

Quandle coloring conditions and zeros of the Alexander polynomials of Montesinos links

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Abstract

We give a simple condition for the existence of a nontrivial quandle coloring on a Montesinos link, which describes the distribution of the zeros of the Alexander polynomial. By this condition, we prove that the real parts of the zeros of the Alexander polynomial of any link that admits an alternating pretzel diagram are greater than -1 : Hoste's conjecture holds for such a link. Furthermore, we show the existence of infinitely many counterexamples for Hoste's conjecture.

1 Introduction

The Alexander polynomial is one of the most famous classical invariants for knots and links, which can be defined in a purely topological way. By contrast, the property “alternating” is rather diagrammatic one; although a topological characterization of alternating knots and links was given independently by [4] and [6], their characterization uses spanning surfaces, which is not necessarily orientable, and then it seems irrelevant to problems of Alexander polynomials or their roots.

Nevertheless, based on his computer experiments, Hoste conjectured in 2002 that the real parts of the roots of the Alexander polynomial Δ_K of any alternating knot K will be greater than -1 , which is called Hoste's conjecture. We should remark that a real root of Δ_K is positive by the alternating property ([2], [14]) of Δ_K . In the case of 2-bridge knots, the positive result is known: if K is a 2-bridge knot and t is a root of Δ_K , we have $-3 < \operatorname{Re} t < 6$ by [13] and $-3/2 < \operatorname{Re} t < 3 + 2\sqrt{2}$ independently by [10] and [15], and finally the conclusive inequality $\operatorname{Re} t > -1$ was obtained by the author [8]. Hoste's conjecture has interested many researchers in the distribution of the roots of Alexander polynomials, rather than themselves or their coefficients, but it is in many cases difficult to detect or estimate where Alexander polynomials have roots.

In this paper, we give a criterion for determining whether a complex number t is a root of the Alexander polynomial Δ_L of a Montesinos link L . More precisely, a rational function c_τ is defined for each oriented rational tangle τ , and then we can judge whether the Montesinos link L composed of rational tangles $\tau_1, \dots, \tau_k$ admits a nontrivial $\mathbb{C}_t$ -coloring from the sum or product (depending on the way of the orientation of L) of $c_{\tau_i}(t)$; see Theorems 3.5 and 3.10 for details. Here,

$\mathbb{C}_t$ is a quandle and it is known that there exists a nontrivial $\mathbb{C}_t$ -coloring if and only if $\Delta_L(t) = 0$.

We can apply this criterion to the problem of Hoste's conjecture to obtain two results. As the first one, Hoste's conjecture holds for the links that have alternating pretzel diagrams (Theorems 4.1 and 4.3). We remark that this holds for such links with any orientation. However, when we consider all alternating Montesinos links, the conjecture does not hold; this is the second result. In fact, we show the existence of infinitely many counterexamples for Hoste's conjecture (Theorem 5.4). Although real parts of counterexamples are only slightly less than -1 , we also see that the set of roots in counterexamples is dense on an open subset in $\mathbb{C}$.

Our proof of the first result is based on the "one-sided" property of $c_\tau(t)$; for example in the case where the criterion uses the sum of $c_\tau(t)$, supposing $\operatorname{Re} t \leq -1$, we can find an open half plane P ($\not\ni 0$) such that $c_\tau(t) \in P$ for every pretzel tangle τ with positive fraction and appropriate orientation. Then, the sum of such $c_\tau(t)$ is also a member of P and especially is not equal to 0. Theorem 3.10 concludes that $\Delta_L(t) \neq 0$ for the corresponding link L .

A family of rational tangles with positive fraction often has the one-sided property, but this is not always true; this is a key to finding counterexamples, i.e. the second result. Suppose that for some t ($\operatorname{Re} t < -1$) and a family of rational tangles, the one-sided property does not hold. Then, we use many copies of them to construct a Montesinos link so that it almost admits a nontrivial $\mathbb{C}_t$ -coloring. By a perturbation of t , we show that the link actually has a nontrivial $\mathbb{C}_t$ -coloring.

Although the original conjecture turned out to be false, we still have many interesting problems. For example, is there a lower bound for the real parts? If there is, what is the greatest lower bound? The author has no idea how to work on these problems, but in this paper we show the existence of a lower bound for the alternating Montesinos links (Theorem 6.1). Here we give a lower bound -2 , though this is not the greatest one and we do not pursue it further.

This paper is organized as follows. In Section 2, we recall a relationship between quandle colorings and roots of Alexander polynomials. In Section 3, we give the coloring conditions, which are the core of this paper. Then, we prove Hoste's conjecture for pretzel links in Section 4 and show the existence of counterexamples in Section 5. Section 6 is devoted to find a lower bound for the real parts of roots.

Remark. Although Theorem 5.4 shows the existence of infinitely many counterexamples, it does not give any concrete one. In fact, the author proved this theorem and gave a talk on it at the workshop "Branched Coverings, Degenerations, and Related Topics 2018" at Hiroshima University in March, 2018, but he did not have an explicit counterexample at that time. However, soon after the workshop, following the strategy of the talk, Mikami Hirasawa and Masaaki Suzuki found counterexamples from numerical computations, and by using another family of candidate knots, they and the author finally gave concrete counterexamples with a rigorous proof; for this final result, see [5].

Acknowledgments

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2 Preliminaries

For $t \in \mathbb{C} \setminus \{0\}$, we define a binary operation $*$ on $\mathbb{C}$ as

$$x * y = tx + (1 - t)y \quad \text{for } x, y \in \mathbb{C}.$$

We find $(\mathbb{C}, *)$ to be a quandle (see e.g. [3] for a definition) and denote it by $\mathbb{C}_t$. A $\mathbb{C}_t$ -coloring on a diagram D of an oriented link L is a map $\mathcal{C} : \{\text{the over arcs of } D\} \rightarrow \mathbb{C}_t$ such that

$$\mathcal{C}(x) * \mathcal{C}(y) = \mathcal{C}(z) \quad \text{at every crossing point} \quad \begin{array}{c} y \\ \downarrow \\ x \text{ --- } z \end{array}.$$

In particular, a constant map is said to be trivial. It is known (e.g. [7]) that D admits a nontrivial $\mathbb{C}_t$ -coloring if and only if $\Delta_L(t) = 0$. Here Δ_L denotes the Alexander polynomial of L , though we do not recall its definition (see e.g. [11]).

In the same way, we define a binary operation $*$ on the Laurent polynomial ring $\mathbb{C}[T^{\pm 1}]$ (replace t with T) and we obtain a quandle $\mathbb{C}_T$. A $\mathbb{C}_T$ -coloring is also defined similarly.

3 Parametrization and a coloring condition

We recall that a *Montesinos link* is a link composed of rational tangles as in Figure 1. In this paper, we do not call the tangle $[\infty]$ a rational tangle. Given an oriented Montesinos link, we can deform it if necessary to assume that it is

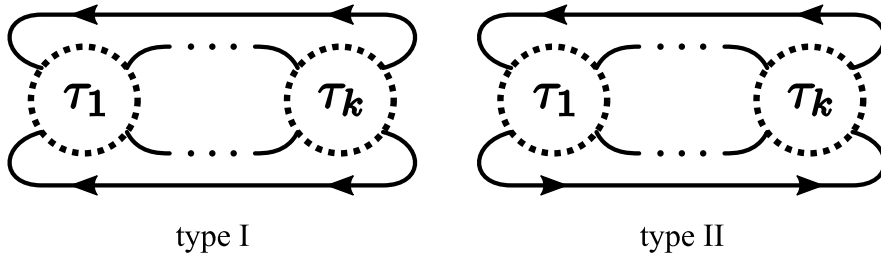


Figure 1: Montesinos links

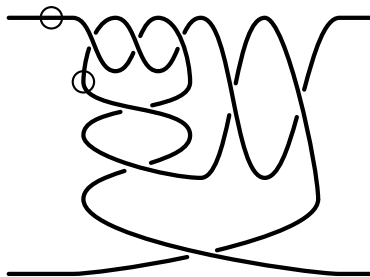


Figure 2: Two colors determine the whole coloring

oriented in one of the two ways described in Figure 1; we call it a Montesinos link of *type I/II* if it is oriented as in the left/right figure. Similarly, a rational tangle appearing in a Montesinos link of type I/II is said to be of type I/II: in the case of type I, the two left strings direct inward; in the case of type II, one of them goes inward and the other outward.

A remarkable property on quandle coloring of rational tangles is that a coloring is uniquely determined by the color of two arcs: For example, when we associate arbitrary elements of a quandle to the two marked arcs in Figure 2, they extend uniquely to be a whole coloring of the tangle. In the case of $\mathbb{C}_t$ -colorings ($t \in \mathbb{C} \setminus \{0, 1\}$), this means that there exists only one nontrivial coloring up to affine transformations. A similar fact holds also for $\mathbb{C}_T$, where we have to consider divisions by common factors. In the following, we regard such a division as an affine transformation.

This observation (see also Lemma 3.2 below) implies that, in the case of $\mathbb{C}_t$ -colorings, the colors of the right strings are often determined by those of the left. Then, the leftmost colors of a Montesinos link are transformed by the rational tangles to be rightmost colors, and if they accord, a whole coloring is completed. Thus, we obtain a condition for the existence of a nontrivial coloring by examining how the rational tangles change colors. This is an outline of Sections 3.1 and 3.2 below.

Before studying each type, we prepare lemmas, which hold for rational tangles of both types.

Lemma 3.1. *For any rational tangle ($\neq [\infty]$), there exists a $\mathbb{C}_T$ -coloring $\mathcal{C}$ such that $\mathcal{C}|_{T=t}$ is a nontrivial $\mathbb{C}_t$ -coloring for any $t \in \mathbb{C} \setminus \{0\}$.*

Proof. Let $\mathcal{C}_0$ be any nontrivial $\mathbb{C}_T$ -coloring. By an affine transformation, we may assume the color of some arc to be $0 \in \mathbb{C}_T$. Let $f \in \mathbb{C}[T^{\pm 1}]$ be the greatest common divisor of the image of $\mathcal{C}_0$ and put $\mathcal{C} = \mathcal{C}_0/f$. We easily find $\mathcal{C}$ to be a desired coloring. $\square$

In the following, we say a $\mathbb{C}_T$ -coloring of Lemma 3.1 to be *prime*.

Lemma 3.2. *Take a nontrivial $\mathbb{C}_T$ -coloring of a rational tangle as in Figure 3, where $x, y, x', y' \in \mathbb{C}_T = \mathbb{C}[T^{\pm 1}]$. Then, we have $x \neq y$ and $x' \neq y'$.*

Proof. By affine transformations, we may assume the coloring to be prime (Lemma 3.1). Then, by substituting -1 for T , we obtain a nontrivial $\mathbb{C}_{-1}$ -coloring. As we find from a definition of the fraction of a rational tangle (see e.g. [9]), $x(-1) - y(-1) \neq 0$, which implies that $x \neq y$. We find that $x' \neq y'$ similarly. $\square$

Lemma 3.3. *Let τ be a rational tangle. If there exists a nontrivial $\mathbb{C}_t$ -coloring ($t \in \mathbb{C} \setminus \{0, 1\}$) which assigns different colors to the left (resp. right) strings, then for any $x, y \in \mathbb{C}_t$ there exists a unique $\mathbb{C}_t$ -coloring of τ which colors the left (resp. right) strings with x and y .*

Proof. The set $\text{Col}_{\mathbb{C}_t}(\tau)$ of the $\mathbb{C}_t$ -colorings on τ is a $\mathbb{C}$ -vector space of dimension 2. The restriction to the left (resp. right) strings defines a homomorphism $r : \text{Col}_{\mathbb{C}_t}(\tau) \rightarrow \mathbb{C}^2$. Since the existences of the nontrivial coloring and a trivial one show that $\dim \text{Im } r \geq 2$, r is an isomorphism. $\square$

3.1 For Montesinos links of type I

Let τ be a rational tangle of type I. We take a nontrivial $\mathbb{C}_T$ -coloring of τ and let the four colors be x, y, x', y' as in Figure 3 ($x, y, x', y' \in \mathbb{C}[T^{\pm 1}]$). By Lemma 3.2, we have $x \neq y$ and then we define a rational function c_τ^I as

$$c_\tau^I = c_\tau^I(T) = \frac{x' - y'}{x - y} \in \mathbb{C}(T).$$

Lemma 3.2 also implies that $c_\tau^I \neq 0$. We can easily check that c_τ^I is invariant under an affine transformation, and hence c_τ^I is independent of the choice of the nontrivial coloring.

Remark 3.4. Suppose that $t \in \mathbb{C} \setminus \{0, 1\}$ is not a pole (resp. zero) of c_τ^I . In the definition above, we can take a prime $\mathbb{C}_T$ -coloring and then find that $x(t) \neq y(t)$ (resp. $x'(t) \neq y'(t)$). In this case, arbitrary left (resp. right) colors uniquely determine the right colors by Lemma 3.3.

Theorem 3.5. *Let L be a Montesinos link of type I composed of k rational tangles $\tau_1, \dots, \tau_k$. Let $\mathcal{P}$ (resp. $\mathcal{Z}$) be the set of the poles (resp. zeros) of $c_{\tau_1}^I, \dots, c_{\tau_k}^I$. Then, for $t \in \mathbb{C} \setminus \{0, \pm 1\}$, we have the following.*

- (i) *If $t \notin \mathcal{P} \cup \mathcal{Z}$, then there exists a nontrivial $\mathbb{C}_t$ -coloring on L if and only if*

$$c_{\tau_1}^I(t) \cdots c_{\tau_k}^I(t) = 1.$$

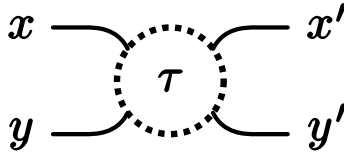


Figure 3: Coloring on a rational tangle

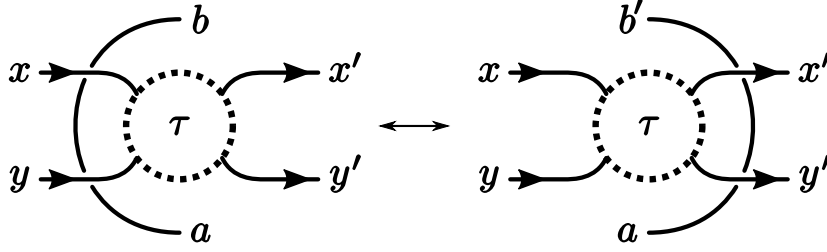


Figure 4: Isotopic tangles

- (ii) If $t \notin \mathcal{P}$ and $t \in \mathcal{Z}$, or if $t \in \mathcal{P}$ and $t \notin \mathcal{Z}$, then there does not exist a nontrivial $\mathbb{C}_t$ -coloring on L .
- (iii) If $t \in \mathcal{P} \cap \mathcal{Z}$, then there exists a nontrivial $\mathbb{C}_t$ -coloring on L .

In Theorem 3.5, let us express $c_{\tau_i}^I$ as the irreducible fraction f_i/g_i of Laurent polynomials. Then Theorem 3.5 is simply stated as follows: For $t \in \mathbb{C} \setminus \{0, \pm 1\}$, there exists a nontrivial $\mathbb{C}_t$ -coloring on L if and only if $f_1(t) \cdots f_k(t) = g_1(t) \cdots g_k(t)$.

Lemma 3.6. *Let τ be a (not necessarily rational) tangle oriented and $\mathbb{C}_t$ -colored ($t \in \mathbb{C} \setminus \{0, 1\}$) as in Figure 3. Then we have $x + ty = x' + ty'$.*

Proof. We consider the oriented $\mathbb{C}_t$ -colored tangles of Figure 4, where $a, b, b' \in \mathbb{C}_t$. An isotopy takes one of them to the other and then we have $b = b'$. Since $b = (a * y) * x$ and $b' = (a * y') * x'$, we have

$$\begin{aligned} t(ta + (1-t)y) + (1-t)x &= t(ta + (1-t)y') + (1-t)x', \\ (1-t)(x + ty) &= (1-t)(x' + ty'). \end{aligned}$$

Since $t \neq 1$, we have $x + ty = x' + ty'$, as required. We remark that this proof essentially used an idea of shadow coloring (see e.g. [3]). $\square$

Proof of Theorem 3.5. Let τ be the tangle composed of $\tau_1, \dots, \tau_k$ and consider a $\mathbb{C}_t$ -coloring as in Figure 5. By the definition of L , it defines a $\mathbb{C}_t$ -coloring of L if and only if $x_1 = x_{k+1}$ and $y_1 = y_{k+1}$. Since $t \neq -1$, an affine transformation allows us to assume that $x_i + ty_i = 0$ for some i (and hence all by Lemma 3.6).

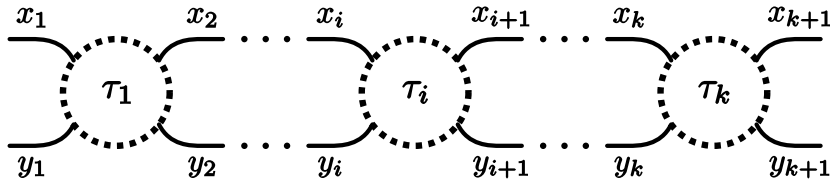


Figure 5: Coloring on a Montesinos link

(i) By Lemma 3.3 (also see Remark 3.4), the top-left color x_1 (remark that $y_1 = -t^{-1}x_1$ by the assumption) determines the whole coloring of τ . Suppose that $x_1 \neq 0$. For $i = 1, \dots, k$, assuming that $x_i \neq 0$, we have

$$x_{i+1} = \frac{(1+t^{-1})x_{i+1}}{(1+t^{-1})x_i}x_i = \frac{x_{i+1}-y_{i+1}}{x_i-y_i}x_i = c_{\tau_i}^I(t) \cdot x_i,$$

where the second equality follows from the assumption and the third from the definition of $c_{\tau_i}^I$. Since $t \notin \mathcal{Z}$, x_{i+1} is then not equal to 0. Thus, we have

$$x_{k+1} = c_{\tau_1}^I(t) \cdots c_{\tau_k}^I(t) \cdot x_1$$

and find that $x_1 = x_{k+1}$ (and hence $y_1 = y_{k+1}$) if and only if $c_{\tau_1}^I(t) \cdots c_{\tau_k}^I(t) = 1$.

(ii) Suppose that $t \notin \mathcal{P}$ and $t \in \mathcal{Z}$. By cyclic permutations of the tangles, we may assume that $c_{\tau_1}^I(t) = 0$. Then, as in the proof of (i), x_1 determines the whole coloring uniquely and in particular $x_2 = c_{\tau_1}^I(t) \cdot x_1 = 0$; by the assumption, y_2 is also 0. For any $x_1 \neq 0$, we now have the coloring which colors τ_1 in this nontrivial way and which is trivial on the other tangles, and this is the uniquely determined coloring. Thus, we have $x_{k+1} = 0$ and hence this coloring does not extend to a coloring of L . The other case, where $t \in \mathcal{P}$ and $t \notin \mathcal{Z}$, is shown in the same way.

(iii) Using cyclic permutations, we assume that for some i and j ($i < j$), t is a pole of $c_{\tau_i}^I$, is a zero of $c_{\tau_j}^I$, and is not a pole or a zero of $c_{\tau_{i+1}}^I, \dots, c_{\tau_{j-1}}^I$. We take a nontrivial x_{i+1} and then we have a unique nontrivial coloring on $\tau_i, \dots, \tau_j$, where $x_i = x_{j+1} = 0$. By giving the trivial coloring “0” to the other tangles, we obtain a nontrivial coloring of τ such that $x_1 = x_{k+1} = 0$, as required. $\square$

Remark 3.7. In Theorem 3.5, we assume t not to be -1 . In the case where $t = -1$, we can examine the colorability as in the case of type II. Since $c_{\tau}^{\text{II}}(-1)$ (c_{τ}^{II} is defined in the next section) is equal to the fraction of τ , the result is only a classical one: there exists a nontrivial coloring, i.e. $\Delta_L(-1) = 0$, if and only if the sum of the fractions of the rational tangles is equal to 0.

3.2 For Montesinos links of type II

Let τ be a rational tangle of type II. We take a nontrivial $\mathbb{C}_T$ -coloring of τ and let the four colors be x, y, x', y' as in Figure 3. Then we define a rational function c_{τ}^{II} as

$$c_{\tau}^{\text{II}} = c_{\tau}^{\text{II}}(T) = (-t)^{\epsilon_{\tau}} \cdot \frac{(x' + y')/2 - (x + y)/2}{x - y} \in \mathbb{C}(T),$$

where ϵ_{τ} is defined to be 0 (resp. 1) if the upper-left string directs inward (resp. outward). Since c_{τ}^{II} is invariant under an affine transformation, this is independent of the choice of the nontrivial coloring. We remark that $c_{\tau}^{\text{II}}(-1)$ is the fraction of the rational tangle τ ; in this sense, c_{τ}^{II} is a generalization of the fraction.

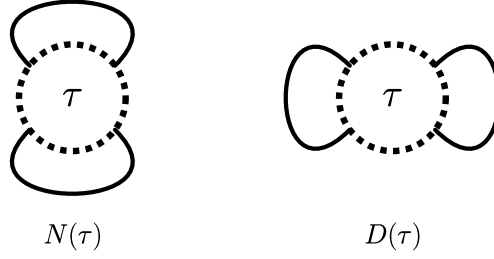


Figure 6: The numerator $N(\tau)$ and the denominator $D(\tau)$

Remark 3.8. As in Remark 3.4, if t is not a pole of c_τ^Π , then the right colors are uniquely determined by any left colors. Furthermore, Lemma 3.11 below implies the other direction: the left colors are uniquely determined by any right colors.

Remark 3.9. For a rational tangle τ of type II, we have two ways of closing it to obtain links: the numerator $N(\tau)$ and the denominator $D(\tau)$, as illustrated in Figure 6. Taking a prime $\mathbb{C}_T$ -coloring in the definition above, we find that t is a zero (resp. pole) of c_τ^Π if and only if $\Delta_{N(\tau)}(t) = 0$ (resp. $\Delta_{D(\tau)}(t) = 0$).

Theorem 3.10. *Let L be a Montesinos link of type II composed of k rational tangles $\tau_1, \dots, \tau_k$. Let $\mathcal{P}_i$ denote the set of the poles of $c_{\tau_i}^\Pi$, and let $\mathcal{P}$ be the union $\bigcup_{i=1}^k \mathcal{P}_i$. Then, for $t \in \mathbb{C} \setminus \{0, 1\}$, we have the following.*

- (i) *If $t \notin \mathcal{P}$, then there exists a nontrivial $\mathbb{C}_t$ -coloring on L if and only if*

$$c_{\tau_1}^\Pi(t) + \dots + c_{\tau_k}^\Pi(t) = 0.$$

- (ii) *If $t \in \mathcal{P}_i$ for exactly one i , then there does not exist a nontrivial $\mathbb{C}_t$ -coloring on L .*
- (iii) *If $t \in \mathcal{P}_i$ for more than one i , then there exists a nontrivial $\mathbb{C}_t$ -coloring on L .*

Lemma 3.11. *Let τ be an oriented (not necessarily rational) tangle with a $\mathbb{C}_t$ -coloring ($t \in \mathbb{C} \setminus \{0, 1\}$). We assume that one of the left strings has the inward orientation and the other has the outward orientation. If the upper (resp. under) left goes inward, we put $\epsilon_\tau = 0$ (resp. 1). Similarly, we define ϵ'_τ according to the orientations of the right strings. Let x, y, x', y' be the four colors as in Figure 3. Then we have $(-t)^{-\epsilon_\tau}(x - y) = (-t)^{-\epsilon'_\tau}(x' - y')$.*

Proof. In the same way as Lemma 3.6. □

Proof of Theorem 3.10. Let τ be the tangle composed of $\tau_1, \dots, \tau_k$ and consider a $\mathbb{C}_t$ -coloring as in Figure 5. We define ϵ_{τ_i} as in the definition of $c_{\tau_i}^\Pi$, and denote it by ϵ_i for simplicity. Then we find from Lemma 3.11 that $(-t)^{-\epsilon_i}(x_i - y_i) =$

$(-t)^{-\epsilon_1}(x_1 - y_1)$ for $i = 1, \dots, k$, and that $x_{k+1} - y_{k+1} = x_1 - y_1$.

(i) We put $z_i = (x_i + y_i)/2$ for $i = 1, \dots, k$. As explained in Remark 3.8, z_1 and $x_1 - y_1$ determine the whole coloring of τ . Since putting $x_1 - y_1 = 0$ gives a trivial coloring, we assume that $x_1 - y_1 = (-t)^{-\epsilon_1}$ by using an affine transformation. Then we have $x_i - y_i = (-t)^{\epsilon_i}$ and hence

$$z_{i+1} = (-t)^{\epsilon_i} \cdot \frac{z_{i+1} - z_i}{x_i - y_i} + z_i = c_{\tau_i}^{\text{II}}(t) + z_i.$$

Then we have

$$z_{k+1} = c_{\tau_1}^{\text{II}}(t) + \dots + c_{\tau_k}^{\text{II}}(t) + z_1.$$

Therefore this coloring extends to a coloring on L if and only if $c_{\tau_1}^{\text{II}}(t) + \dots + c_{\tau_k}^{\text{II}}(t) = 0$.

(ii) Taking a prime $\mathbb{C}_T$ -coloring and substituting t for T , we have a nontrivial $\mathbb{C}_t$ -coloring on τ_i , where $x_i = y_i$ and $x_{i+1} = y_{i+1}$, and any $\mathbb{C}_t$ -coloring on τ_i is obtained from it by an affine transformation. Then, by applying Lemma 3.3 to τ_i rotated by $\pi/4$ and to the other $k - 1$ tangles, we find that any x_i and x_{i+1} determine the whole coloring of τ uniquely, where the tangles $\tau_1, \dots, \tau_{i-1}$ (resp. $\tau_{i+1}, \dots, \tau_k$) are trivially colored with x_i (resp. x_{i+1}). The coloring extends to a coloring on L if and only if $x_i = x_1 = x_k = x_{i+1}$, but then it is a trivial coloring.

(iii) Suppose that $t \in \mathcal{P}_i \cap \mathcal{P}_j$ ($i < j$). As with (ii), there exist nontrivial $\mathbb{C}_t$ -colorings on τ_i and τ_j such that $x_i = y_i = x_{j+1} = y_{j+1}$ and $x_{i+1} = y_{i+1} = x_j = y_j$. We trivially color $\tau_1, \dots, \tau_{i-1}, \tau_{j+1}, \dots, \tau_k$ with x_i and $\tau_{i+1}, \dots, \tau_{j-1}$ with x_{i+1} to obtain a nontrivial coloring on τ , which extends to be a nontrivial coloring on L . $\square$

Before proceeding to the following sections, we show a useful lemma on c_{τ}^{II} :

Lemma 3.12. *Let τ be a rational tangle of type II. Then we have $c_{\tau}^{\text{II}} = c_{-\tau}^{\text{II}}$, where $-\tau$ denotes the rational tangle τ with reversed orientation.*

Proof. Let ϵ_{τ} and ϵ'_{τ} be as in Lemma 3.11. It is sufficient to show the lemma when $\epsilon_{\tau} = 1$. We recall the fact (see e.g. [9]) that a flip (π -rotation) of an unoriented rational tangle is isotopic to the original one. Then the horizontal flip of τ is isotopic to $-\tau$ and we have a nontrivial $\mathbb{C}_T$ -coloring of $-\tau$ as shown in Figure 7, where by definition

$$c_{\tau}^{\text{II}} = \frac{(x' + y')/2 - (x + y)/2}{x - y}.$$

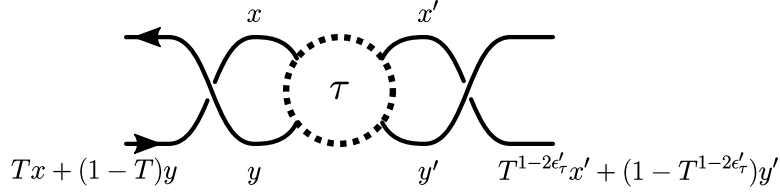


Figure 7: The horizontal flip of a tangle

Then we have

$$\begin{aligned}
c_{-\tau}^{\text{II}} &= -T \cdot \frac{\frac{y' + T^{1-2\epsilon'_\tau}x' + (1-T^{1-2\epsilon'_\tau})y'}{2} - \frac{y + Tx + (1-T)y}{2}}{y - (Tx + (1-T)y)} \\
&= \frac{\frac{x' + y'}{2} - \frac{x + y}{2} - (1-T^{1-2\epsilon'_\tau})\frac{x' - y'}{2} + (1-T)\frac{x - y}{2}}{x - y} \\
&= c_\tau^{\text{II}} + (x - y)^{-1} \left((1-T)\frac{x - y}{2} - (1-T^{1-2\epsilon'_\tau})(-T)^{\epsilon'_\tau} \frac{x - y}{2} \right) \\
&= c_\tau^{\text{II}},
\end{aligned}$$

where the third equality follows from the definition of Lemma 3.11 and c_τ^{II} recalled above, and the last is obtained from an elementary calculation; recall that ϵ'_τ is either 0 or 1. $\square$

4 Hoste's conjecture for pretzel links

In this section, we show that Hoste's conjecture holds for the links which admit alternating pretzel diagram. Indeed, it also holds for the pretzel links of type I which are alternating, where we do not assume diagrams of them to be alternating and pretzel simultaneously.

4.1 For pretzel links of type I

Here we show Hoste's conjecture for pretzel links of type I:

Theorem 4.1. *Let L be a pretzel link of type I and suppose that L admits an alternating diagram. Then, the real parts of the roots of the Alexander polynomial of L is greater than -1 .*

To apply Theorem 3.5 to our problem, we prepare a lemma:

Lemma 4.2. *Let $\sigma_1(m)$ and $\sigma_2(m)$ be the rational tangles of type I shown in Figure 8. Then we have*

$$c_{\sigma_1(m)}^{\text{I}} = \frac{m(1-T) - T}{m(1-T) + 1}, \quad c_{\sigma_2(m)}^{\text{I}} = \frac{m(1-T) + 1}{m(1-T^{-1}) + 1}.$$

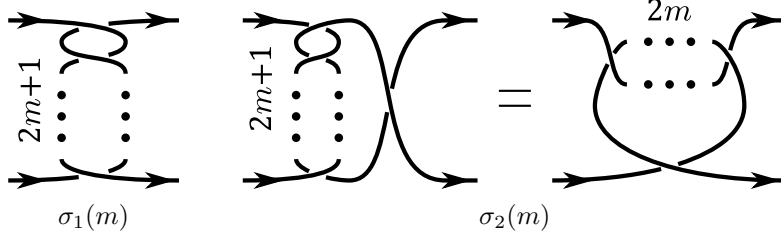


Figure 8: Rational tangles of type I

Proof. We first consider $\sigma_1(m)$. We put colors $x, x' \in \mathbb{C}_T$ ($x \neq x'$) to the upper strings, and then the colors are determined in order. After two half-twists, the colors are changed to be $x + (1 - T^{-1})(x - x')$ and $x' + (1 - T^{-1})(x - x')$ as shown in Figure 9 (left). Repeating m times, we obtain the colors $x + m(1 - T^{-1})(x - x')$ and $x' + m(1 - T^{-1})(x - x')$ and finally we find the bottom colors to be $T^{-1}x' + (1 - T^{-1})x + m(1 - T^{-1})(x - x')$ and $x + m(1 - T^{-1})(x - x')$; see Figure 9 (right). Thus,

$$\begin{aligned} c_{\sigma_1(m)}^I &= \frac{x' - x - m(1 - T^{-1})(x - x')}{x - T^{-1}x' - (1 - T^{-1})x - m(1 - T^{-1})(x - x')} \\ &= \frac{(x - x')(-1 - m(1 - T^{-1}))}{(x - x')(T^{-1} - m(1 - T^{-1}))} \\ &= \frac{m(1 - T) - T}{m(1 - T) + 1}, \end{aligned}$$

as required. We can compute $c_{\sigma_2(m)}$ similarly; otherwise,

$$\begin{aligned} c_{\sigma_2(m)}^I &= c_{\sigma_1(-m-1)}^I \cdot c_{\sigma_1(0)}^I \\ &= \frac{-(m+1)(1-T)-T}{-(m+1)(1-T)+1} \cdot (-T) \\ &= \frac{m(1-T)+1}{m(1-T^{-1})+1} \end{aligned}$$

since the formula shown above also holds for negative m . $\square$

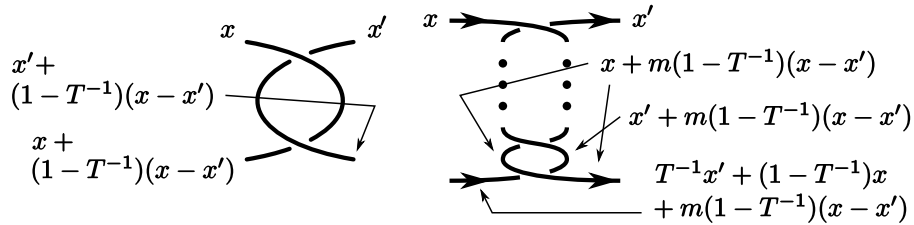


Figure 9: A coloring on $\sigma_1(m)$

Proof of Theorem 4.1. According to the classification of Montesinos links (e.g. [1]) and minimal diagrams of them ([12]), we may assume L to be composed of rational tangles $\sigma_1(m)$ and $\sigma_2(m)$, by taking the mirror image if necessary. Let t be a complex number with real part ≤ -1 , and we show that $\Delta_L(t) \neq 0$. By the alternating property ([2],[14]), we may assume t not to be real and then suppose that $\text{Im } t < 0$. For convenience, we put $s = -t$; then $\text{Im } s > 0$. Using Lemma 4.2, we have

$$\begin{aligned} |c_{\sigma_1(m)}^{\text{I}}(t)| &= \left| \frac{m(1+s) + s}{m(1+s) + 1} \right| > \left| \frac{m(1+s) + \text{Re } s}{m(1+s) + 1} \right| \\ &\geq \left| \frac{m(1+s) + 1}{m(1+s) + 1} \right| = 1, \end{aligned}$$

where the inequalities follows since $\text{Re } m(1+s) > 0$ and $\text{Im } m(1+s) > 0$. Similarly,

$$|c_{\sigma_2(m)}^{\text{I}}(t)| = \left| \frac{m+1+ms}{m+1+ms^{-1}} \right| > \left| \frac{m+1+m\bar{s}^{-1}}{m+1+ms^{-1}} \right| = 1.$$

In the notation of Theorem 3.5, we now have $|c_{\tau_i}^{\text{I}}(t)| > 1$ for every i ; in particular, $c_{\tau_1}^{\text{I}}(t) \cdots c_{\tau_k}^{\text{I}}(t) \neq 1$. Then Theorem 3.5 shows the nonexistence of $\mathbb{C}_t$ -coloring on L , which means that $\Delta_L(t) \neq 0$, as required. $\square$

4.2 For pretzel links of type II

The aim of this section is to prove Hoste's conjecture for pretzel links of type II:

Theorem 4.3. *Let L be a Montesinos link of type II and suppose that L admits an alternating pretzel diagram. Then, the real parts of the roots of the Alexander polynomial of L is greater than -1 .*

To show this theorem, let us calculate c_{τ}^{II} for some rational tangles:

Lemma 4.4. *Let $\sigma_3(m)$ and $\sigma_4(m)$ be the rational tangles of type II shown in Figure 10. Then we have*

$$c_{\sigma_3(m)}^{\text{II}} = \frac{1}{m(1-T^{-1})}, \quad c_{\sigma_4(m)}^{\text{II}} = \frac{1+T}{2} \cdot \frac{1+(-T)^m}{1-(-T)^m}.$$

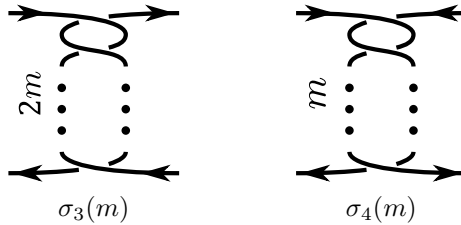


Figure 10: Rational tangles of type II

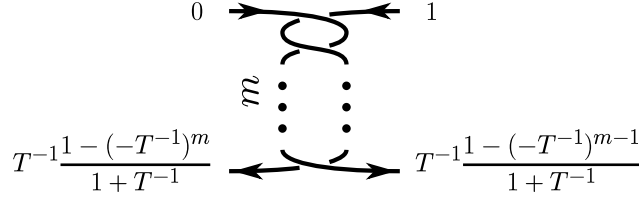


Figure 11: A coloring on $\sigma_4(m)$

Proof. We have already computed a $\mathbb{C}_T$ -coloring on $\sigma_3(m)$ in the proof of Lemma 4.2; remark that $\sigma_3(m)$ is a part of $\sigma_1(m)$. Then we have

$$\begin{aligned} c_{\sigma_3(m)}^{\text{II}} &= \frac{\frac{x' + x' + m(1 - T^{-1})(x - x')}{2} - \frac{x + x + m(1 - T^{-1})(x - x')}{2}}{x - x - m(1 - T^{-1})(x - x')} \\ &= \frac{x' - x}{m(1 - T^{-1})(x' - x)} \\ &= \frac{1}{m(1 - T^{-1})}, \end{aligned}$$

where we assume $x \neq x'$. A calculation on $\sigma_4(m)$ is similar; we have a coloring on $\sigma_4(m)$ as shown in Figure 11 (by an induction) and then

$$\begin{aligned} c_{\sigma_4(m)}^{\text{II}} &= \frac{\frac{1}{2} \left(1 + T^{-1} \frac{1 - (-T^{-1})^{m-1}}{1 + T^{-1}} \right) - \frac{1}{2} \cdot T^{-1} \frac{1 - (-T^{-1})^m}{1 + T^{-1}}}{-T^{-1} \frac{1 - (-T^{-1})^m}{1 + T^{-1}}} \\ &= \frac{1 + (-T^{-1})^m}{-2 \cdot \frac{T^{-1} + (-T^{-1})^{m+1}}{1 + T^{-1}}} \\ &= \frac{1 + T}{2} \cdot \frac{1 + (-T)^m}{1 - (-T)^m}, \end{aligned}$$

as required. $\square$

Proof of Theorem 4.3. By the assumption, we may assume L to be composed of $\sigma_3(m)$, $-\sigma_3(m)$, $\sigma_4(m)$, and $-\sigma_4(m)$; we take the mirror image, if necessary. As in the proof of Theorem 4.1, we put $s = -t$ for fixed $t \in \mathbb{C}$ and suppose that $\text{Re } s \geq 1$ and $\text{Im } s > 0$. By Lemma 4.4,

$$c_{\sigma_3(m)}^{\text{II}}(-s) = \frac{1}{m(1 + s^{-1})}, \quad c_{\sigma_4(m)}^{\text{II}}(-s) = \frac{s - 1}{2} \cdot \frac{1 + s^{-m}}{1 - s^{-m}}.$$

We define a half plane P as

$$P = \{x \in \mathbb{C} \mid \text{Re } x/(s - 1) > 0\}.$$

We easily find that P includes the first quadrant and that $c_{\sigma_3}(m)(-s)$ lies in the first quadrant; hence $c_{\sigma_3(m)}^{\text{II}}(-s) \in P$. Furthermore, we find

$$\arg \frac{1+s^{-m}}{1-s^{-m}} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

for any $m \in \mathbb{Z}_{>0}$ since $|s^{-1}| < 1$, and then $c_{\sigma_4(m)}^{\text{II}}(-s) \in P$. Lemma 3.12 shows that $c_{-\sigma_3(m)}^{\text{II}}(t)$ and $c_{-\sigma_4(m)}^{\text{II}}(t)$ are also contained in P . Thus, in the notation of Theorem 3.10, $c_{\tau_i}^{\text{II}}(t) \in P$ for every i and therefore $c_{\tau_1}^{\text{II}}(t) + \cdots + c_{\tau_k}^{\text{II}}(t) \in P$. In particular, the sum is not equal to 0. Now Theorem 3.10 prove that there does not exist a nontrivial $\mathbb{C}_t$ -coloring on L and hence $\Delta_L(t) \neq 0$, as required. $\square$

5 Counterexamples for Hoste's conjecture

In this section, we show the existence of infinitely many counterexamples for Hoste's conjecture.

5.1 Counterexamples of type I

Let $\tau(n)$ be the rational tangle of type I of Figure 12 and $L(n, a, b)$ the Montesinos link composed of a copies of $\tau(n)$ and b copies of the half twist $\sigma_1(1)$; see Figure 13. We remark that the crossing number of $L(n, a, b)$ is $(n+2)a + b$ and that the number of the components of $L(m, a, b)$ is

$$\begin{cases} 1 & \text{if } (n-1)a + b \text{ is odd,} \\ 2 & \text{if } (n-1)a + b \text{ is even.} \end{cases}$$

As is shown below, some of the links $L(n, a, b)$ are counterexamples.

Lemma 5.1. *We have $c_{\tau(n)}^{\text{I}} = \frac{(-T)^{n+1} + (-T)^n - 2}{2(-T)^n - 1 + T^{-1}}$.*

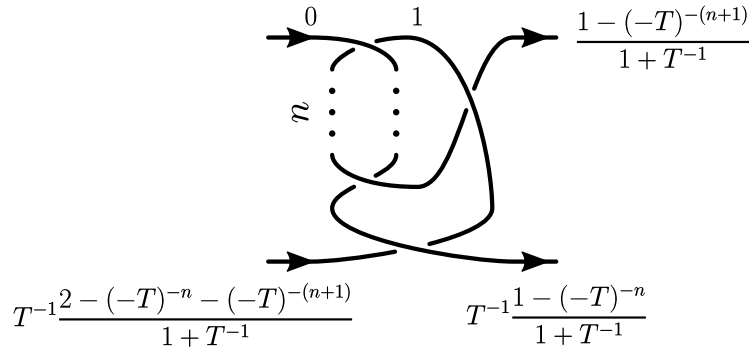


Figure 12: A rational tangle $\tau(n)$ and a coloring

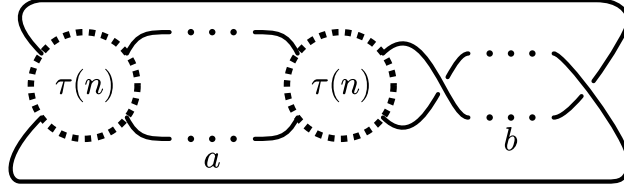


Figure 13: $L(n, a, b)$

Proof. We put colors 0 and 1 to two of the arcs and find this to be uniquely extended to a $\mathbb{C}_T$ -coloring on $\tau(n)$ as shown in Figure 12; remark that $\sigma_4(n)$ is a part of $\tau(n)$ and we can use the calculation in the proof of Lemma 4.4. By definition, we have

$$c_{\tau(n)}^I = \frac{\frac{1 - (-T)^{-(n+1)}}{1 + T^{-1}} - T^{-1} \frac{1 - (-T)^{-n}}{1 + T^{-1}}}{-T^{-1} \frac{2 - (-T)^{-n} - (-T)^{-(n+1)}}{1 + T^{-1}}} = \frac{(-T)^{n+1} + (-T)^n - 2}{2(-T)^n - 1 + T^{-1}},$$

as required. $\square$

We should remark that the real part of any zero or pole of $c_{\tau(n)}^I$ is greater than -1 ; if $t \in \mathbb{C} \setminus \{-1\}$ has the real part ≤ -1 , we have $|1 - t| > 2$ and then

$$|(-t)^{n+1} + (-t)^n| = |t|^n \cdot |1 - t| > 2, \quad |2(-t)^n| > |1 - t| \cdot |t|^{-1} = |1 - t^{-1}|.$$

For convenience, we put $c_n(s) = c_{\tau(n)}^I(-s) = \frac{s^{n+1} + s^n - 2}{2s^n - 1 - s^{-1}}$. If, as in the proof of Theorem 4.1, $|c_n(s)| > 1$ held for s with real part ≥ 1 , the same proof would proceed. However,

Lemma 5.2. *There exist $s \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$ such that $\operatorname{Re} s > 1$ and $|c_n(s)| < 1$.*

Proof. Since the condition $|c_n(s)| < 1$ is open, it is sufficient to find such s with real part equal to 1. Here, we have

$$c_n(s) = 1 + \frac{(s-1)(s^n - s^{-1})}{2(s^n - s^{-1}) - (1 - s^{-1})} = 1 + \frac{1}{\frac{2}{s-1} - \frac{1}{s^{n+1}-1}}$$

and then, assuming $\operatorname{Re} s = 1$, we find that $|c_n(s)| < 1$ if and only if $|s^{n+1} - 2| < 1$; transform the (open) unit disc by $(-\bullet - 1)^{-1} + (\text{an imaginary number})^{-1} + 1$. By taking sufficiently large n and putting, say, $s = 1 + i \tan 2\pi/(n+1)$, we have $|s^{n+1} - 2| = (\cos 2\pi/(n+1))^{-(n+1)} - 2 < 1$ and then $|c_n(s)| < 1$. $\square$

Remark 5.3. In this proof, $|c_n(1 + i \tan 2\pi/(n+1))| < 1$ holds for $n \geq 18$. In fact, $(n, s) = (17, 1 + 0.35i)$ satisfies the condition, according to Masaaki Suzuki. A numerical calculation also shows that there does not exist such s for $n \leq 16$ (a hard calculation could show this, but here we do not pursue it).

Theorem 5.4. *The closure of the set*

$$\left\{ t \in \mathbb{C} \left| \begin{array}{l} \operatorname{Re} t < -1, \\ \text{there exist } n, a \in \mathbb{Z}_{>0} \text{ and } b \in \mathbb{Z}_{\geq 0} \text{ such that} \\ (n-1)a + b \notin 2\mathbb{Z} \text{ and } \Delta_{L(n,a,b)}(t) = 0 \end{array} \right. \right\}$$

includes the nonempty open set

$$\bigcup_{n>0} \{t \in \mathbb{C} \mid \operatorname{Re} t < -1, |c_n(-t)| < 1\}.$$

The condition $(n-1)a + b \notin 2\mathbb{Z}$ makes $L(n, a, b)$ a (one-component) knot. Then,

Corollary 5.5. *There exist infinitely many alternating Montesinos knots K such that the real part of a zero of Δ_K is less than -1 .*

Proof of Theorem 5.4. We take any $t_0 \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$ such that $\operatorname{Re} t_0 < -1$ and $|c_n(-t_0)| < 1$. We put $s_0 = -t_0$ for convenience. By Theorem 3.5, $t \in \mathbb{C}$ with $\operatorname{Re} t < -1$ is a zero of $\Delta_{L(n,a,b)}$ if and only if $(c_n(s))^a s^b = 1$, where $s = -t$. Thus, it is sufficient to find such s in any neighborhood of s_0 .

We take a logarithm of $c_n(s_0)$ and decompose it as

$$\log c_n(s_0) = \alpha \cdot \log s_0 + \beta \cdot 2\pi i \quad (\alpha, \beta \in \mathbb{R});$$

remark that $\mathbb{C} = \log s_0 \mathbb{R} \oplus 2\pi i \mathbb{R}$ since $|s_0| > 1$. By Lemma 5.7 below, we can perturb s_0 to s in any neighborhood of s_0 so that $\log c_n(s) \in \log s \mathbb{Q} \oplus 2\pi i \mathbb{Q}$. We express the decomposition as

$$\log c_n(s) = -p_1/q_1 \cdot \log s + p_2/q_2 \cdot 2\pi i \quad (p_1, p_2, q_1, q_2 \in \mathbb{Z}, q_1, q_2 > 0),$$

where $(p_1, q_1) = (p_2, q_2) = 1$. By comparing the real parts of both sides, we find that p_1/q_1 and hence p_1 are positive. Furthermore, an appropriate choice of s makes p_1 and q_2 odd, and q_1 even. We put $a = q_1 q_2$ and $b = p_1 q_2$. Then we have $a \cdot \log c_n(s) + b \cdot \log s \in 2\pi i \mathbb{Z}$, i.e. $(c_n(s))^a s^b = 1$, and $(n-1)a + b$ is odd, as required. $\square$

Remark 5.6. This proof does not give an explicit counterexample. However, According to Mikami Hirasawa and Masaaki Suzuki, numerical computations find many counterexamples in a form of $L(n, a, b)$; $L(17, 36, 0)$ is the one with the minimum crossing number 684 among them, and if we restrict the candidate to the knots, $L(18, 37, 0)$ is the simplest one, which has the crossing number 740.

Lemma 5.7. *Let f and g be holomorphic function defined on a connected open subset $U \subset \mathbb{C}$, and suppose that $\operatorname{Re} g(z) \neq 0$ at every $z \in U$. We define $\mathbb{R}$ -valued functions a and b so that $f(z) = a(z)g(z) + b(z)i$. If not both a and b are constant, the map $\varphi := (a, b) : U \rightarrow \mathbb{R}^2$ is open.*

Proof. We put $f = f_1 + f_2 i$ and $g = g_1 + g_2 i$; f_1, f_2, g_1, g_2 are $\mathbb{R}$ -valued. By definition $f_1 = a g_1$ and $f_2 = a g_2 + b$, and then we have

$$\begin{aligned} \partial_x f_1 &= g_1 \partial_x a + a \partial_x g_1, & \partial_x f_2 &= g_2 \partial_x a + a \partial_x g_2 + \partial_x b, \\ \partial_y f_1 &= g_1 \partial_y a + a \partial_y g_1, & \partial_y f_2 &= g_2 \partial_y a + a \partial_y g_2 + \partial_y b, \end{aligned}$$

where $z = x + iy$, $\partial_x = \partial/\partial x$, and $\partial_y = \partial/\partial y$. Then the Jacobian of φ is calculated as follows:

$$\begin{aligned}
J(\varphi) &= \begin{vmatrix} \partial_x a & \partial_y a \\ \partial_x b & \partial_y b \end{vmatrix} \\
&= \begin{vmatrix} \partial_x a & \partial_y a \\ \partial_x f_2 - a\partial_x g_2 - g_2\partial_x a & \partial_y f_2 - a\partial_y g_2 - g_2\partial_y a \end{vmatrix} \\
&= \begin{vmatrix} (\partial_x f_1 - a\partial_x g_1)/g_1 & (\partial_y f_1 - a\partial_y g_1)/g_1 \\ \partial_x f_2 - a\partial_x g_2 & \partial_y f_2 - a\partial_y g_2 \end{vmatrix} \\
&= \frac{1}{g_1} \begin{vmatrix} \partial_x f_1 - a\partial_x g_1 & \partial_y f_1 - a\partial_y g_1 \\ -\partial_y f_1 + a\partial_y g_1 & \partial_x f_1 - a\partial_x g_1 \end{vmatrix} \\
&= \frac{1}{g_1} ((\partial_x f_1 - a\partial_x g_1)^2 + (\partial_y f_1 - a\partial_y g_1)^2) \\
&= \frac{1}{g_1^3} \left((g_1 \partial_x f_1 - f_1 \partial_x g_1)^2 + (g_1 \partial_y f_1 - f_1 \partial_y g_1)^2 \right),
\end{aligned}$$

where the fourth equality follows from Cauchy-Riemann equations. Thus, the map φ is not locally diffeomorphic at $z \in U$ if and only if

$$(\operatorname{Re} f(z))g'(z) = f'(z)(\operatorname{Re} g(z)). \quad (1)$$

We here assert that the set

$$S := \{z \in U \mid \text{the condition (1) holds}\}$$

does not have an accumulation point in itself (or equivalently in U) if not both a and b are constant. To show this, we fix a point $z_0 \in S$. By replacing f with $f - a(z_0)g - b(z_0)i$, we may assume $f(z_0) = 0$. Then, it is sufficient to show that z_0 is not an accumulation point of S if f is not constantly 0. Let the Taylor series of f at $z = z_0$ be $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$. Then the condition (1) is rewritten as

$$\left(\operatorname{Re} \sum_{k=0}^{\infty} a_k(z - z_0)^k \right) g'(z) = \left(\sum_{k=1}^{\infty} k a_k(z - z_0)^{k-1} \right) (\operatorname{Re} g(z)) \quad (2)$$

around $z = z_0$. Suppose that $a_0 = \dots = a_{m-1} = 0$ and $a_m \neq 0$; here $m \neq 0$ since $a_0 = a(z_0) = 0$. Then, in a neighborhood of z_0 , we have

$$\begin{aligned}
\left| \left(\operatorname{Re} \sum_{k=0}^{\infty} a_k(z - z_0)^k \right) g'(z) \right| &\leq |z - z_0|^m \left| \sum_{k=m}^{\infty} a_k(z - z_0)^{k-m} \right| |g'(z)| \\
&\leq C |z - z_0|^m
\end{aligned} \quad (3)$$

for some constant C . Furthermore, since $a_m \neq 0$ and $\operatorname{Re} g(z_0) \neq 0$, we have

$$\begin{aligned} & \left| \left(\sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \right) (\operatorname{Re} g(z)) \right| \\ &= |z - z_0|^{m-1} \left| m a_m + \sum_{k=m+1}^{\infty} k a_k (z - z_0)^{k-m} \right| |\operatorname{Re} g(z)| \\ &\geq c |z - z_0|^{m-1} \end{aligned} \tag{4}$$

for some constant c . By (3) and (4), the condition (2) does not hold for $z \neq z_0$ if z is sufficiently close to z_0 . This means that there exists a neighborhood $V \subset U$ of z_0 such that $V \cap S = \{z_0\}$, which concludes the assumption.

We assume a or b is not constant. Since φ is locally diffeomorphic on $U \setminus S$, it is sufficient to show φ to be open at any $z_0 \in S$. Let $V \subset U$ be any neighborhood of z_0 . We take a closed disc D in $\mathbb{R}^2$ such that $\varphi(z_0) \in \operatorname{int} D$ and the connected component $\tilde{D}$ of $\varphi^{-1}(D)$ having z_0 as a member is compact and included in V . By shrinking D if necessary, we may assume that $\tilde{D} \cap S = \{z_0\}$. Since φ is continuous, we find that $\varphi(\partial \tilde{D}) \subset \partial D$. Now we can consider the mapping degree of $\varphi|_{\tilde{D}} : (\tilde{D}, \partial \tilde{D}) \rightarrow (D, \partial D)$, and it is not zero since the sign of $J(\varphi)$ is constant on $\tilde{D} \setminus \{z_0\}$. In particular, $\varphi(\tilde{D}) = D$ and hence $\operatorname{int} D \subset \varphi(V)$. $\square$

5.2 Counterexamples of type II

Here, we see that infinitely many counterexamples for Hoste's conjecture are also found among the Montesinos knots of type II. However, since we have achieved the goal of this section, we do not give a detailed proof of results.

Let us consider Montesinos links composed of the rational tangles shown in Figure 14, in addition to $\sigma_3(m)$ and $\sigma_4(m)$ introduced in Section 4. Such a link is pretzel and has an alternating diagram. As in Lemma 4.4, we can calculate c_{τ}^{II} of them (in fact, we can use Lemma 4.4 to compute them more easily):

Lemma 5.8. *We have*

$$c_{\tau_1(n)}^{\text{II}} = \frac{1-T}{2} - \frac{1}{n(1-T^{-1})}, \quad c_{\tau_2(n)}^{\text{II}} = \frac{-T - (-T)^n}{1 - (-T)^n}.$$

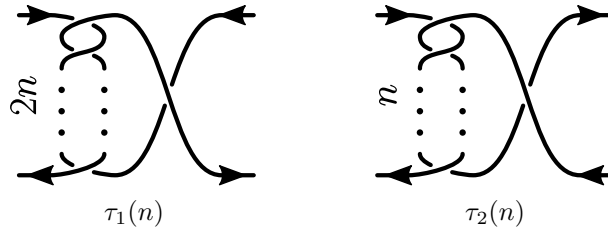


Figure 14: Other rational tangles of type II

In the proof of Theorem 4.3, we found a half plane that contains $c_{\sigma_3(m)}^{\text{II}}(t)$ and $c_{\sigma_4(m)}^{\text{II}}(t)$. However, the addition of $\tau_2(n)$ eliminates such a half plane:

Lemma 5.9. *There exist $m, n \in \mathbb{Z}_{>0}$ and $t \in \mathbb{C}$ such that $\text{Re } t < -1$, and the triangle spanned by $c_{\sigma_4(1)}^{\text{II}}(t), c_{\sigma_4(m)}^{\text{II}}(t), c_{\tau_2(n)}^{\text{II}}(t)$ contains $0 \in \mathbb{C}$ as an interior point.*

For example, we take sufficiently large m, n so that $m/n \sim 3/4$, and put $t \sim -(1 + i \tan 2\pi/n)$ to show the lemma. We have an analogous lemma for $\tau_1(n)$ and we can construct counterexamples as follows, though we omit it here.

We fix (m, n, t) obtained in Lemma 5.9. Then we have $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$ such that $\alpha c_{\sigma_4(1)}^{\text{II}}(t) + \beta c_{\sigma_4(m)}^{\text{II}}(t) + \gamma c_{\tau_2(n)}^{\text{II}}(t) = 0$. By a perturbation of t , we can make α, β, γ rational, and then we find nonnegative integers p, q, r such that $p c_{\sigma_4(1)}^{\text{II}}(t) + q c_{\sigma_4(m)}^{\text{II}}(t) + r c_{\tau_2(n)}^{\text{II}}(t) = 0$, where we assume that not all of p, q, r are 0 and that $p + q$ is even. Then, the Montesinos link composed of p copies of $\sigma_4(1)$, q copies of $\sigma_4(m)$, and r copies of $\tau_2(n)$ is a counterexample of Hoste's conjecture by Theorem 3.10.

In fact, we can construct counterexample in a more concrete way: In the construction above, we take t from a quadratic field $\mathbb{Q}(\sqrt{-d})$ ($d \in \mathbb{Z}_{>0}$). Then, we can find rational α, β, γ and then p, q, r without a perturbation of t since c_{τ}^{II} is a rational function. Thus, we have

Theorem 5.10. *The set*

$$\left\{ t \in \mathbb{C} \left| \begin{array}{l} \text{Re } t < -1, \\ \text{there exists an alternating Montesinos link } L \\ \text{of type II such that } \Delta_L(t) = 0 \end{array} \right. \right\}$$

includes $\left(\bigcup_{d \in \mathbb{Z}_{>0}} \mathbb{Q}(\sqrt{-d}) \right) \cap U$, where U is the nonempty open set

$$\bigcup_{m, n \in \mathbb{Z}_{>0}} \left\{ t \in \mathbb{C} \left| \begin{array}{l} \text{Re } t < -1, \\ (m, n, t) \text{ satisfies the conditions of Lemma 5.9} \end{array} \right. \right\}.$$

Remark 5.11. In [5], we found many explicit counterexamples in this manner, where we further assumed $p = 0$ and $r = 1$; if further m is odd and n is even, then the candidate link is a knot. The one with minimal crossing number is obtained when $(m, n, q) = (51, 64, 14)$ and the crossing number is 778.

We can also find counterexamples which are knots. We find m, n, t, p, q, r as above and we further assume that m and n are odd. We remark that this gives a 2-component link. Here, we take sufficiently large N and further add $\sigma_1(N)$ when composing the Montesinos link. In this case, we can find t' near t such that $p c_{\sigma_4(1)}^{\text{II}}(t') + q c_{\sigma_4(m)}^{\text{II}}(t') + r c_{\tau_2(n)}^{\text{II}}(t') + c_{\sigma_3(N)}^{\text{II}}(t') = 0$. Since what we obtained is a knot, we have

Theorem 5.12. *The closure of the set*

$$\left\{ t \in \mathbb{C} \left| \begin{array}{l} \operatorname{Re} t < -1, \\ \text{there exists an alternating Montesinos knot } K \\ \text{of type II such that } \Delta_K(t) = 0 \end{array} \right. \right\}$$

includes the nonempty open set

$$\bigcup_{m,n \in \mathbb{Z}_{>0}} \left\{ t \in \mathbb{C} \left| \begin{array}{l} \operatorname{Re} t < -1, \\ (m, n, t) \text{ satisfies the conditions of Lemma 5.9} \end{array} \right. \right\}.$$

However, the author does not know whether we can assume L to be a knot in Theorem 5.10.

6 A bound of roots

As we saw in the previous section, the original version of Hoste's conjecture does not hold. Then, it seems natural to ask how less the real part of a zero of the Alexander polynomial of an alternating knot can be, or, more simply, whether there exists a lower bound for them. In this section, we consider this problem for Montesinos links and show the existence of a lower bound:

Theorem 6.1. *Let L be an alternating Montesinos link. Then, the real part of any root of the Alexander polynomial is greater than -2 .*

6.1 Calculation of c_τ^I and c_τ^{II}

We first recall the treatment of rational tangles and colorings on them in [8].

We say a rational tangle oriented as in Figure 15 to be of *type o*. Such a rational tangle τ is expressed by an oriented 3-braid β , and a coloring of τ is determined by the two left colors of β . When the fraction of τ is positive, we can decompose β into elements β_n ($n = \pm 1, \pm 2, \dots$) shown in Figure 16, and then the transformation of colors given by β is the product of elementary transformations.

Here, we color the two arcs with 0, 1 and examine the ratio z/x of the output colors. Remark that $y = x + z$. The transformation of the ratio by β_n is described as a Möbius transformation, and hence $z/x = g_1 \circ \dots \circ g_k(0)$ for some Möbius transformations g_i .

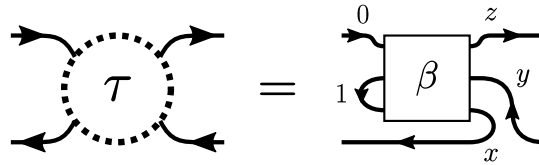


Figure 15: A rational tangle τ of type o

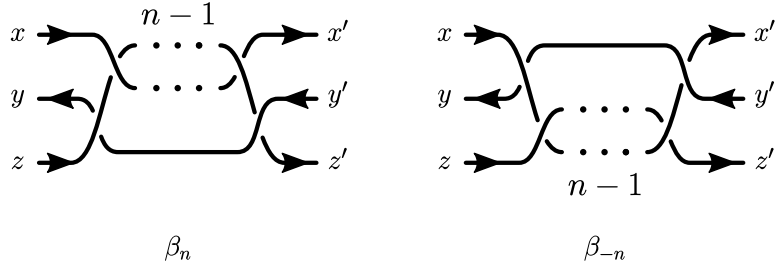


Figure 16: Elementary tangles

In [8], we examine $\mathbb{C}_t$ -colorings in this manner, but we can consider $\mathbb{C}_T$ -colorings in the same way. We define Möbius transformations f_n ($n = \pm 1, \pm 2, \dots$) on $\mathbb{C}(T) \cup \{\infty\} = \widehat{\mathbb{C}(T)}$ as

$$f_n(\xi) = \frac{\xi + a_n}{\xi/a_{n+1} + 1}, \quad \text{where } a_n = \frac{(-T)^n + T}{(-T)^n - 1},$$

for $\xi \in \widehat{\mathbb{C}(T)}$. We should note that $f_{-1}(\xi) = \xi - T + 1$. Then, for a nontrivial $\mathbb{C}_T$ -coloring on β_n as shown in Figure 16, we have $-z'/x' = f_n(-z/x)$. We should remark that $c_\tau^{\text{II}} = -z/x$ in Figure 15. Thus, we obtained the following lemma:

Lemma 6.2 ([8]). *We put*

$$\mathcal{P}^o = \left\{ c_\tau^{\text{II}} \mid \begin{array}{l} \tau \text{ is a rational tangle of type o} \\ \text{with nonnegative fraction} \end{array} \right\}.$$

Then, we have

$$\mathcal{P}^o = \bigcup_{k \geq 0} \{f_{i_1} \circ \dots \circ f_{i_k}(0) \mid i_1, \dots, i_k \in \mathbb{Z} \setminus \{0\}\}.$$

Furthermore, we put

$$\begin{aligned} \mathcal{P}^{\text{I}} &= \{c_\tau^{\text{I}} \mid \tau \text{ is a rational tangle of type I with nonnegative fraction}\}, \\ \mathcal{P}^{\text{II}} &= \{c_\tau^{\text{II}} \mid \tau \text{ is a rational tangle of type II with nonnegative fraction}\}. \end{aligned}$$

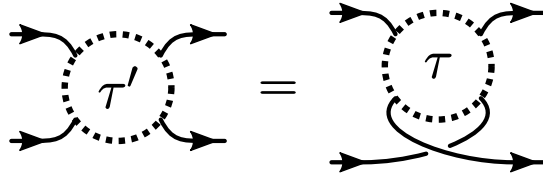


Figure 17: Type I from type o

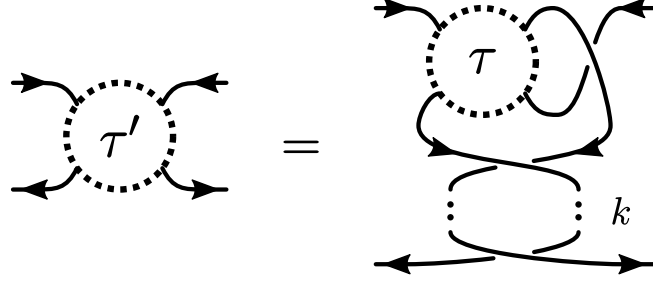


Figure 18: Type II from type o

We easily find that any rational tangle τ' of type I with fraction in $[0, 1)$ is obtained from a rational tangle τ of type o with nonnegative fraction by a half twist on the lower strings as shown in Figure 17. Using a nontrivial $\mathbb{C}_T$ -coloring on τ , we have $c_{\tau'}^I = \frac{\xi + 1}{-T^{-1}\xi + 1}$, where $\xi = c_{\tau}^{\text{II}}$. We recall the formula $c_{\sigma_1(0)}^I = -T$ to obtain

Lemma 6.3. *We have*

$$\mathcal{P}^I = \left\{ (-T)^k \frac{\xi + 1}{-T^{-1}\xi + 1} \mid \xi \in \mathcal{P}^o, k \in \mathbb{Z}_{\geq 0} \right\}.$$

Similarly, any rational tangle τ' oriented as in Figure 18 with positive fraction is obtained from a rational tangle τ as shown in the figure, where $k \geq 0$, and then we color τ to have

$$\begin{aligned} c_{\tau'}^{\text{II}} &= \frac{T - \xi + 1 - T^{-1} \frac{1 - (-T)^{-k+1}}{1 + T^{-1}} (1 + \xi) - 1 + T^{-1} \frac{1 - (-T)^{-k}}{1 + T^{-1}} (1 + \xi)}{2 \left(-1 + T^{-1} \frac{1 - (-T)^{-k}}{1 + T^{-1}} (1 + \xi) \right)} \\ &= \frac{1 + T^{-1}}{2} \cdot \frac{T - \xi - (-T)^{-k} (1 + \xi)}{-1 - T^{-1} + T^{-1} (1 - (-T)^{-k}) (1 + \xi)} \\ &= -\frac{T + 1}{2} \cdot \frac{(1 + (-T)^{-k})\xi + (-T + (-T)^{-k})}{(1 - (-T)^{-k})\xi - (T + (-T)^{-k})}, \end{aligned}$$

where $\xi = -z/x = c_{\tau}^{\text{II}}$. We define Möbius transformations g_k as

$$g_k(\xi) = -\frac{T + 1}{2} \cdot \frac{(1 + (-T)^{-k})\xi + (-T + (-T)^{-k})}{(1 - (-T)^{-k})\xi - (T + (-T)^{-k})}$$

for $\xi \in \widehat{\mathbb{C}(T)}$. By Lemma 3.12 and the formula $c_{\sigma_4(1)}^{\text{II}} = (1 - T)/2$,

Lemma 6.4. *We have*

$$\mathcal{P}^{\text{II}} = \mathcal{P}^o \cup \{g_k(\xi) \mid \xi \in \mathcal{P}^o, k \in \mathbb{Z}_{\geq 0}\}.$$

6.2 Proof of Theorem 6.1

In this section, we give a proof of Theorem 6.1. The theorem follows from the following propositions:

Proposition 6.5. *Suppose that $t \in \mathbb{C}$ has the real part less than or equal to -2 . Then, for any $\xi \in \mathcal{P}^\circ \setminus \{0\}$, we have*

$$\left| \frac{\xi(t) + 1}{-t^{-1}\xi(t) + 1} \right| > 1.$$

Proposition 6.6. *Suppose that $t \in \mathbb{C}$ has the real part less than or equal to -2 , and put*

$$P = \{z \in \mathbb{C} \mid \operatorname{Re} -t^{-1}z > 0\}.$$

Then, we have $\mathcal{P}^\circ \setminus \{0\} \subset P$, and for any $\xi \in \mathcal{P}^\circ$ and $k \in \mathbb{Z}_{\geq 0}$, $g_k(\xi)|_{T=t} \in P$.

Proof of Theorem 6.1. By [12], we may assume L to be composed of rational tangles $\tau_1, \dots, \tau_k$ with fractions of the same sign. Since taking the mirror image does not affect its Alexander polynomial, we assume that they have the positive fraction. Let $t \in \mathbb{C}$ have the real part ≤ -2 .

In the case of type I, by Lemma 6.3, there exist $\xi_i \in \mathcal{P}^\circ$ and $l_i \in \mathbb{Z}_{\geq 0}$ such that $c_{\tau_i}^I(t) = (-t)^{l_i}(\xi_i + 1)/(-t^{-1}\xi_i + 1)$, and hence it follows from Proposition 6.5 that $|c_{\tau_i}^I(t)| > 1$ for $i = 1, \dots, k$. Thus, we have $|c_{\tau_1}^I(t) \cdots c_{\tau_k}^I(t)| > 1$ and especially $c_{\tau_1}^I(t) \cdots c_{\tau_k}^I(t) \neq 1$; Theorem 3.5 completes the theorem in this case.

In the other case, by Lemma 6.4 and Proposition 6.6, we find that $c_{\tau_i}^{II}(t) \in P$. Then the sum $c_{\tau_1}^{II}(t) + \cdots + c_{\tau_k}^{II}(t)$ is also in P and hence is not 0. The theorem follows from Theorem 3.10. $\square$

In the following, we show Propositions 6.5 and 6.6. As in these propositions, we take $t \in \mathbb{C}$ such that $\operatorname{Re} t \leq -2$.

For convenience, we put $s = -t$ (then $\operatorname{Re} s \geq 2$) and suppose that $\operatorname{Im} s > 0$. We also set $r = |s|$, $\theta = \arg s \in (0, \pi/2)$, and $s = s' + is''$ ($s', s'' \in \mathbb{R}$). In the following, we do not explicitly denote the substitution of t to T ; a_n means $a_n(t)$ and f_n is a Möbius transformation on $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$. We sometimes use notations like $\alpha^{1/2}$ for $\alpha \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$; it means the square root of α with positive real part. Furthermore, for such α , we denote the argument in $(-\pi, \pi)$ by $\arg \alpha$.

Let f be an affine transformation on $\hat{\mathbb{C}}$ defined as

$$f(\xi) = \frac{z + b}{z/c + 1} \quad \text{for } z \in \hat{\mathbb{C}},$$

where $b, c \in \mathbb{C}$ and we assume that $\operatorname{Re} b, \operatorname{Re} c > 0$. Elementary calculations show that the fixed points of f are $\pm(bc)^{1/2}$ and that $(bc)^{1/2}$ is the sink point. The line

$$(bc)^{1/2}i\mathbb{R} \cup \{\infty\} = \{z \in \mathbb{C} \mid |z - (bc)^{1/2}| = |z + (bc)^{1/2}|\} \cup \{\infty\}$$

is mapped by f to the circle

$$\left\{ z \in \mathbb{C} \mid \left| \frac{z - (bc)^{1/2}}{z + (bc)^{1/2}} \right| = \frac{|(b/c)^{1/2} - 1|}{|(b/c)^{1/2} + 1|} \right\}.$$

Let us consider f_n for $n \neq 0, \pm 1$. For $m \geq 2$, we have

$$\begin{aligned} \operatorname{Re} a_m &= \operatorname{Re} \left(1 - \frac{s-1}{s^m-1} \right) > 1 - \left| \frac{s-1}{s-1} \right| = 0, \\ \operatorname{Re} a_{-m+1} &= \operatorname{Re} \frac{s^m-1}{s^{m-1}-1} = \operatorname{Re} \left(s + \frac{s-1}{s^{m-1}-1} \right) \geq s' - 1 > 0, \end{aligned}$$

where the first and second inequalities follow from Lemma 6.7 below; we can apply the argument about f to our problem. Here, we define open sets P_n and D_n ($n \neq 0, \pm 1$) of $\mathbb{C}$ as follows:

$$\begin{aligned} P_n &:= \{ z \in \mathbb{C} \mid |z - (a_n a_{n+1})^{1/2}| < |z + (a_n a_{n+1})^{1/2}| \} \\ &= \{ z \in \mathbb{C} \mid \operatorname{Re} z / (a_n a_{n+1})^{1/2} > 0 \}, \\ D_n &:= f_n(P_n) = \left\{ z \in \mathbb{C} \mid \left| \frac{z - (a_n a_{n+1})^{1/2}}{z + (a_n a_{n+1})^{1/2}} \right| < \frac{|(a_n/a_{n+1})^{1/2} - 1|}{|(a_n/a_{n+1})^{1/2} + 1|} \right\}. \end{aligned}$$

Lemma 6.7. *For any positive integer m , we have $r^m < \left| \frac{s^{m+1}-1}{s-1} \right| < r^{m+1}$.*

In particular, $|s^m - 1| > |s - 1|$ if $m \geq 2$.

Proof. We have

$$\left| \frac{s^{m+1}-1}{s-1} \right| = |s^m + \dots + 1| \leq r^m + \dots + 1 = \frac{r^{m+1}-1}{r-1} < r^{m+1}$$

as required, where the last inequality follows from the assumption $r \geq 2$. If $m = 1$, the other claim $r < |s + 1|$ holds since $\operatorname{Re} s \geq 2 > 0$. If $m \geq 2$, it is equivalent to the condition $|1 + s^{-1} + \dots + s^{-m}| < 1$, which is checked as follows:

$$\begin{aligned} |1 + s^{-1} + \dots + s^{-m}| &\geq |1 + s^{-1} + s^{-2}| - |s^{-3} + \dots + s^{-m}| \\ &\geq 1 + \frac{\cos \theta}{r} + \frac{\cos 2\theta}{r^2} - r^{-3} \cdot \frac{1 - r^{2-m}}{1 - r^{-1}} \\ &\geq 1 + \frac{2}{r^2} + \frac{2 \cdot 4/r^2 - 1}{r^2} - r^{-3} \cdot 2 \\ &= 1 + r^{-2}(1 - 2r^{-1}) + 8r^{-4} \\ &> 1. \end{aligned}$$

To see the second inequality, we should note that $\operatorname{Re} s^{-1} = r^{-1} \cos \theta$ and $\operatorname{Re} s^{-2} = r^{-2} \cos 2\theta$. Here, the third follows from the assumption $r \cos \theta \geq 2$. $\square$

To show Proposition 6.5, we prepare three lemmas, which are shown in Section 6.3:

Lemma 6.8. For any $m, n \in \mathbb{Z} \setminus \{0, \pm 1\}$, $\overline{D_m}$ is included in P_n .

Lemma 6.9. We put

$$U_0 = \left\{ z \in \mathbb{C} \mid \left| \frac{z+1}{s^{-1}z+1} \right| > 1 \right\}.$$

Then, for any $m \in \mathbb{Z} \setminus \{0, \pm 1\}$, $\overline{D_m}$ is included in U_0 .

Lemma 6.10. We put

$$U_1 = \bigcap_{n=2}^{\infty} (P_n \cap P_{-n}), \quad U_2 = U_0 \cap U_1.$$

Then, we have $f_{-1}(\overline{U_1}) \subset U_2$ and $f_1(\overline{U_2} \setminus \{0\}) \subset U_2$.

Assuming these lemmas, we prove the first proposition:

Proof of Proposition 6.5. By Lemmas 6.8 and 6.9, $\overline{D_m} \subset U_2$ for any $m \in \mathbb{Z} \setminus \{0, \pm 1\}$. Then, it follows from Lemma 6.10 that

$$f_m(\overline{U_2}) \subset U_2 \text{ if } m \neq 0, 1, \quad f_1(U_2) \subset U_2. \quad (5)$$

For $\xi \in \mathcal{P}^o \setminus \{0\}$, by Lemma 6.2 there exist $k \in \mathbb{Z}_{>0}$ and $i_1, \dots, i_k \in \mathbb{Z} \setminus \{0\}$ such that $\xi(t) = f_{i_1} \circ \dots \circ f_{i_k}(0)$, where we regard f_{i_j} as an affine transformation on $\hat{\mathbb{C}}$. Since $0 \in \overline{U_2}$, we find from (5) that $\xi(t)$ lies on $U_2 \subset U_0$ as required, or $i_1 = \dots = i_k = 1$; remark that $f_1(0) = 0$. In the latter case, the corresponding rational tangle of type I is $[0]$ and $\xi = 0 \in \mathbb{C}(T)$; we do not need to consider this case. $\square$

By definition, U_1 is the intersection of open half planes, whose boundaries contain $0 \in \mathbb{C}$, and then there exist angles θ_0 and θ_1 such that

$$\overline{U_1} = \{z \in \mathbb{C} \mid \arg z \in [\theta_0, \theta_1]\} \cup \{0\}. \quad (6)$$

Since $\lim_{n \rightarrow \infty} (a_n a_{n+1})^{1/2} = 1$ and $\lim_{n \rightarrow -\infty} (a_n a_{n+1})^{1/2} = s$, we find that

$$\arg s - \pi/2 < \theta_0 < 0 \quad \text{and} \quad \theta < \theta_1 < \pi/2. \quad (7)$$

Here, we used Lemma 6.8 to show that $\theta_0 < 0$ and $\theta < \theta_1$.

To prove the second proposition, we prepare another lemma, which is also shown in Section 6.3.

Lemma 6.11. We have $|(\xi + 1)/(\xi + s)| < 1$ for any $\xi \in \mathcal{P}^o \setminus \{0\}$.

Proof of Proposition 6.6. As we saw in the proof of Proposition 6.5, we have $\xi \in U_2$ for any $\xi \in \mathcal{P}^o \setminus \{0\}$. Furthermore, it follows from (7) that $U_2 \subset P$. Thus we have the first part $\mathcal{P}^o \setminus \{0\} \in P$.

Let us show the second part. Since $g_0(\xi) = \xi + (s+1)/2$, the case where $k = 0$ follows from the first part. Suppose that $k \leq 1$. We should notice that

$$g_k(\xi) = \frac{s-1}{2} \cdot \frac{(1+s^{-k})\xi + (s+s^{-k})}{(1-s^{-k})\xi + (s-s^{-k})} = \frac{s-1}{2} \cdot \frac{1+s^{-k}\frac{\xi+1}{\xi+s}}{1-s^{-k}\frac{\xi+1}{\xi+s}}.$$

We put $\xi' = (\xi+1)/(\xi+s)$. By Lemma 6.11, $|\xi'| < 1$ and then $|s^{-k}\xi'| < 1/2$. Therefore we find that

$$\tan \left| \arg \frac{1+s^{-k}\xi'}{1-s^{-k}\xi'} \right| < \tan \left| \arg \frac{1+i/2}{1-i/2} \right| = \frac{4}{3}.$$

Furthermore, since $|s^{-1} - 1/4| \leq 1/4$, we have $\sin(\arg(1-s^{-1})) \leq 1/3$, which means that $\tan(\arg(1-s^{-1})) \leq 1/(2\sqrt{2})$. Because $4/3 \cdot 1/(2\sqrt{2}) < 1$, we have

$$\left| \arg \frac{g_k(\xi)}{s} \right| \leq \arg \frac{s-1}{s} + \left| \arg \frac{1+s^{-k}\xi'}{1-s^{-k}\xi'} \right| < \frac{\pi}{2},$$

from which it follows that $g_k(\xi) \in P$. $\square$

6.3 Proof of lemmas

Here we show Lemmas 6.8, 6.9, 6.10, and 6.11.

Proof of Lemma 6.8. We first estimate $(a_n a_{n+1})^{1/2}$, especially its argument. If $n \geq 3$, we have

$$|a_n a_{n+1} - 1| = \left| \frac{1-s^2}{s^{n+1}-1} \right| \leq \left| \frac{s^2-1}{s^4-1} \right| \leq \frac{1}{r^2-1}, \quad (8)$$

where the first inequality follows from Lemma 6.7. In particular, we have

$$|\sin(\arg a_n a_{n+1})| \leq \frac{1}{r^2-1}. \quad (9)$$

Similarly, we have

$$|a_{-n} a_{-n+1} - s^2| = \left| \frac{s^{n+1}-1}{s^{n-1}-1} - s^2 \right| = \left| \frac{s^2-1}{s^{n-1}-1} \right| \leq 1 \quad (10)$$

and hence

$$|\sin(\arg a_{-n} a_{-n+1} - 2\theta)| \leq \frac{1}{r^2} < \frac{1}{r^2-1}. \quad (11)$$

By a direct calculation, we have $a_2 a_3 = s^2/(s^2+s+1)$ and $a_{-2} a_{-1} = s^2+s+1$. Then, we easily find that

$$\arg(a_2 a_3)^{1/2} > 0, \quad \arg(a_{-2} a_{-1})^{1/2} < \theta. \quad (12)$$

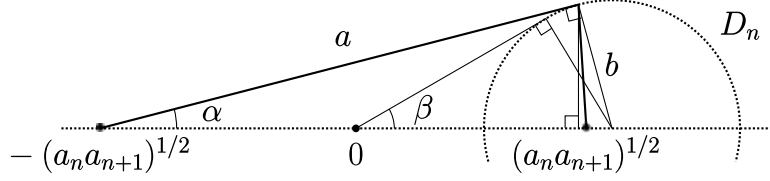


Figure 19: $\sin \beta = 2 \sin \alpha \leq b/a = \frac{|(a_n/a_{n+1})^{1/2} - 1|}{|(a_n/a_{n+1})^{1/2} + 1|}$

Next, we consider $(a_n/a_{n+1})^{1/2}$. We remark that

$$a_n/a_{n+1} = \frac{(s^{n-1} - 1)(s^{n+1} - 1)}{(s^n - 1)^2} = 1 - \frac{s^{n-1}(s - 1)^2}{(s^n - 1)^2}.$$

If $n \geq 2$, we have

$$\left| \frac{s^{n-1}(s - 1)^2}{(s^n - 1)^2} \right| \leq \left| \frac{s^{n-1}}{s^n - 1} \right| \leq \left| \frac{s - 1}{s^2 - 1} \right| = \frac{1}{|s + 1|},$$

where the inequalities follow from Lemma 6.7. Since $a_n/a_{n+1} = a_{-n}/a_{-n+1}$, this also holds for $n \leq -2$. Then, for any $n \in \mathbb{Z} \setminus \{0, \pm 1\}$,

$$\frac{|(a_n/a_{n+1})^{1/2} - 1|}{|(a_n/a_{n+1})^{1/2} + 1|} = \left| \frac{a_n/a_{n+1} - 1}{((a_n/a_{n+1})^{1/2} + 1)^2} \right| \leq \frac{9}{25|s + 1|}. \quad (13)$$

Here, we have the inequality as follows:

$$\left| \left(\frac{a_n}{a_{n+1}} \right)^{1/2} + 1 \right| \geq 1 - \frac{1}{|s + 1|} + 1 \geq \frac{5}{3}$$

since $\operatorname{Re}(s + 1) \geq 3$. By (13), we have $\left| \sin \left(\arg \frac{z + (a_n a_{n+1})^{1/2}}{(a_n a_{n+1})^{1/2}} \right) \right| \leq \frac{9}{25|s + 1|}$ for $z \in \overline{D_n}$, and then it follows that

$$\left| \sin \left(\arg \frac{z}{(a_n a_{n+1})^{1/2}} \right) \right| \leq \frac{18}{25|s + 1|} \quad \text{for } z \in \overline{D_n}; \quad (14)$$

see Figure 19.

By inequalities (9), (11), (12), and (14), we have

$$\begin{aligned} \cos \left(\arg \frac{z}{(a_n a_{n+1})^{1/2}} \right) &> \cos \theta - \frac{1}{r^2 - 1} - \frac{18}{25|s + 1|} \\ &\geq \frac{2}{r} - \frac{1}{r + 1} - \frac{18}{25r} \\ &> 0 \end{aligned}$$

for any $m, n \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $z \in \overline{D_m}$. By the definition of P_n , this means that $\overline{D_m} \subset P_n$. $\square$

Proof of Lemma 6.9. Since

$$\mathbb{C} \setminus U_0 = \{z \in \mathbb{C} \mid |z+1| \leq r^{-1}|z+s|\},$$

this is included in the closed disc of center -1 and radius $|s-1|/(r-1)$. Here, we have

$$\begin{aligned} \left(\frac{|s-1|}{r-1}\right)^2 &\leq \frac{|1+i\sqrt{r^2-4}|^2}{(r-1)^2} = 1 + \frac{2r-4}{(r-1)^2} \\ &= 1 + \frac{2}{(r-2)+2+(r-2)^{-1}} \leq 1 + \frac{2}{2+2} = \frac{3}{2}. \end{aligned}$$

Thus, the real part of any point of $\mathbb{C} \setminus U_0$ is less than or equal to $\sqrt{6}/2 - 1$.

Suppose $m \geq 3$. By (13) and (8) we have

$$\begin{aligned} \operatorname{Re} z &\geq \operatorname{Re} (a_m a_{m+1})^{1/2} - 2|a_m a_{m+1}|^{1/2} \cdot \frac{9}{25|s+1|} \\ &\geq 1 - \frac{1}{r^2-1} - \frac{2r^2}{r^2-1} \cdot \frac{9}{25|s+1|} \\ &\geq \frac{2}{3} - \frac{2 \cdot 2^2}{2^2-1} \cdot \frac{9}{25 \cdot 3} = \frac{26}{75} \\ &> \frac{\sqrt{6}}{2} - 1, \end{aligned}$$

for $z \in \overline{D_m}$, which implies that $\overline{D_m} \subset U_0$. When $m = 2$, we have

$$|1 - a_2 a_3| = \left| \frac{s+1}{s^2+s+1} \right| = \left| \frac{1}{(s+1) + (s+1)^{-1} - 1} \right| < \frac{1}{3-1} = \frac{1}{2}. \quad (15)$$

Here, $|(s+1) + (s+1)^{-1}| > 3$ is shown as follows:

$$\begin{aligned} |(s+1) + (s+1)^{-1}|^2 &= r'^2 + r'^{-2} + 2 \cos 2\theta' = r'^2 + r'^{-2} + 4 \cos^2 \theta' - 2 \\ &\geq r'^2 + r'^{-2} + 36r'^{-2} - 2 \geq 2\sqrt{37} - 2 > 3^2, \end{aligned}$$

where we put $r' = |s+1|$, $\theta' = \arg(s+1)$; then by the assumption $r' \cos \theta' \geq 3$. Further, since for $\theta \in \mathbb{R}$

$$|(1 + e^{i\phi}/3)^2 - 1| = |2e^{i\phi}/3 + e^{2i\phi}/9| \geq 2/3 - 1/9 > 1/2,$$

we find that $|1 - (a_2 a_3)^{1/2}| < 1/3$. Thus, for $z \in \overline{D_2}$, we have

$$\begin{aligned} \operatorname{Re} z &\geq \operatorname{Re} (a_2 a_3)^{1/2} - 2|a_2 a_3|^{1/2} \cdot \frac{9}{25|s+1|} \\ &> \frac{2}{3} - 2 \cdot \frac{4}{3} \cdot \frac{3}{25} = \frac{26}{75} > \frac{\sqrt{6}}{2} - 1. \end{aligned}$$

Therefore $\overline{D_2} \subset U_0$.

For $m \geq 3$, we find that $|(a_{-m}a_{-m+1})^{1/2} - s| < r^{-3}$ from (10), as we found $|1 - (a_2a_3)^{1/2}| < 1/3$ from (15) above. Then, by (14), we have

$$\begin{aligned} \operatorname{Re} z &\geq s' - |(a_{-m}a_{-m+1})^{1/2} - s| - |a_{-m}a_{-m+1}|^{1/2} \cdot \left| \sin \left(\arg \frac{z}{(a_m a_{m+1})^{1/2}} \right) \right| \\ &\geq 2 - \frac{1}{r^3} - \left(r + \frac{1}{r^3} \right) \frac{18}{25|s+1|} \geq 2 - \frac{1}{8} - \left(r + \frac{1}{8} \right) \frac{18}{25r} \\ &\geq \frac{15}{8} - \frac{18}{25} - \frac{1}{8} \cdot \frac{18}{25} > \frac{\sqrt{6}}{2} - 1 \end{aligned}$$

for $z \in \overline{D_{-m}}$ and hence $\overline{D_{-m}} \subset U_0$. Moreover, we have $\operatorname{Re}(a_{-2}a_{-1})^{1/2} > s'$ since $|a_{-2}a_{-1}|^{1/2} \geq r$ by Lemma 6.7 and $0 < \arg(a_{-2}a_{-1})^{1/2} < \arg s$. Then, for $z \in \overline{D_{-2}}$,

$$\begin{aligned} \operatorname{Re} z &\geq s' - |a_{-2}a_{-1}|^{1/2} \cdot \left| \sin \left(\arg \frac{z}{(a_m a_{m+1})^{1/2}} \right) \right| \\ &\geq 2 - \sqrt{r^2 + r + 1} \cdot \frac{18}{25r} \geq 2 - \frac{18}{25} \cdot \frac{\sqrt{7}}{2} \\ &> \frac{\sqrt{6}}{2} - 1, \end{aligned}$$

from which it follows that $\overline{D_{-2}} \subset U_0$. $\square$

Proof of Lemma 6.10. We first show that $f_{-1}(\overline{U_1}) \subset U_2$. Here, the transformation f_{-1} is by definition the shift by $s+1$, and $\arg(s+1) \in (0, \arg s)$ and (7) show that $s+1 \in \operatorname{int} U_1$. Then, it follows from the description (6) that $f_{-1}(\overline{U_1}) \subset U_1$. Furthermore, the assumption $s' \geq 2$ shows that $f_{-1}(\overline{U_1})$ is contained in $\{z \in \mathbb{C} \mid \operatorname{Re} s \geq 3\}$. Since

$$U_0 = \{z \in \mathbb{C} \mid |z+1| > r^{-1}|z+s|\},$$

$|z+1|$ for $z \in \partial U_0$ is less than or equal to $\frac{|s-1|}{r-1}$. We remark that

$$\frac{|s-1|}{r-1} \leq \frac{r+1}{r-1} = 1 + \frac{2}{r-1} \leq 3,$$

from which it follows that $\mathbb{C} \setminus U_0 \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \leq 2\}$. Thus, we have $f_{-1}(\overline{U_1}) \subset U_0$ and then $f_{-1}(\overline{U_1}) \subset U_2$.

To show that $f_1(\overline{U_1} \setminus \{0\}) \subset U_2$, we put $g(z) = z^{-1}$ for $z \in \hat{\mathbb{C}}$. We easily have

$$\begin{aligned} g \circ f_1 \circ g^{-1}(z) &= z + \frac{s+1}{s}, \\ g(U_0) &= \{z \in \mathbb{C} \mid |z+s^{-1}| < |z+1|\}, \\ g(\operatorname{int} U_1) &= \{z \in \mathbb{C} \mid \arg z \in (-\theta_1, -\theta_0)\}. \end{aligned}$$

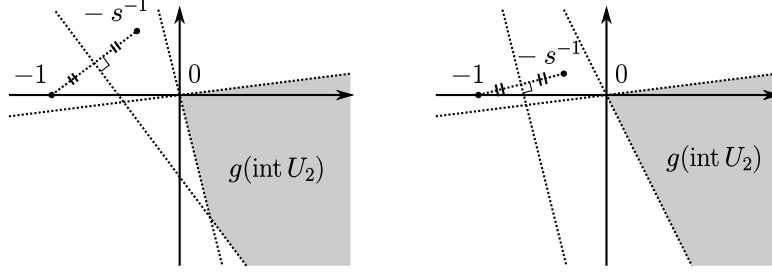


Figure 20: $g(\text{int } U_2)$ is the intersection of half planes

We should note that $\partial(g(U_0))$ is the line $-(1 + s^{-1})/2 + i(1 - s^{-1})\mathbb{R}$. Then $g(\text{int } U_2)$ is one of the regions into which the three lines divide $\mathbb{C}$ as shown in Figure 20. Here, we have $\arg(s + 1)/s > \arg i(s^{-1} - 1)$ because

$$\text{Im} \frac{(s + 1)/s}{i(s^{-1} - 1)} = \text{Im} \left(1 + \frac{2}{s - 1} \right) i > 0.$$

Since $\arg(s + 1)/s \in (-\theta_1, -\theta_0)$ by (7), we have $\overline{g \circ f_1 \circ g^{-1} \circ g(\text{int } U_2)}$, where the closure is taken in $\mathbb{C}$, and this means that $f_1(\overline{U_2} \setminus \{0\}) \subset U_2$. $\square$

Proof of Lemma 6.11. We define half planes as

$$\begin{aligned} P' &= \{z \in \mathbb{C} \mid |z + 1| < |z + s|\}, \\ P'' &= \{z \in \mathbb{C} \mid \text{Re } z/(s - 1) > 0\}. \end{aligned}$$

The lemma claims that $\xi \in P'$. Since $P'' = P' + (s + 1)/2$, we estimate the distances d as

$$\begin{aligned} d(1, \partial P') &= 2 \cos(\arg(s - 1)) + \frac{|s - 1|}{2} \geq \frac{2}{|s - 1|} + \frac{|s - 1|}{2} \geq 2, \\ d(s, \partial P') &= 2 \cos(\arg(s - 1)) + \frac{3|s - 1|}{2} \geq \frac{2}{|s - 1|} + \frac{3|s - 1|}{2} \geq 2\sqrt{3}. \end{aligned}$$

Let $n \geq 2$ be an integer. By (8) and an estimation in the proof of Lemma 6.9, we find $|1 - (a_n a_{n+1})^{1/2}| < 1/3$. Then, by (13), we have

$$|1 - z| \leq \frac{1}{3} + \frac{4}{3} \cdot 2 \cdot \frac{9/(25|s + 1|)}{1 - 9/(25|s + 1|)} \leq \frac{1}{3} + \frac{8}{3} \cdot \frac{3}{22} = \frac{23}{33} < 2$$

for $z \in \overline{D_n}$. Thus, $\overline{D_n}$ is included in P' . Next, by (10), we have $|(a_{-n} a_{-n+1})^{1/2} - s| < 1$ when $n \geq 3$. Also when $n = 2$, this holds as follows:

$$|(a_{-2} a_{-1})^{1/2} - s| = |(s^2 + s + 1)^{1/2} - s| = \left| \frac{s + 1}{(s^2 + s + 1)^{1/2} + s} \right| \leq \frac{r + 1}{2r} < 1,$$

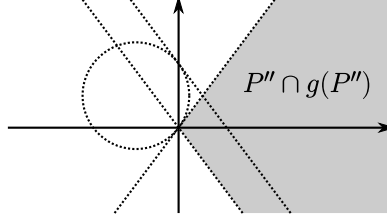


Figure 21: $(s+1)/s \in P'' \cap g(P'')$

where the first inequality follows from Lemma 6.7. Using (13) again, we have

$$\begin{aligned} |z-s| &\leq 1 + (r+1) \cdot 2 \cdot \frac{9/(25|s+1|)}{1-9/(25|s+1|)} < 1 + 2(r+1) \cdot \frac{9/25}{r-9/25} \\ &\leq 1 + 6 \cdot \frac{9/25}{2-9/25} = \frac{95}{41} < 2\sqrt{3} \end{aligned}$$

for $z \in \overline{D_{-n}}$ ($n \geq 2$), which concludes that $\overline{D_n} \subset P'$ for any $n \in \mathbb{Z} \setminus \{0, \pm 1\}$.

Next, we put $P^+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. We assert that $f_{\pm 1}(P' \cap P^+) \subset P' \cap P^+$. The case of f_{-1} is easy to see, since $f_{-1}(z) = z + s + 1$ and $s + 1 \in P'' \cap P^+$. To show the other case, we put $g(z) = z^{-1}$ as in the proof of Lemma 6.10. Then we have

$$\begin{aligned} g(P') &= \{z \in \mathbb{C} \mid |z+1| < r|z+s^{-1}|\}, \\ g(P'') &= \{z \in \mathbb{C} \mid \operatorname{Re} z(s-1) > 0\}, \end{aligned}$$

and $g(P^+) = P^+$. Since $\operatorname{Re}(s-1)(s+1)/s = \operatorname{Re}(s-s^{-1}) > 0$, $(s+1)/s \in g(P'')$. Furthermore, we have $(s+1)/s \in P''$ since

$$\frac{s+1}{s(s-1)} = \frac{1}{s-2s/(s+1)} = \frac{1}{s-2+2(s+1)^{-1}}.$$

Thus, the assertion $f_1(P' \cap P^+) \subset P' \cap P^+$ follows; see Figure 21.

By Lemma 6.2, there exist integers k and $i_1, \dots, i_k$ such that $f_{i_1} \circ \dots \circ f_{i_k}(0) = \xi$. Suppose that $i_{j+1} = \dots = i_k = 1$ and $i_j \neq 1$. Then $\xi = f_{i_1} \circ \dots \circ f_j(0)$ and $f_j(0) \in P' \cap P^+$. Since $f_{i_1}, \dots, f_{i_{j-1}}$ preserve $P' \cap P^+$, we find $\xi \in P' \cap P^+$. In particular, we have $|\xi+1| < |\xi+s|$ as required. $\square$

References

- [1] Burde, G., Zieschang, H., *Knots*. De Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 1985.
- [2] Crowell, R., *Genus of alternating link types*. Ann. of Math. **69** (1959), 258–275.

- [3] Elhamdadi, M., Nelson, S., *Quandles—an introduction to the algebra of knots*. Student Mathematical Library, 74. American Mathematical Society, Providence, 2015.
- [4] Greene, J. E., *Alternating links and definite surfaces*. With an appendix by A. Juhász and M. Lackenby. *Duke Math. J.* **166** (2017), 2133–2151.
- [5] Hirasawa, M., Ishikawa, K., Suzuki, M., *Alternating knots with Alexander polynomials having unexpected zeros*, preprint.
- [6] Howie, J. A., *A characterisation of alternating knot exteriors*. *Geom. Topol.* **21** (2017), 2353–2371.
- [7] Inoue, A., *Quandle homomorphisms of knot quandles to Alexander quandles*. *J. Knot Theory Ramifications* **10** (2001), 813–821.
- [8] Ishikawa, K., *Hoste’s conjecture for the 2-bridge knots*, to appear in *Proc. Amer. Math. Soc.*
- [9] Kauffman, L. H., Lambropoulou, S. *Classifying and applying rational knots and rational tangles*. *Physical knots: knotting, linking, and folding geometric objects in $\mathbb{R}^3$* (Las Vegas, NV, 2001), 223–259, *Contemp. Math.*, **304**, Amer. Math. Soc., Providence, RI, 2002.
- [10] Koseleff, P.-V., Pecker, D., *On Alexander-Conway polynomials of two-bridge links*. *J. Symbolic Comput.* **68** (2015), 215–229.
- [11] Lickorish, W. B. R., *An introduction to knot theory*. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.
- [12] Lickorish, W. B. R., Thistlethwaite, M. B., *Some links with nontrivial polynomials and their crossing-numbers*. *Comment. Math. Helv.* **63** (1988), 527–539.
- [13] Lyubich, L., Murasugi, K., *On zeros of the Alexander polynomial of an alternating knot*. *Topology Appl.* **159** (2012), 290–303.
- [14] Murasugi, K., *On the genus of the alternating knot. II*. *J. Math. Soc. Japan* **10** (1958) 235–248.
- [15] Stoimenow, A., *Hoste’s conjecture and roots of link polynomials*, to appear in *Ann. Combinatorics*. Stoimenow, A., *Hoste’s conjecture and roots of link polynomials*. *Ann. Comb.* **22** (2018) 393–431.

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