

# Non-expressibility of a class of finite graphs

Akito Tsuboi  
Institute of Mathematics,  
University of Tsukuba

## 1 Introduction

A graph is an  $R$ -structure in which  $R$  is irreflexive and symmetric. In this article, we will use a compactness argument to examine the expressibility of a class of finite graphs. First let us explain a simple example as follows: Let  $\mathcal{C}$  be the class of all finite circles, i.e., all graphs  $H$  of the form:

- $H = \{h_1, \dots, h_n\}$ ;
- $R^H = \{h_1h_2, h_2h_3, \dots, h_{n-1}h_n, h_nh_1\}$ , where  $hk$  denotes  $(h, k), (k, h)$ .

Then  $\mathcal{C}$  is not an elementary class, because finiteness cannot be represented by a sentence or even by a set of sentences. Furthermore, there is no  $R$ -sentence  $\varphi$  such that, for any finite graphs  $G$ ,

$$G \models \varphi \iff G \in \mathcal{C}.$$

For showing this, we can use a compactness argument: Suppose that there were such a sentence  $\varphi$ . Let  $T$  be the theory consisting of the following sentences:

1. Graph axioms;
2. Every node has exactly two neighbors;
3. There is no (finite) cycles.

Since every finite part of  $T$  is satisfied by a graph in  $\mathcal{C}$ , by compactness, there is a countable infinite graph  $G_0 \models T \cup \{\varphi\}$ . By extending  $G_0$ , if necessary, we can assume  $G_0$  is a disjoint union of countably many  $\mathbb{Z}$ -chains. Notice that every graph consisting of two circles does not belong to  $\mathcal{C}$ . So, again by compactness, there is a countable infinite graph  $G_1 \models T \cup \{\neg\varphi\}$ .  $G_1$  is also assumed to be a disjoint union of countably many  $\mathbb{Z}$ -chains. So, we must conclude  $G_0 \cong G_1$ , a contradiction.

In this paper, concerning finite graphs, we consider a different type of (non-)expressibility.

## 2 Preliminaries

Let  $L = \{R(*,*)\}$  and  $L^* = L \cup \{X_i(*) : i < n\}$ , where  $R$  is a binary predicate symbol, and  $X_i$ 's are unary predicate symbols. Let  $T$  be a finite set of  $L$ -sentences and  $\varphi(X_0, \dots, X_{n-1})$  an  $L^*$ -sentence.

**Definition 1.**  $PC_{fin}(\varphi, T)$  is the class of  $L$ -reducts of finite models of  $T \cup \{\varphi\}$ . If  $T$  is the axiom for graphs, we simply write  $PC_{fin}(\varphi)$  for  $PC_{fin}(\varphi, T)$ .  $PC$  stands for ‘pseudo elementary class.’

**Example 2.** 1. Let  $\mathcal{C}$  be the class of all non-connected finite graphs. Then, there is a sentence  $\varphi$  such that  $PC_{fin}(\varphi) = \mathcal{C}$ . Let  $\varphi$  be the sentence asserting that (i) both  $X_0$  and  $\neg X_0$  are non-empty, and (ii) there is no edge between  $X_0$  and  $\neg X_0$ . Then clearly  $\varphi$  satisfies the required condition.

2. Let  $\mathcal{C}$  be the class of all finite graphs with a cycle. Then, there is a sentence  $\varphi$  such that  $PC_{fin}(\varphi) = \mathcal{C}$ .

Now another important point will be explained below. In the structure  $\mathbb{N}$ , by a coding method, finite sets are represented. In other words,  $\mathbb{N}$  can be considered as a model of finite set theory. So, we assume  $\mathbb{N} = (\mathbb{N}, 0, 1, +, \cdot, <, \in)$ , where  $\in$  is the membership relation. Finite graphs are objects in  $\mathbb{N}$ . Let  $G$  be a finite graph with the code  $a_G$ , i.e.  $G = \{g \in \mathbb{N} : \mathbb{N} \models g \in a_G\}$ . The connectedness of  $G$  can be expressed by a sentence in  $\mathbb{N}$  as follows:  $G$  is connected  $\iff$  there is a coded function  $f : [0, n] \rightarrow G$  such that (i)  $\text{ran}(f) = G$ , and (ii)  $R(f(i), f(i+1))$  ( $\forall i < n$ ).

Let  $\mathbb{N}^* \succ \mathbb{N}$  be a recursively saturated countable model. In  $\mathbb{N}^*$ , a coded set  $\{x \in \mathbb{N}^* : x \in a\}$ , where  $a \in \mathbb{N}^*$ , is not necessarily finite. Let  $\text{con}(x)$  be

the formula expressing (in  $\mathbb{N}$ ) that the graph coded by  $x$  is connected, and let  $a \in \mathbb{N}^*$  be an element with  $\mathbb{N}^* \models \text{con}(a)$ . The graph coded by  $a$  is not connected in general, although it is connected in the sense of  $\mathbb{N}^*$ .

### 3 Non-expressibility

As an application of compactness argument to finite graphs, we show the following proposition, which is due to Fagin [1].

**Proposition 3.** *Let  $\mathcal{C}$  be the class of all finite connected graphs. Then there is no  $L^*$ -sentence  $\varphi = \varphi(X_0, \dots, X_{n-1})$  with  $\mathcal{C} = PC_{fin}(\varphi)$ .*

*Sketch of Proof.* A more detailed proof of a more general result be given in our forthcoming paper. Suppose that there were a sentence  $\varphi(X_0, \dots, X_{n-1})$  with  $\mathcal{C} = PC_{fin}(\varphi)$ . Let  $C_n$  be the circle graph with the universe  $[0, n-1]$  and the edges  $R(i, i+1)$  ( $i < n-1$ ) and  $R(n-1, 0)$ . Now we work in  $\mathbb{N}^*$ . Let  $n^*$  be a nonstandard number, and let  $G = C_{n^*}$ . Since  $C_n \models \varphi$  for all  $n$ , we have some coded sets  $D_0, \dots, D_{n-1}$  such that

$$G \models \varphi(D_0, \dots, D_{n-1}). \quad (1)$$

By the recursive saturation, there are two points  $a < b \in \mathbb{N}^*$  such that  $\text{tp}_{\mathbb{N}^*}(a/n^*, d_0, \dots, d_{n-1}) = \text{tp}_{\mathbb{N}^*}(b/n^*, d_0, \dots, d_{n-1})$ , where  $d_i$  is the code of  $D_i$ . We define a new graph  $G'$  by:

1. The universe of  $G'$  is the same as  $G$ , hence  $|G'| = |G| = [0, n^* - 1]$ ;
2.  $R^{G'} = R^G \setminus \{a(a+1), b(b+1)\} \cup \{a(b+1), b(a+1)\}$

Each of  $G$  and  $G'$  is a disjoint union of  $\mathbb{Z}$ -chains with coloring by  $X_i$ 's. By our construction,  $G$  and  $G'$  are definable (and hence both are coded in  $\mathbb{N}^*$ ). Moreover, they are isomorphic as  $\{R, X_0, \dots, X_{n-1}\}$ -structures, since they have the same  $\mathbb{Z}$ -chains (counting multiplicity) with coloring. Hence we have:

**Claim A.**  $G \cong_{\{R, X_0, \dots, X_{n-1}\}} G'$ . *This isomorphism, say  $\sigma$ , is not definable in  $\mathbb{N}^*$ . But, each  $D_i$  is  $\sigma$ -invariant.*

On the other hand,  $G'$  is not connected in the sense of  $\mathbb{N}^*$  (in fact, it is a disjoint union of two circles), hence we have:

**Claim B.**  $G' \models \neg\varphi(B_0, \dots, B_{n-1})$ , for all coded sets  $B_i$ 's.

The two claims above together with (1) yield a contradiction.  $\square$

## References

- [1] Fagin, R. Monadic generalized spectra. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 21 (1975), 123 – 134.
- [2] Libkin, L. *Elements of finite model theory*. Berlin: Springer, (2015).
- [3] Kaye, R. W. *Models of peano arithmetic*; Clarendon Press: Oxford, 2011.
- [4] Our forthcoming paper.