'_i = M(\kappa, \mu, \lambda)V_i; i = 0, 1, 2, 3. 0 \leq \kappa, \mu, \lambda \leq 2\pi\}$ gives the vertices in a general position.

$$\{z_0, z_1, z_2, z_3\} (\kappa = 0, \mu = 0, \lambda = 0) \quad \text{and} \quad \{z'_i = \frac{x'_{1i} + ix'_{2i}}{1 - x'_{3i}}; V'_i = \begin{pmatrix} x'_{1i} \\ x'_{2i} \\ x'_{3i} \end{pmatrix}, i =$$

$0, 1, 2, 3., 0 \leq \kappa, \mu, \lambda \leq 2\pi, x'_{3i} \neq 0\}$ are the stereographic projections of the vertices of the tetrahedron in a general position.

(CHECK) It is verified, with random numbers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for κ, μ and λ , that V'_0, V'_1, V'_2, V'_3 are on the sphere S by checking $x'^2_{1i} + x'^2_{2i} + x'^2_{3i} = 1$ for $V_i, 0 \leq \forall i \leq 3$, and that they are the vertices of a regular tetrahedron by checking the angle between

OV_i and OV_j is equal to $\theta = \pi + \tan^{-1} \frac{1}{\sqrt{2}}$, $0 \leq \forall i, j \leq 3$, $i \neq j$, with the built-in Mathematica symbol *VectorAngle*.

4. Let Z, Z' denote the stereographic projections of z, z' , and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive (9).

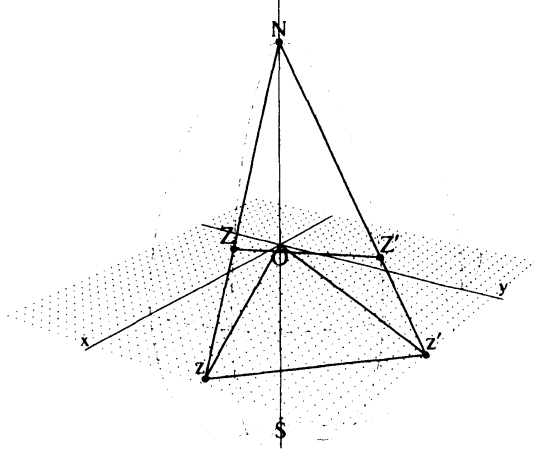


FIG. 15: exercise 4.

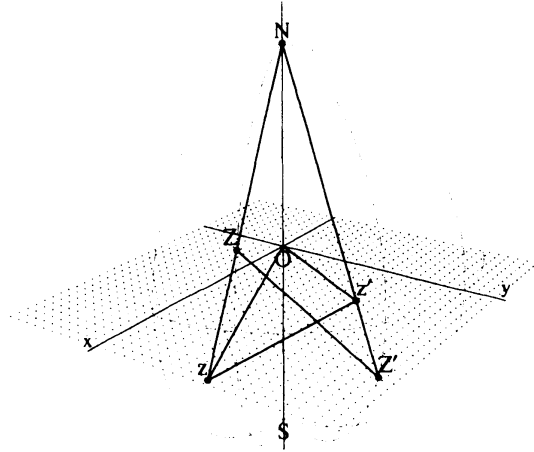


FIG. 16: exercise 4.

As shown in FIG.7 and FIG.8,

$$NZ = \sqrt{2 - 2x_3}, \quad Nz = \frac{\sqrt{2}}{\sqrt{1 - x_3}}, \quad NZ' = \sqrt{2 - 2x'_3}, \quad Nz' = \frac{\sqrt{2}}{\sqrt{1 - x'_3}}.$$

Therefore

$$\frac{Nz}{NZ'} = \frac{Nz'}{NZ} = \frac{1}{\sqrt{(1 - x_3)(1 - x'_3)}}.$$

Hence $\triangle NZZ'$ and $\triangle Nzz'$ are indirectly similar. Consequently, from (7), (9) is derived.

4 CONCLUSION

Through this work, Mathematica allowed me to have vivid mental pictures of the subjects with interactive and dynamic visualization, and to experiment with the ideas on the subjects.

When I tried to use Mathematica for some subjects, I was led to think about and understand the subjects deeply. Not only using the finished programs but also the programming processes facilitated my understanding of the subjects.

References

- [1] Ahlfors, L.V., *Complex Analysis*, 3rd edition, McGraw-Hill, 1979.