

# On the convergence of a three-dimensional crystalline motion to Gauss curvature flow

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## Abstract

We consider an approximation of the Gauss curvature flow in  $\mathbb{R}^3$  by so-called crystalline motion. Here, the Gauss curvature flow makes a *smooth strictly convex surface* shrink with the outward normal velocity equals to the Gauss curvature with negative sign. The crystalline motion was introduced by Taylor [15] and Angenent & Gurtin [1] to analyze crystal growth mathematically. The most typical crystalline motion of polygon in  $\mathbb{R}^2$  makes each edge of a polygon keep the same direction but move with *the normal speed inversely proportional to its length*. Although such motion is very restrictive at first glance, it is very useful not only in the mathematical theory of crystal growth but also as a numerical method for free boundary problems. In two dimensional case, there are already many researches on the relation between the crystalline motion of polygonal curves and the curvature driven motion of curves (e.g. [12]).

We extend the most typical two dimensional crystalline motion to a three dimensional one whose Wulff shape is a convex polyhedron ( $\tilde{W}^k$ ). Here the Wulff shape represents the anisotropy of the problem. This motion makes each side of a polyhedron move with *the normal speed inversely proportional to its area*. We prove this crystalline motion converges to the Gauss curvature flow in  $\mathbb{R}^3$  under the assumptions that the polyhedra  $\tilde{W}^k$  converges to the unit ball  $B^3$  in the Hausdorff distance and are symmetric with respect to the origin.

K. Ishii and H.M. Soner[12] showed the convergence of the two dimensional crystalline motion to the curve shortening flow by a kind of perturbed test function methods. We employ their method to prove our result under aid from the theory of Minkowski problem (e.g. [14]).

## 1 Introduction

In this paper, we consider an approximation of three dimensional Gauss curvature flow of smooth convex surfaces by using so-called crystalline algorithm.

First we explain the crystalline algorithm. To investigate the crystal growth mathematically, Taylor[15] and Angenent & Gurtin[1] introduced the crystalline

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curvature flow and the crystalline motion of specific kind of polygons which gives the solution to the crystalline curvature flow. Crystalline algorithm is an approximation method for some kind of moving boundary problems by using this crystalline motion of the polygons.

Let us explain the crystalline mean curvature flow of closed hypersurfaces. Let  $\gamma$  be a positive, continuous, and homogeneous of degree one function on  $\mathbb{R}^d$  ( $d = 2, 3$ ), which is called surface energy density, and define the surface energy of closed hypersurface  $S \subset \mathbb{R}^d$  by

$$I(S) = \int_S \gamma(\nu) dS.$$

Here,  $\nu$  denotes the unit normal vector field on  $S$ . Then, the gradient flow of  $I$  is called anisotropic mean curvature flow. And when  $\gamma(p) = |p|$ , this flow is nothing but the classical mean curvature flow. Let us define the *Wulff shape* for  $\gamma$  by

$$\tilde{W} = \{x \in \mathbb{R}^d \mid \langle x, \nu \rangle \leq \gamma(\nu)\},$$

which represents the anisotropy of the problem. Here,  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^d$ . In the case where this shape is a convex polygon, the energy  $I$  is called *crystalline surface energy* and the gradient flow is called *crystalline mean curvature flow*.

Hereafter, we only deal with the case where the polygons and the closed curves are all convex, for simplicity. The solution to the two dimensional crystalline curvature flow is given by so-called crystalline motion. This is the motion of admissible polygons. A polygon is called *admissible* with respect to the Wulff shape  $\tilde{W}$  if and only if the set of all outward unit normal vector of the polygon coincides with the one of  $\tilde{W}$  and each pair of normal vectors of adjacent edges of the polygon is adjacent in  $\tilde{W}$ . The crystalline motion which is proposed by Taylor is the motion of admissible polygons whose normal velocity  $v_j$  is proportional to

$$\kappa_j = \frac{\tilde{L}_j}{L_j}.$$

Here,  $v_j$  and  $L_j$  are the outward normal velocity and the length of the  $j$ th edge of the admissible polygon, respectively, and  $\tilde{L}_j$  is the length of the  $j$ th edge of the Wulff shape  $\tilde{W}$ , respectively. And the quantity  $\kappa_j$  is called the *crystalline curvature* of the  $j$ th edge of the admissible polygon. We note that during the evolution the admissibility is preserved. Hence, this motion makes each edge of a polygon keep the same direction but move with the *normal speed inversely proportional to its length*. We also note that this motion is governed by a system of ordinary differential equations for  $L_j$ .

There are already many researches on the two dimensional crystalline motion. It is known that as the number of edges of  $\tilde{W}$  goes to infinity and  $\tilde{W}$  converges to a circle, the two dimensional crystalline motion converges to the curvature flow of plane curves (see [5, 9, 10, 6, 7, 12, 17]). Especially, in [7, 12] the convergence between crystalline motion and curvature flow is proved in the

case where the curves are not necessarily convex. In other words, we can approximate the curvature flow by crystalline motion. Such a way of approximation is called *crystalline algorithm*. Numerical schemes based on this algorithm are also studied and the class of problems which can be treated by this algorithm is extended ([17, 18], etc.).

Here, we would like to say about a good nature of crystalline algorithm as a numerical scheme for moving boundary problems. Generally speaking, to compute the solution for moving boundary problems by discretizing the boundary curves directly is not easy task, since it often causes numerical instability like concentration of the points. In Fig.1, we plot the result of computation for free boundary problems which are governed by the evolution law  $v = -|H|^{\alpha-1}H$  by a simple numerical scheme. Here  $v$  and  $H$  are the outward normal velocity and the curvature of the free boundary, respectively, and  $\alpha$  is a positive parameter. In this figure, the most outside curves are the initial curves. We can observe numerical instabilities which we mentioned. Although several methods are proposed preventing such instability, these methods often employ artificial tricks like distribution of points.

We would like to claim that the crystalline algorithm is a good method from this point of view. We plot the computation results for the same problem as Fig.1 by the crystalline algorithm in Fig.2. Here, we particularly note that the crystalline algorithm do not need any artificial technique like redistribution of the partition points to prevent the instability and the convergence of the algorithm is proved for the problem of Fig.2[17, 18].

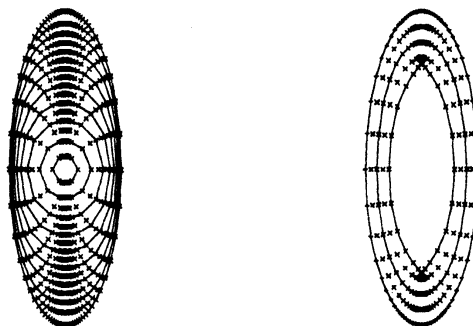


Figure 1: A simple method:  $\alpha = 1$ (left),  $\alpha = 1/3$ (right).

It is an interesting question that whether the three dimensional version of crystalline algorithm can be constructed. However, crystalline algorithm for higher dimensional mean curvature flow is not success, yet. In three dimensional case, it is not clear that the crystalline mean curvature flow can be solved in the what class of polyhedra. Moreover, for the crystalline mean curvature flow, the comparison principle does not hold in general, while the convergence results of the two dimensional case crucially depend on the comparison principle. For

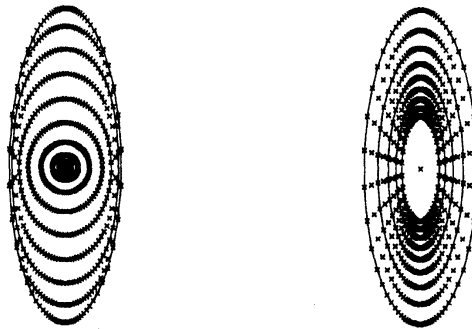


Figure 2: Crystalline algorithm:  $\alpha = 1$ (left),  $\alpha = 1/3$ (right).

more precise information about these things, we refer [8, 2, 3].

Hence, in this research, we consider three dimensional Gauss curvature flow and the approximation of it by a crystalline algorithm.

The organization of the paper is as follows: In §2 we shall introduce the Gauss curvature flow and a generalization of it. In §3, the most typical crystalline motion in two dimension is extended to a three dimensional crystalline motion. We shall also explain the wellposedness of this extended problem in this section. Our main result will be explained in the final section §4.

## 2 Gauss curvature flow

Let us explain the Gauss curvature flow of smooth convex surface  $\Gamma(t) \subset \mathbb{R}^3$ . This flow makes  $\Gamma(t)$  shrink with the outward normal velocity equals to the Gauss curvature. Let  $v$  and  $\kappa$  be the outward normal velocity and the Gauss curvature of  $\Gamma(t)$ , respectively. The support function of the surface  $\Gamma(t)$  is defined by

$$h(\nu, t) = \sup\{\langle \nu, p \rangle \mid p \in \Gamma(t)\},$$

where  $\nu$  denotes the outward unit normal vector of the surface  $\Gamma(t)$ . Using these notation, the evolution law for the Gauss curvature flow can be described by

$$v = \frac{\partial h}{\partial t} = -\kappa. \quad (1)$$

For smooth convex surfaces  $\Gamma_0$ , the existence and the uniqueness of the solution to the Gauss curvature flow is shown in [16, 4]. More precisely, the following theorem holds.

**Theorem 1** Let  $\Gamma_0$  be a smooth, strictly convex, and closed surface. There exists a unique solution  $\Gamma(t)$  for (1) with initial surface  $\Gamma_0$ . Moreover,  $\Gamma(t)$  remains smooth and strictly convex until a finite time, say  $T$ , and  $\Gamma(t)$  shrinks to

a point at this time  $T$ . The extinction time  $T$  is given by  $T = V(\Omega_0)/(3V(B^3))$ , where  $\Omega_0$  is the set which is enclosed by  $\Gamma_0$  and  $V$  denotes the volume.

**Remark 1** We can also consider a generalization of (1):

$$v = \frac{\partial h}{\partial t} = -\kappa^\alpha. \quad (2)$$

Here,  $\alpha$  is a positive constant. For any  $\alpha$  and any smooth convex surfaces  $\Gamma_0$ , the existence and the uniqueness of the solution to (2) are also established in [4]. Moreover, the solution surface disappear in finite time, say  $T$ .

### 3 Three dimensional crystalline motion

In this section, we extend the crystalline motion of convex admissible polygons in the plane to the one of convex polyhedra. Our three dimensional crystalline motion is defined as follows: Let the *Wulff shape*  $\tilde{W}$  be an  $N$  faceted convex polyhedron. Let  $\tilde{h}_j$ ,  $\nu_j$ , and  $\tilde{A}_j$  the support, the unit outward normal vector, and the area of the  $j$ th facet of  $\tilde{W}$ , respectively. We set  $\tilde{h} = (\tilde{h}_j)_{\{1 \leq j \leq N\}}$ . For this  $\tilde{W}$ , an  $N$ -faceted polyhedron  $\Omega$  and its boundary  $\Gamma$  is called  $\tilde{W}$ -admissible if and only if the outward normal vector of the  $j$ -th facet, say  $\Gamma_j$ , of  $\Gamma$  is  $\nu_j$  for all  $j$ . See figure ?? . The  $\tilde{W}$ -crystalline Gauss curvature flow is the motion of the  $\tilde{W}$ -admissible  $\Gamma(t)$ , which is the boundary of  $\tilde{W}$ -admissible polyhedra  $\Omega(t)$ , whose evolution law is given by

$$v_j = \frac{dh_j}{dt} = -\tilde{h}_j \frac{\tilde{A}_j}{A_j}. \quad (3)$$

Here,  $h_j$ ,  $A_j$ , and  $v_j$  denote the support, area, and the outward normal velocity of the  $j$ th facet of  $\Gamma(t)$ .

Although the comparison principle does not hold in general for the three dimensional crystalline mean curvature flow, we can prove this principle for the  $\tilde{W}$ -crystalline Gauss curvature flow (3). This fact plays an important role in the proof of our main result.

**Lemma 1** Let  $\tilde{W}$  be a convex polyhedron in  $\mathbb{R}^3$  including the origin as its interior point, and  $\Gamma'(t)$  and  $\Gamma(t)$  solutions to  $\tilde{W}$ -crystalline flow for  $t \in [0, T)$ . Then,  $\Gamma'(0) \subset \Gamma(0) \cup \Omega(0)$  implies  $\Gamma'(t) \subset \Gamma(t) \cup \Omega(t)$  for all  $t \in [0, T)$ . Here,  $\Omega(t)$  is the open set enclosed by  $\Gamma(t)$ .

For this problem (3), we can prove the following theorem.

**Theorem 2** Let  $\Gamma_0$  be  $\tilde{W}$ -admissible convex  $N$  faceted polyhedron,  $\Omega_0$  the boundary of it. There exists unique solution to the problem (3) with initial surface  $\Gamma_0$ . Moreover,  $V(\Gamma(t))$  vanishes at a finite time, say  $T$ . This  $T$  is given by  $T = V(\Omega_0)/(3V(\tilde{W}))$ .

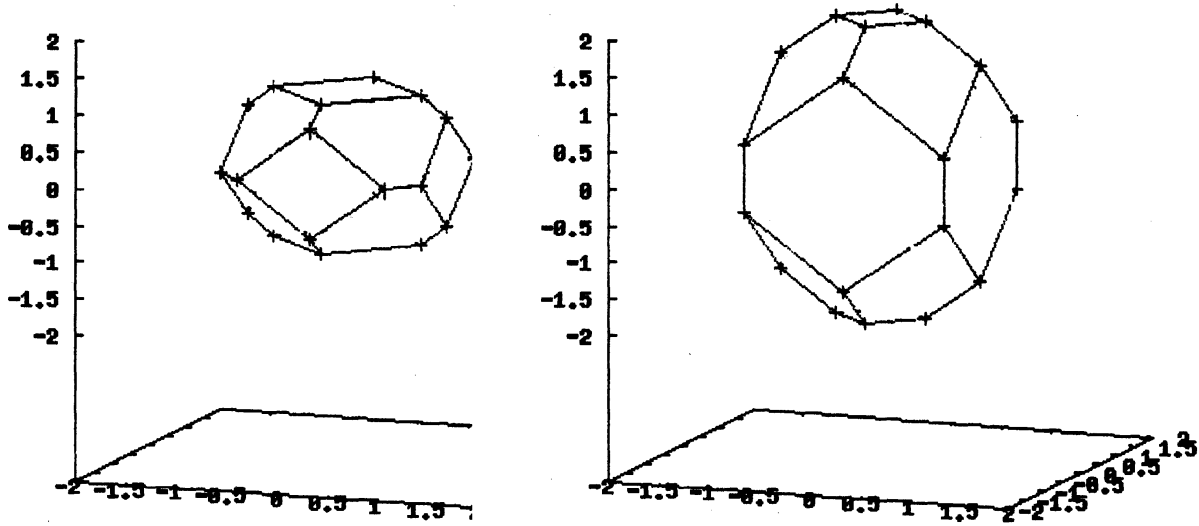


Figure 3: Example of a Wulff shape  $\tilde{W}$ (left) and a  $\tilde{W}$  admissible polygon(right)

Proof of the lemma and the theorem can be found in [19].

**Remark 2** We can also consider a generalization of (3):

$$v_j = \frac{dh_j}{dt} = -\tilde{h}_j \left( \frac{\tilde{A}_j}{A_j} \right)^\alpha. \quad (4)$$

Here  $\alpha$  is a positive constant. For this problem (4), we can also prove the comparison lemma and the existence and the uniqueness of solution.

We also note that for the solution  $\Gamma(t)$  of this problem, the volume  $V(\Gamma(t))$  vanishes in finite time but  $\Gamma(t)$  does not necessarily shrinks to a point. For example, let us consider the solution to the problem (4) which starts from a rectangular parallelepiped under the condition that the Wulff shape is a cube. We assume that the rectangular parallelepiped has the symmetry,  $h_1 = h_4$  and

$h_2 = h_3 = h_5 = h_6$ . Then the problem can be reduced to the following system of ordinary differential equations:

$$\begin{aligned}\frac{dh_1}{dt} &= -\frac{1}{h_2^{2\alpha}}, \\ \frac{dh_2}{dt} &= -\frac{1}{h_1^\alpha h_2^\alpha}.\end{aligned}$$

Since  $h_1^{-\alpha} h_1 = h_2^{-\alpha} h_2$  holds, we obtain

$$h_1^{1-\alpha}(t) - h_2^{1-\alpha}(t) = h_1^{1-\alpha}(0) - h_2^{1-\alpha}(0),$$

where  $\dot{\phantom{x}}$  denotes the derivative with respect to time  $t$ . Hence, for  $0 < \alpha < 1$ , if  $h_1^{1-\alpha}(0) - h_2^{1-\alpha}(0) > 0$  then  $\Gamma(t)$  shrinks to a line segment and if  $h_1^{1-\alpha}(0) - h_2^{1-\alpha}(0) < 0$  then  $\Gamma(t)$  shrinks to a plane segment.

For two dimensional crystalline motion, such degeneracy of the extinction is already known and extensively studied in [13].

## 4 Main result

Now we consider the approximation of the Gauss curvature flow by a sequence of the crystalline Gauss curvature flow. Hereafter,  $k$  denotes the parameter which indicates the approximation, and the larger  $k$  corresponds to the better approximation. Let  $\tilde{W}^k$  be an  $N^k$  faceted convex polyhedron which is symmetric with respect to the origin. For this  $\tilde{W}^k$ , we have a  $\tilde{W}^k$ -crystalline Gauss curvature flow. Let  $\Gamma^k(t)$  be the solution of this flow with initial surface  $\Gamma_0^k$  and  $\Omega^k(t)$  the  $\tilde{W}^k$ -admissible convex  $N^k$  faceted polyhedron which is enclosed by  $\Gamma^k(t)$ . Under several assumption, we can prove the convergence of  $\Gamma^k(t)$  to a solution  $\Gamma(t)$  of the Gauss curvature flow as  $k$  goes to infinity. Let  $B^3$  be  $\{P \in \mathbb{R}^3 \mid |P| \leq 1\}$  and  $d_H$  the Hausdorff distance.

We assume the following four things.

- (A1) The convex  $N^k$  faceted polyhedron  $W^k$  is symmetric with respect to the origin.
- (A2)  $\lim_{k \rightarrow \infty} d_H(\tilde{W}^k, B^3) = 0$ .
- (A3)  $\Gamma_0^k$  is  $\tilde{W}^k$ -admissible convex  $N^k$  faceted polyhedron.
- (A4)  $\lim_{k \rightarrow \infty} d_H(\Gamma_0^k, \Gamma_0) = 0$ .

**Theorem 3** We assume (A1),(A2),(A3),(A4). Let  $\Gamma(t)$  be the solution to (1) with initial data  $\Gamma_0$ ,  $T$  its extinction time,  $\Gamma^k(t)$  the solution to the  $\tilde{W}^k$ -crystalline Gauss curvature flow with initial data  $\Gamma_0^k$ . Then for any  $\epsilon > 0$ , we have

$$\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T-\epsilon} d_H(\Gamma^k(t), \Gamma(t)) = 0.$$

We briefly comment on the proof of this result. Precise description will be found in [19]. By the aid from the theory of Minkowski problem ([14]), we can construct a nice sequence of  $\tilde{W}^k$ -admissible polyhedra which converges to ellipsoid.

**Lemma 2** For positive numbers  $a$  and  $b$ , we set

$$E = E(a, b) = \{(x, y, z) \mid ax^2 + by^2 + z^2 \leq 1\}.$$

For any  $k \in \mathbb{N}$ , there uniquely exists a  $\tilde{W}^k$ -admissible polyhedron  $E^k$  symmetric with respect to the origin such that

$$\kappa^E(\nu_i^k) = \frac{\tilde{A}_i^k}{A_i^{E^k}} \quad (5)$$

holds for all  $1 \leq i \leq N^k$ . Moreover,

$$\lim_{k \rightarrow \infty} d_H(E^k, E) = 0 \quad (6)$$

holds. Here,  $\nu_i^k$  denotes the outward normal vector of the  $i$ -th side of  $\tilde{W}^k$ ,  $\kappa^E(\nu)$  Gauss curvature of  $E$  at the point where the outward normal vector is  $\nu$ ,  $\tilde{A}_i^k$  the area of the  $i$ -th side of  $\tilde{W}^k$ ,  $A_i^{E^k}$  the area of the  $i$ -th side of  $E^k$ , respectively.

Using this sequence, we can prove that the upper and lower semicontinuous envelopes of  $\{\Omega^k(t)\}$ ,

$$\begin{aligned} \hat{\Omega}(t) &= \bigcap_{\varepsilon > 0, N \in \mathbb{N}} \text{cl} \left( \bigcup_{|s-t| \leq \varepsilon, s \geq 0, k \geq N} (\Gamma^k(s) \cup \Omega^k(s)) \right), \\ \underline{\Omega}(t) &= \bigcup_{\varepsilon > 0, N \in \mathbb{N}} \text{int} \left( \bigcap_{|s-t| \leq \varepsilon, s \geq 0, k \geq N} \Omega^k(s) \right), \end{aligned}$$

are weak sub and super solutions in viscosity sense. Using a kind of perturbed test function methods, which is employed by K. Ishii and H.M. Soner[12] to prove the convergence of two dimensional crystalline algorithm, we can obtain the result above.

**Remark 3** We can also prove the convergence between (2) and (4) under the same assumptions (A1) to (A4). The proof is a simple modification of proof of the main result.

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