

Growth estimates of generalized eigenfunctions and principle of limiting absorption

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1. Introduction

In this note we present a unified approach to growth estimates of generalized eigenfunctions and principle of limiting absorption for the Schrödinger operators. The results are applicable to short-range, long-range, oscillating long-range and exploding potentials.

As an example we consider the Schrödinger operator $L = -\Delta + c(x)$ with von Neumann-Wigner type potential

$$c(x) = \frac{c \sin br}{r} + c_2(x), \quad x \in \mathbf{R}^n,$$

where $b, c > 0$, $r = |x|$ and $c_2(x)$ is a real valued short-range potential: $c_2(x) = o(r^{-1-\delta})$ ($0 < \delta \leq 1$). Obviously, L is selfadjoint and $\sigma_e(L) = [0, \infty)$. As for the growth estimates of generalized eigenfunctions

$$-\Delta u + c(x)u = \lambda u, \quad \lambda > 0, \tag{1}$$

the following results is known. Assume that the support of solution u is not compact.

Kato [1]: Let $\lambda > c^2/4$, where $c = \limsup_{r \rightarrow \infty} r|c(x)|$. Then for any $\epsilon > 0$

$$\lim_{r \rightarrow \infty} r^{c/\sqrt{\lambda} + \epsilon} \int_{S_r} \{|\partial_r u|^2 + |u|^2\} dS = \infty.$$

Thus, $(c^2, \infty) \subset \sigma_c(L)$ if L has a unique continuation property.

Mochizuki-Uchiyama [2]: Let $\lambda > bc/\gamma$ for $0 < \gamma \leq 2$. Then

$$\liminf_{r \rightarrow \infty} r^{\gamma/2} \int_{S_r} \{|\partial_r u|^2 + |u|^2\} dS > 0.$$

Thus, $\left(\frac{bc}{2}, \infty\right) \subset \sigma_c(L)$ if L has a unique continuation property.

For solution of the stationary equation

$$-\Delta u + c(x)u - \zeta u = f(x), \quad \zeta \in \{\zeta \in C; \operatorname{Re} \zeta > 0, \pm \operatorname{Im} \zeta > 0\}, \quad (2)$$

we define the vector function $\theta = \theta(x, \zeta)$ by

$$\theta(x, \zeta) = \nabla u + \tilde{x}K(x, \zeta)u, \quad \tilde{x} = x/r,$$

where

$$K(x, \zeta) = -i\sqrt{k(x, \zeta)} + \frac{n-1}{2r} + \frac{\partial_r k(x, \zeta)}{4k(x, \zeta)}$$

with $k(x, \zeta) = \zeta - \eta(\zeta)\frac{c \sin br}{r}, \quad \eta(\zeta) = \frac{4\zeta}{4\zeta - b^2}.$

This function is introduced in Mochizuki-Uchiyama [3] to define the radiation condition for (2) and to show, under the above results of [2], the principle of limiting absorption in

$$\left(\frac{b^2}{4} + \frac{bc}{\min\{2, 4\delta\}}, \infty\right).$$

Jäger-Rajto [4]: Let $|\lambda - b^2/4| > bc/2$. If solution u of (1) has no compact support, then

$$\liminf_{r \rightarrow \infty} \int_{S_r} |\theta(x, \lambda \pm i0)|^2 dS > 0.$$

Not only growth estimates of generalized eigenfunctions, this is directly applied to to show the principle of limiting absorption in

$$\left(0, \frac{b^2}{4} - \frac{bc}{\min\{2, 4\delta\}}\right) \cup \left(\frac{b^2}{4} + \frac{bc}{\min\{2, 4\delta\}}, \infty\right); \quad (3)$$

Mochizuki [5], [6]: Let I be any interval in this set and $0 < \epsilon_0 \leq 1$. We define

$$\Gamma_{\pm} = \Gamma_{\pm}(I, \epsilon_0) = \{\zeta = \lambda \pm i\epsilon; \lambda \in I, 0 < \epsilon < \epsilon_0\}.$$

For positive function $\xi = \xi(r)$ we define the weighted L^2 -space $L_{\xi}^2 = L_{\xi}^2(\mathbf{R}^n)$ with norm

$$\|f\|_{\xi}^2 = \int \xi(r)|f(x)|^2 dx.$$

Let $\mu = \mu(r) = (1+r)^{-1-\delta}$ and $\varphi = \varphi(r) = \delta^{-1}(1+r)^{\delta}$. The principle then is derived as follows: Let $R(\zeta) = (L - \zeta)^{-1}$, $\zeta \in \Gamma_{\pm}$, be the resolvent of L . Then $R(\zeta)$ continuously extended to $\bar{\Gamma}_{\pm}$ as an operator from $L_{\mu^{-1}}^2$ to L_{μ}^2 , and we have

$$\sup_{\zeta \in \Gamma_{\pm}} \|R(\zeta)f\|_{\mu} \leq C\|f\|_{\mu^{-1}}, \quad C = C(\Gamma_{\pm}) > 0.$$

Moreover, $u = R(\zeta)f$ satisfies the radiation condition $\|\theta(\cdot, \lambda \pm i0)\|_{\varphi'} < \infty$.

This result is dissatisfactory in the sense that the set (3) vanishes if δ goes to 0.

One purpose of this talk is to improve (3) to the set independent of $\delta > 0$ as follows

$$\left(0, \frac{b^2}{4} - \frac{bc}{2}\right) \cup \left(\frac{b^2}{4} + \frac{bc}{2}, \infty\right).$$

Moreover, we can treat general second order elliptic operators in exterior domain which also cover some exploding potential $c(x) \rightarrow -\infty$ as $r \rightarrow \infty$.

Main tasks will be done under a modification of the radiation conditions.

2. Results

Let $\Omega \subset \mathbf{R}^n$ ($n \geq 2$) be an exterior domain with smooth boundary $\partial\Omega$. We consider in Ω the boundary value problem

$$Lu - \zeta u = f(x) \text{ in } \Omega, \quad \mathcal{B}u = 0 \text{ on } \partial\Omega; \quad (4)$$

$$L = -\Delta_{a,b} + c(x) = - \sum_{j,k=1}^n \{\partial_j + ib_j(x)\} a_{jk}(x) \{\partial_k + ib_k(x)\} + c(x)$$

and $\mathcal{B}u|_{\partial\Omega} = 0$ is the Dirichlet or Robin boundary condition. Here $\zeta \in \mathbf{C}$, $\partial_j = \partial/\partial x_j$ and $i = \sqrt{-1}$. The coefficients are all real and sufficiently smooth, $A = (a_{jk}(x))$ is uniformly positive definite and $c(x) \geq -C(1+r^\alpha)$ ($\alpha < 2$). Then L determines a selfadjoint operator in $L^2(\Omega)$ with domain

$$\mathcal{D}(L) = \{u \in H_{\text{loc}}^2(\overline{\Omega}) \cap L^2(\Omega); -\Delta_{a,b}u + cu \in L^2(\Omega), \mathcal{B}u|_{\partial\Omega} = 0\}.$$

Let $\mu = \mu(r) > 0$ be a decreasing weight function verifying

$$(\mu.1) \quad \mu(r) = o(r^{-1}), \text{ decreasing and } \int_0^\infty \mu(r) dr < \infty.$$

[Assumptions]

$$(A.1) \quad \nabla^\ell \{a_{jm}(x) - \delta_{jm}\} = O(r^{-\ell+1}\mu) \quad (\ell = 0, 1, 2),$$

(oscillating long-range potentials) $c(x) = c_0(r) + c_1(x) + c_2(x)$ where

$$(A.2)_o \quad \partial_r^\ell c_0(r) = O(r^{-1}), \quad \partial_r^2 c_0(r) + ac_0(r) = O(\mu) \text{ for some } a \geq 0,$$

$$(A.3)_o \quad c_1(x) = O(r\mu), \quad \nabla c_1^\ell(x) = O(\mu) \quad (\ell = 1, 2),$$

$$(A.4)_o \quad \nabla \times b(x), \quad c_2(x) = O(\mu).$$

(exploding potentials) $c(x) = c_0(r) + c_1(x) + c_2(x)$ where

$$(A.2)_e \quad 1 \leq -c_0(r) \leq C(1 + r^\alpha) \quad (0 < \alpha < 2), \quad c_0(r) \rightarrow -\infty \quad (r \rightarrow \infty),$$

$$-\frac{\beta}{r} \leq \frac{\partial_r c_0(r)}{2c_0(r)} \leq \frac{1}{r} \quad (0 < \beta < 1), \quad \frac{\partial_r^2 c_0(r)}{c_0(r)} = O(r^{-1}),$$

$$(A.3)_e \quad \frac{c_1(x)}{c_0(r)} = O(r\mu), \quad \frac{\nabla^\ell c_1(x)}{c_0(r)} = O(\mu) \quad (\ell = 1, 2),$$

$$(A.4)_e \quad \frac{\nabla \times b(x)}{\sqrt{-c_0(r)}}, \quad \frac{c_2(x)}{\sqrt{-c_0(r)}} = O(\mu).$$

Remark 1. Oscillating long-range potential $c_0(r)$ is generalized to $c_0(x)$ if we require

$$\tilde{\nabla} \partial_r^\ell c_0(x) = O(\mu) \quad (\ell = 0, 1), \quad \text{where } \tilde{\nabla} = \nabla - \tilde{x} \partial_r.$$

This condition is satisfied e.g. by $c_0(x) = \frac{x_1 \sin br}{r^2}$.

2. For general exploding potential $c(x) = \tilde{c}(x) + c_2(x)$ satisfying $(A.2)_e$, put

$$c_0(r) = \frac{1}{|S_1|} \int_{S_1} \tilde{c}(r\tilde{x}) dS_{\tilde{x}}.$$

Then $c_1(x) = \tilde{c}(x) - c_0(r)$ may verify $(A.3)_e$ under the additional assumption

$$\tilde{\nabla} \partial_r^\ell \tilde{c}(x) = O(r^{-\ell}\mu) \quad (\ell = 0, 1).$$

For oscillating long-range potentials we choose an interval $I = [\lambda_1, \lambda_2]$ to satisfy

$$\lambda_1 > \frac{a}{4} + E^+ \quad \text{or} \quad 0 < \lambda_1 < \lambda_2 < \frac{a}{4} - E^-, \quad E^\pm = \limsup_{r \rightarrow \infty} \left[\pm \frac{1}{2} r \partial_r c_0(x) \right].$$

For exploding potentials I is any interval in $\mathbf{R}$. Put $\Gamma_\pm = \{\zeta = \lambda \pm i\epsilon; \lambda \in I, 0 < \epsilon \leq \epsilon_0\}$. For $(x, \zeta) \in \Omega \times \bar{\Gamma}_\pm$ let

$$k(x, \zeta) = \frac{\zeta - \eta(\zeta)c_0(r) - c_1(x)}{\tilde{x} \cdot A\tilde{x}}, \quad \eta(\zeta) = \frac{4\zeta}{4\zeta - a}$$

(in exploding case $\eta(\zeta) \equiv 1$). Then the following estimates hold for $(x, \zeta) \in \Omega'_{R_1} \times \bar{\Gamma}_\pm$ if R_1 is chosen sufficiently large.

$$(K.1) \quad 0 < C_0 \leq \operatorname{Re} k(x, \zeta) \leq C(1 + r^\alpha), \quad |\operatorname{Im} k(x, \zeta)| \leq C|\operatorname{Im} \zeta|,$$

$$(K.2) \quad -\frac{\beta}{r} \leq \operatorname{Re} \frac{\partial_r k(x, \zeta)}{2k(x, \zeta)} \leq \frac{1}{r} + O(\mu) \quad \text{for some } \beta \in (0, 1),$$

$$(K.3) \quad \frac{\nabla^{\ell+1} k(x, \zeta)}{k(x, \zeta)} = O(r^{-1}), \quad \frac{\tilde{\nabla} \partial_r^\ell k(x, \zeta)}{k(x, \zeta)} = O(\mu), \quad \ell = 0, 1,$$

as $r \rightarrow \infty$ uniformly in $\zeta \in \Gamma_\pm$.

$$(K.4) \quad c(x) - \zeta + \tilde{x} \cdot A\tilde{x} \left\{ k(x, \zeta) + \frac{\partial_r^2 k(x, \zeta)}{4k(x, \zeta)} \right\} = O(\mu)$$

as $r \rightarrow \infty$ uniformly in $\zeta \in \bar{\Gamma}_\pm$.

For solution $u \in H_{\text{loc}}^2$ of (4) let

$$K(x, \zeta) = -i\sqrt{k(x, \zeta)} + \frac{n-1}{2r} + \frac{\partial_r k(x, \zeta)}{4k(x, \zeta)}$$

and we define the vector function $\theta = \theta(x, \zeta)$ by

$$\theta(x, \zeta) = \nabla_b u + \tilde{x} K(x, \zeta) u \quad \text{where } \nabla_b = \nabla + ib(x).$$

Theorem 1 *Under the above Assumption, let $u \in H_{\text{loc}}^2(\bar{\Omega})$ solves the eigenvalue problem*

$$-\Delta_{a,b} u + cu - \lambda u = 0 \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial\Omega \quad (5)$$

with $\lambda \in I$. If the support of u is not compact, then it satisfies

$$\liminf_{t \rightarrow \infty} \int_{S_t} \frac{1}{\sqrt{k(x, \lambda)}} |\tilde{x} \cdot A\theta(x, \lambda \pm i0)|^2 dS > 0.$$

Assume that there exists a positive decreasing function $\mu_0(r) \leq \mu(r)$ such that the functions

$$\varphi_0(r) = \left(\int_r^\infty \mu_0(s) ds \right)^{-1}, \quad \varphi(r) = \left(\int_r^\infty \mu(s) ds \right)^{-1}$$

satisfy for $r > R_1$

$$(\mu.2) \quad \varphi'_0(r) \leq \varphi'(r) \quad \text{and} \quad \frac{\varphi'_0(r)}{\varphi_0(r)} \leq \frac{1}{r} + \min \left\{ 0, \operatorname{Re} \frac{\partial_r k(x, \zeta)}{2k(x, \zeta)} \right\}.$$

Definition 1 The solution of (4) is said to satisfy the radiation condition if

$$\int \mu_0(r) |\sqrt{k(x, \zeta)}| |u(x, \zeta)|^2 dx < \infty, \quad \int \frac{\varphi'_0(r)}{|\sqrt{k(x, \zeta)}|} |\tilde{x} \cdot A\theta(x, \zeta)|^2 dx < \infty.$$

A solution of (4) which also satisfies the radiation condition is called a radiative solution.

Let $\zeta \in \Gamma_{\pm}$. Then the resolvent $R(\zeta) = (L - \zeta)^{-1}$ forms a bounded operator in $L^2(\Omega)$ which depends continuously on ζ . Moreover, if $f \in L^2_{(\mu_0|\sqrt{k}|)^{-1}}(\Omega)$, then $u = R(\zeta)f$ is shown to satisfy the above radiation condition.

Theorem 2 *Under the above Assumption, let $\zeta \in \Gamma_{\pm}$ and $f \in L^2_{(\mu_0|\sqrt{k}|)^{-1}}$. Then there exists $C = C(\Gamma_{\pm}) > 0$ such that*

$$\sup_{\zeta \in \Gamma_{\pm}} \|R(\zeta)f\|_{\mu_0|\sqrt{k}|} \leq C\|f\|_{(\mu_0|\sqrt{k}|)^{-1}},$$

and as an operator from $L^2_{(\mu_0|\sqrt{k}|)^{-1}}$ to $L^2_{\mu_0|\sqrt{k}|}(\Omega)$, $R(\zeta)$ is extended continuously to $\bar{\Gamma}_{\pm}$. Moreover, $u = R(\lambda \pm i0)f$ becomes an (outgoing (+) or incoming (-)) radiative solution of (4) with $\zeta = \lambda$.

Remark 3. In case of exploding potentials, similar results is obtained by Yamada [7] under slightly stringent conditions on the coefficients. In his case the radiation conditions are, as in the case of [3], defined by

$$\|u\|_{\mu|\sqrt{k}|} < \infty, \quad \|\tilde{x} \cdot \theta\|_{\varphi'} < \infty$$

3. A quadratic identity

For the sake of simplicity we restrict ourselves to the equation with $a_{jk}(x) = \delta_{jk}$:

$$-\Delta_b u + c(x)u - \zeta u = f(x) \quad \text{in } \mathbf{R}^n, \quad (6)$$

where $\Delta_b = \nabla_b \cdot \nabla_b$ with $\nabla_b = \nabla + ib(x)$.

For solution u of (6) we put

$$u_{\sigma} = e^{\sigma} u, \quad f_{\sigma} = e^{\sigma} f \quad \text{and} \quad \theta_{\sigma} = \nabla_b u_{\sigma} + \tilde{x} K u_{\sigma},$$

where $\sigma = \sigma(r)$ is a positive function of $r > 0$. (6) is rewritten as

$$-\nabla_b \cdot \theta_{\sigma} + (K + 2\sigma')\tilde{x} \cdot \theta_{\sigma} + q_{K,\sigma} u = f_{\sigma}, \quad (7)$$

$$q_{K,\sigma} = q_K + \sigma'' + \frac{n-1}{r}\sigma' - \sigma'^2 - 2K\sigma' \quad \text{with}$$

$$q_K = c(x) - \zeta + \partial_r K + \frac{n-1}{r}K - K^2.$$

For a smooth weight function $\Phi = \Phi(x) > 0$, let us consider the real part of the equation (6) multiplied by $\Phi \tilde{x} \cdot \overline{\theta_\sigma}$. The integrating by parts over $B_{R,t} = \{x \in \mathbf{R}^n; R < |x| < t\}$ give the following identity:

$$\begin{aligned} & - \left[\int_{S_t} - \int_{S_R} \right] \Phi \left\{ |\tilde{x} \cdot \theta_\sigma|^2 - \frac{1}{2} |\theta_\sigma|^2 \right\} dS + \operatorname{Re} \int_{\Omega_{R,t}} \Phi \left[\frac{1}{r} \{ |\theta_\sigma|^2 - |\tilde{x} \cdot \theta_\sigma|^2 \} \right. \\ & \quad + \left(K - \frac{n-1}{2r} \right) |\theta_\sigma|^2 + 2\sigma' |\tilde{x} \cdot \theta_\sigma|^2 + \frac{\nabla \Phi}{\Phi} \cdot \theta_\sigma (\tilde{x} \cdot \overline{\theta_\sigma}) - \frac{\partial_r \Phi}{2\Phi} |\theta_\sigma|^2 \\ & \quad \left. + \mathcal{B}(u_\sigma, \theta_\sigma) + (q_{K,\sigma} - q_K) u_\sigma (\tilde{x} \cdot \overline{\theta_\sigma}) \right] dx = \operatorname{Re} \int_{\Omega_{R,t}} \Phi f_\sigma (\tilde{x} \cdot \overline{\theta_\sigma}) dx, \end{aligned} \quad (8)$$

where

$$\mathcal{B}(u_\sigma, \theta_\sigma) = i u_\sigma (\nabla \times b) \cdot (\tilde{x} \times \overline{\theta_\sigma}) + u_\sigma (\tilde{\nabla} K \cdot \overline{\theta_\sigma}) + q_{K,\sigma} u_\sigma (\tilde{x} \cdot \overline{\theta_\sigma}).$$

Lemma 1 *Under the above Assumptions we have*

$$|\mathcal{B}(u_\sigma, \theta_\sigma)| = O(\mu) |k(x, \zeta)|^{1/2} |u_\sigma| |\theta_\sigma| \quad \text{as } r \rightarrow \infty.$$

4. Outline of the proof of Theorem 1

We choose $0 < \delta < 1 - \beta$ and put

$$\varphi_0(x, \lambda) = \frac{1}{\sqrt{k(x, \lambda)}}, \quad \varphi(x, \lambda) = \frac{r^{2-\delta} \sqrt{k_0(r, \lambda)}^{2-\delta}}{\sqrt{k(x, \lambda)}},$$

where $k_0(r, \lambda) = \lambda - \eta(\lambda) c_0(r)$. Note that

$$\frac{\partial_r k(x, \lambda)}{k(x, \lambda)} - \frac{\partial_r k_0(r, \lambda)}{k_0(r, \lambda)} = O(\mu). \quad (9)$$

We define the two functionals of solution u of the homogeneous equation (5).

$$F_0(t) = \int_{S_t} \varphi_0 \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} dS,$$

$$F_{\sigma,\tau}(t) = \int_{S_t} \varphi \left\{ |\tilde{x} \cdot \theta_\sigma|^2 - \frac{1}{2} |\theta_\sigma|^2 + \frac{1}{2} (\sigma^2 - \tau) |u_\sigma|^2 \right\} dS$$

where $\sigma = \sigma(r)$ and $\tau = \tau(r)$ are positive smooth functions given later.

Lemma 2 *The weight functions φ_0 and φ verify*

$$\frac{\nabla \varphi_0}{\varphi_0} = - \frac{\partial_r k}{2k} \tilde{x} + O(\mu), \quad (10)$$

$$\frac{\nabla \varphi}{\varphi} = \frac{2-\delta}{r} + (1-\delta) \frac{\partial_r k}{2k} \tilde{x} + O(\mu). \quad (11)$$

Lemma 3 *u be a solution of (5). Then for each $r > R_0$ and $\lambda \in I$ we have*

$$\operatorname{Im} \left[\int_{S_r} \tilde{x} \cdot \nabla_b u_\sigma \overline{u_\sigma} dS \right] = 0.$$

Lemma 4 *Let $r > R_1$. Then for each solution u of (5) we have*

$$\int_{S_r} \varphi_0 k |u_\sigma|^2 dS \leq \int_{S_r} \varphi_0 |\tilde{x} \cdot \theta_\sigma|^2 dS,$$

Proof of Theorem 1, Part 1 In this part we require an additional assumption that there exists a sequence $r_k \rightarrow \infty$ such that $F_0(r_k) > 0$.

We choose $\Phi = \varphi_0$, $\zeta = \lambda \pm i0$, $f = 0$ and $\sigma = 0$ in identity (8). Then noting

$$\operatorname{Re} \left(K - \frac{n-1}{2r} \right) = \frac{\partial_r k}{4k},$$

(10) and Lemmas 1, 4 we have

$$\begin{aligned} \frac{d}{dt} F_0(t) &\geq \int_{S_t} \varphi_0 \left[\left(\frac{1}{r} + \frac{\partial_r k}{2k} \right) (|\theta|^2 - |\tilde{x} \cdot \theta|^2) - O(\mu) |\theta|^2 \right] dS \\ &= \int_{S_1} \varphi_0 \left[\left(\frac{1}{r} + \frac{\partial_r k}{2k} - 2O(\mu) \right) (|\theta|^2 - |\tilde{x} \cdot \theta|^2) \right. \\ &\quad \left. - 2O(\mu) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} \right] dS \geq -2O(\mu(t)) F_0(t) \end{aligned}$$

for $t \geq R_1$ if $R_1 \geq R_0$ is chosen sufficiently large. By assumption there exists $r_n \geq R_1$ and hence we conclude

$$F_0(t) \geq e^{-C \int_{r_n}^\infty \mu(s) ds} F_0(r_n) > 0,$$

which proves Theorem 1 since we have

$$\int_{S_t} \frac{1}{\sqrt{k}} |\tilde{x} \cdot \theta|^2 dS \geq 2F_0(t).$$

Proof of Theorem 1, Part 2 We assume $F_0(t) \leq 0$ in $t > R_0$ and u does not have compact support.

We choose $\Phi = \varphi$, $\zeta = \lambda \pm i0$ and $f = 0$ in identity (8) added by the identity

$$\frac{1}{2} \left[\int_{S_t} - \int_{S_R} \right] \varphi (\sigma'^2 - \tau) |u_\sigma|^2 dS = \operatorname{Re} \int_{B_{R,t}} \varphi \left[(\sigma'^2 - \tau) u_\sigma (\tilde{x} \cdot \overline{\theta_\sigma}) \right]$$

$$+(\sigma'^2 - \tau) \left(\frac{\nabla \varphi}{2\varphi} - \frac{\partial_r k}{4k} \right) |u_\sigma|^2 + \left(\sigma' \sigma'' - \frac{\tau'}{2} \right) |u_\sigma|^2 dx,$$

where $\tau = \tau(r) > 0$, and differentiate both sides by t . Then we have

$$\begin{aligned} \frac{d}{dt} \int_{S_t} \varphi \left\{ |\tilde{x} \cdot \theta_\sigma|^2 - \frac{1}{2} |\theta_\sigma|^2 + \frac{1}{2} (\sigma'^2 - \tau) |u_\sigma|^2 \right\} dS &= \operatorname{Re} \int_{S_t} \varphi \left[\frac{1}{r} \{ |\theta_\sigma|^2 - |\tilde{x} \cdot \theta_\sigma|^2 \} \right. \\ &\quad + \frac{\partial_r k}{4k} |\theta_\sigma|^2 + 2\sigma' |\tilde{x} \cdot \theta_\sigma|^2 + \left(\frac{\nabla \varphi}{\varphi} \right) \cdot \left\{ \theta_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma) - \frac{1}{2} |\theta_\sigma|^2 \right\} \\ &\quad + \mathcal{B}(u_\sigma, \theta_\sigma) + \left(\sigma'' + \frac{n-1}{r} \sigma' - 2\sigma' K \right) u_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma) - \tau u_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma) \\ &\quad \left. + (\sigma'^2 - \tau) \left(\frac{\nabla \varphi}{2\varphi} - \frac{\partial_r k}{4k} \right) |u_\sigma|^2 + \left(\sigma' \sigma'' - \frac{\tau'}{2} \right) |u_\sigma|^2 \right] dS. \end{aligned}$$

Here, by use of (11) we have

$$\begin{aligned} &\bullet \frac{1}{r} \{ |\theta_\sigma|^2 - |\tilde{x} \cdot \theta_\sigma|^2 \} + \frac{\partial_r k}{4k} |\theta_\sigma|^2 + \left(\frac{\nabla \varphi}{\varphi} \right) \cdot \left\{ \theta_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma) - \frac{1}{2} |\theta_\sigma|^2 \right\} \\ &\quad \geq \left(\frac{1-\delta}{r} + (1-\delta) \frac{\partial_r k}{2k} \right) |\tilde{x} \cdot \theta_\sigma|^2 + \left\{ \frac{\delta}{2r} + \delta \frac{\partial_r k}{4k} - O(\mu) \right\} |\theta_\sigma|^2, \\ &\bullet 2\sigma' |\tilde{x} \cdot \theta_\sigma|^2 + \operatorname{Re} \left\{ \left(\sigma'' - 2\sigma' \frac{\partial_r k}{4k} + 2\sigma' i\sqrt{k} \right) u_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma) \right\} \\ &\quad \geq 2\sigma' |\tilde{x} \cdot \theta_\sigma + i\sqrt{k} u_\sigma|^2 + 2\sigma' \operatorname{Im} \{ \sqrt{k} u_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma + i\sqrt{k} u_\sigma) \} \\ &\quad \quad - \frac{\sigma'}{2} |\tilde{x} \cdot \theta_\sigma + i\sqrt{k} u_\sigma|^2 - \frac{\sigma'}{2} \left(\frac{\sigma''}{\sigma'} - \frac{\partial_r k}{2k} \right)^2 |u_\sigma|^2, \\ &\bullet -\tau \operatorname{Re} [u_\sigma (\tilde{x} \cdot \bar{\theta}_\sigma)] + (\sigma'^2 - \tau) \left(\frac{\nabla \varphi}{2\varphi} - \frac{\partial_r k}{4k} \right) |u_\sigma|^2 + \left(\sigma' \sigma'' - \frac{\tau'}{2} \right) |u_\sigma|^2 \\ &\quad \geq -\frac{\sigma'}{2} |\tilde{x} \cdot \theta_\sigma + i\sqrt{k} u_\sigma|^2 - \left(\frac{\tau^2}{2\sigma'} + \frac{\tau'}{2} + \frac{C\tau}{r} \right) |u_\sigma|^2 \\ &\quad \quad + \frac{\sigma'^2}{2} \left(\frac{2-\delta}{r} - O(\mu) - \delta \frac{\partial_r k}{2k} + \frac{2\sigma''}{\sigma} \right) |u_\sigma|^2 \end{aligned}$$

with $C > 0$ chosen to satisfy $\frac{2-\delta}{r} - \delta \frac{\partial_r k}{2k} - O(\mu) \leq \frac{C}{r}$. Moreover, since

$$\operatorname{Re} \int_{S_t} \varphi \mathcal{B}(u_\sigma, \theta_\sigma) dS \leq \int_{S_t} \varphi O(\mu) |\theta_\sigma|^2 dS$$

by Lemmas 1 and 4, it follows that

$$\frac{d}{dt} F_{\sigma, \tau}(t) \geq \int_{S_t} \varphi \left[\left\{ \frac{1-\delta}{r} + (1-\delta) \frac{\partial_r k}{2k} \right\} |\tilde{x} \cdot \theta_\sigma|^2 + \left\{ \frac{\delta}{2r} + \delta \frac{\partial_r k}{4k} - 2O(\mu) \right\} |\theta_\sigma|^2 \right]$$

$$\begin{aligned}
& + \left\{ \sigma' |\tilde{x} \cdot \theta_\sigma + i\sqrt{k}u_\sigma|^2 + 2\sigma' \operatorname{Im} \left[\sqrt{k}u_\sigma (\tilde{x} \cdot \theta_\sigma + i\sqrt{k}u_\sigma) \right] \right\} \\
& - \frac{\sigma'}{2} \left(\frac{\sigma''}{\sigma'} - \frac{\partial_r k}{2k} \right)^2 |u_\sigma|^2 - \left(\frac{\tau^2}{2\sigma'} + \frac{C\tau}{r} + \frac{\tau'}{2} \right) |u_\sigma|^2 \\
& + \frac{\sigma'^2}{2} \left(\frac{2-\delta}{r} - O(\mu) - \delta \frac{\partial_r k}{2k} + \frac{2\sigma''}{\sigma'} \right) |u_\sigma|^2 dS.
\end{aligned}$$

Now, let $m \geq 1$ and $\frac{1}{3} < \gamma < 1 - \delta$ (without loss of generality we can assume $\delta < \frac{2}{3}$ in Theorem 1) and choose $\sigma(r)$ and $\tau(r)$ as follows:

$$\sigma(r) = \frac{m}{1-\gamma} r^{1-\gamma}, \quad \tau(r) = r^{-2\gamma} \log r. \quad (12)$$

Then as $r \rightarrow \infty$,

$$\begin{aligned}
& -\frac{\sigma'}{2} \left(\frac{\sigma''}{\sigma'} - \frac{\partial_r k}{2k} - \frac{o(1)}{r} \right)^2 = mO(r^{-2-\gamma}), \\
& \frac{\sigma'^2}{2} \left(\frac{2-\delta}{r} - \delta \frac{\partial_r k}{2k} - \frac{o(1)}{r} + \frac{2\sigma''}{\sigma'} \right) \\
& \geq m^2 \{2(1-\delta-\gamma) - o(1)\} r^{-1-2\gamma} > 0
\end{aligned} \quad (13)$$

since $1 - \delta > \gamma$, and

$$-\left(\frac{\tau^2}{2\sigma'} + \frac{C\tau}{r} + \frac{\tau'}{2} \right) \geq -C_5 \mu_1,$$

where $\mu_1 = r^{-3\gamma}(\log r)^2 \in L^1([R_1, \infty))$ and $C_5 > 0$ is independent of m and $r \geq R_4$.

Moreover, by Lemma 3

$$\begin{aligned}
& \operatorname{Im} \int_{S_t} \varphi \sqrt{k} u_\sigma (\tilde{x} \cdot \theta_\sigma + i\sqrt{k}u_\sigma) dS \\
& = t^{2-\delta} k_0(t, \lambda)^{(2-\delta)/2} \operatorname{Im} \int_{S_t} u_\sigma \tilde{x} \cdot \nabla u_\sigma dS = 0.
\end{aligned} \quad (14)$$

Summarizing these results, we obtain the following: for any $m \geq 1$, there exists $R_5 \geq R_4$ such that

$$\frac{d}{dt} F_{\sigma, \tau}(t) \geq \int_{S_t} \varphi \left\{ \left(\frac{1-\delta}{r} + (1-\delta) \frac{\partial_r k}{2k} \right) |\tilde{x} \cdot \theta_\sigma|^2 - C_5 \mu_1 |u_\sigma|^2 \right\} dS \geq 0 \quad (15)$$

in $t \geq R_5$. Here we have used Lemma 4 again to show the last inequality.

By assumption that the support of u is not compact, R_5 can be chosen to satisfy

$$\int_{S_{R_5}} |u_\sigma|^2 dS > 0.$$

Then as we see from (13), $F_{\sigma,\tau}(R_5)$ goes to ∞ as $m \rightarrow \infty$. We fix a large m satisfying $F_{\sigma,\tau}(R_5) > 0$ to conclude $F_{\sigma,\tau}(t) > 0$ for $t \geq R_5$.

Finally, we note the identity

$$\begin{aligned} F_{\sigma,\tau}(t) &= e^{2\sigma t^{2-\delta}} k_0(r, \lambda)^{(2-\delta)/2} \left\{ F_0(t) + \sigma' \operatorname{Re} \int_{S_t} \varphi_0(\tilde{x} \cdot \nabla u) \bar{u} dS \right. \\ &\quad \left. + \int_{S_t} \varphi_0 \left(\sigma'^2 - \frac{1}{2} \tau + \sigma' \frac{n-1}{2t} + \sigma' \frac{\partial_r k}{4k} \right) \int_{S(t)} \varphi_1 |u|^2 dS \right\} \end{aligned}$$

In this equation we use

$$\begin{aligned} &\operatorname{Re} \int_{S_t} \varphi_0(\tilde{x} \cdot \nabla u) \bar{u} dS - \frac{1}{2} \frac{d}{dt} \int_{S_t} \varphi_1 |u|^2 dS \\ &= -\frac{1}{2} \int_{S_t} \left\{ \partial_r \varphi_0 + \frac{n-1}{r} \varphi_1 \right\} |u|^2 dS \leq \int_{S_t} O(r^{-1}) \varphi_0 |u|^2 dS, \end{aligned}$$

and note that $F_0(t) \leq 0$ near infinity by assumption. Then since

$$\sigma'^2 - \frac{1}{2} \tau + \sigma' \frac{n-1}{2t} + \sigma' \frac{\partial_r k}{4k} + \sigma' O(t^{-1})$$

becomes negative when t goes large, it follows that

$$\frac{d}{dt} \int_{S_t} \varphi_0 |u|^2 dS > 0$$

for t large enough. This and Lemma 4 establish the conclusion of the Theorem. $\square$

Remark 4. In case of general oscillating potential $c_0(x)$ in Remark 1, we have to divide the proof of *Part 2* in two steps. We choose

$$\varphi_1(x) = \frac{r^\delta \sqrt{k_1(x, \lambda)}^{2-\delta}}{\sqrt{k(x, \lambda)}}, \quad \varphi(x) = \frac{r^{2-\delta} \sqrt{k_1(x, \lambda)}^{2-\delta}}{\sqrt{k(x, \lambda)}}$$

with $k_1(x, \lambda) = \lambda - \eta(\lambda) c_1(x)$, and define

$$\begin{aligned} F_1(t) &= \int_{S_t} \varphi_1 \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} dS, \\ F_{\sigma,\tau}(t) &= \int_{S_t} \varphi \left\{ |\tilde{x} \cdot \theta_\sigma|^2 - \frac{1}{2} |\theta_\sigma|^2 + \frac{1}{2} (\sigma^2 - \tau) |u_\sigma|^2 \right\} dS. \end{aligned}$$

Step 1 $F_0(t) \leq 0$ in $t > R_0$ and u does not have compact support, on the other hand, there exists a sequence $r_p \rightarrow \infty$ such that $F_1(r_p) > 0$.

Step 2 $F_1(t) \leq 0$ in $T > R_0$ and u does not have compact support.

In the proof of *Step 1* the inequality

$$\int_{S_t} \varphi_1 |\tilde{x} \cdot \theta|^2 dS \leq (1 + O(r^{-1})) \int_{S_t} \varphi_1 \{|\theta|^2 - |\tilde{x} \cdot \theta|^2\} dS$$

which follows from the assumption $F_0(t) \leq 0$ plays an important role. On the other hand, in the proof of *Step 2* equation (14) is not expected to hold. Instead, we have

$$\begin{aligned} 2\sigma' \int_{S_t} \varphi \left\{ \operatorname{Im} \left[\pm \sqrt{k} (\tilde{x} \cdot A\tilde{x}) u_\sigma \tilde{x} \cdot A\overline{\theta}_{\sigma,1} \right] + \frac{1}{2} |\tilde{x} \cdot A\theta_{\sigma,1}|^2 \right\} dS \\ \geq -C \int_{S_t} \varphi \sigma' r^{-2} |u_\sigma|^2 dS = -Cm \int_{S_t} \varphi r^{-2-\gamma} |u|^2 dS \end{aligned}$$

since $\varphi \sqrt{k} = r^{2-\delta} \lambda^{(2-\delta)/2} \{1 + O(r^{-1})\}$. Thus, this term can be absorbed in the term corresponding to (13).

5. Outline of the proof of Theorem 2

We choose the weight function $\mu = \mu(r)$ to satisfy $(\mu.1)$ and also the following: There exists $\mu_0(r)$ verifying also $(\mu.1)$ such that

$$(\mu.2) \quad \mu(r) \leq \mu_0(r)$$

and if we put

$$\varphi(r) = \left(\int_r^\infty \mu(\tau) d\tau \right)^{-1} \quad \text{and} \quad \varphi_0(r) = \left(\int_r^\infty \mu_0(\tau) d\tau \right)^{-1}, \quad (16)$$

then it satisfies for $r \geq R_0$

$$(\mu.3) \quad \varphi'_0(r) \leq \varphi'(r) \quad \text{and} \quad \frac{1}{r} - \frac{\varphi'_0(r)}{\varphi_0(r)} \geq \max \left\{ 0, -\operatorname{Re} \frac{\partial_r k}{2k} \right\}.$$

Remark 5. If $\mu = r^{-1-\delta}$ ($0 < \delta \leq 1$) for $r > R_0$, then $\varphi = \delta r^\delta$ and $\varphi' = \delta^2 r^{-1+\delta}$. In this case $(\mu.3)$ is verified from (K.2) if we choose $\mu_0 = r^{-1-\tilde{\delta}}$ with $0 < \tilde{\delta} \leq \min\{\delta, 1 - \beta\}$.

If $\mu = r^{-1}(\log r)^{-1-\delta}$ ($0 < \delta \leq 1$), then $\varphi = \delta(\log r)^\delta$ and $\varphi' = \delta^2 r^{-1}(\log r)^{-1+\delta}$. Thus, we have $\frac{\varphi'}{\varphi} = o(r^{-1})$ and $(\mu.3)$ is satisfied by $\mu_0(r) = \mu(r)$.

Lemma 5 *We have for any $R > 0$,*

$$\frac{\varphi'_0(r)}{\varphi_0(r)} = \mu_0(r) \varphi_0(r) \notin L^1([R, \infty)).$$

Proof By definition $\varphi_0(r) \rightarrow \infty$ as $r \rightarrow \infty$. So, the assertion holds since we have

$$\int_R^r \frac{\varphi'_0(s)}{\varphi_0(s)} ds = \log \left\{ \frac{\varphi_0(r)}{\varphi_0(R)} \right\} \rightarrow \infty \quad \text{as } r \rightarrow \infty. \quad \square$$

Lemma 6 *Let u be a radiative solution of (6).*

(i) *If $\text{Im}\zeta \neq 0$, then we have $u \in L^2(\Omega)$ and*

$$|\text{Im}\zeta| \|u\| \leq \|f\|.$$

(ii) *There exists $C > 0$ such that for any $R \geq R_0$ and $\zeta \in \Gamma_{\pm}$,*

$$\int_{B'_R} \mu_0 |\sqrt{k}| |u|^2 dx \leq C \varphi_0(R)^{-1} \left\{ \|\tilde{x} \cdot \theta\|_{\varphi'_0 \mathcal{H}^{-1}, B'_R}^2 + \|u\|_{\mu_0 \mathcal{H}}^2 + \|f\|_{(\mu_0 \mathcal{H})^{-1}}^2 \right\}$$

Proof By the Green formula

$$\text{Im} \int_{B_r} f \bar{u} dx = -\text{Im} \int_{S_r} (\tilde{x} \cdot \nabla_{\delta} u) \bar{u} dS - \text{Im}\zeta \int_{B_r} |u|^2 dx.$$

This is rewritten as

$$\text{Im}\zeta \int_{B_r} |u|^2 dx - \int_{S_r} \text{Im}K |u|^2 dS = -\text{Im} \left[\int_{S_r} \tilde{x} \cdot \theta \bar{u} dS + \int_{B_r} f \bar{u} dx \right].$$

Note here that $\text{Im}\zeta$ and $-\text{Im}K$ has the same sign when r is large, say for $r \geq R$. $\square$

Lemma 7 *Let u be a radiative solution of (6). Then there exists $C = C(\Gamma_{\pm}) > 0$ such that*

$$\int_{B'_{R+1}} \varphi'_0 |\sqrt{k}|^{-1} |\theta|^2 dx \leq C \left\{ \|u\|_{\mu_0 |\sqrt{k}|, B'_R}^2 + \|f\|_{(\mu_0 |\sqrt{k}|)^{-1}, B'_R}^2 \right\}.$$

Proof In the quadratic identity (8) with $\sigma = 0$ we choose

$$\Phi = \frac{\chi \varphi_0(r)}{|k(x, \zeta)|^{1/2}}. \quad (17)$$

where $\chi = \chi(r)$ is smooth and satisfy $\chi(r) = 0$ ($r < R$) and $= 1$ ($r > R+1$). Then

$$\begin{aligned} & - \left[\int_{S_t} - \int_{S_R} \right] \Phi \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} dS + \text{Re} \int_{B_{R,t}} \Phi \left[\frac{1}{r} \{ |\theta|^2 - |\tilde{x} \cdot \theta|^2 \} \right. \\ & \left. + \left(-i\sqrt{k} + \frac{\partial_r k}{4k} \right) |\theta|^2 + \left(\frac{\varphi'_0}{\varphi_0} - \text{Re} \frac{\partial_r k}{2k} + \frac{\chi'}{\chi} \right) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} \right] \end{aligned}$$

$$-\operatorname{Re} \frac{\tilde{\nabla} k}{2k} \cdot \theta(\tilde{x} \cdot \bar{\theta}) + u(\tilde{\nabla} K \cdot \bar{\theta}) + q_K u(\tilde{x} \cdot \bar{\theta}) \Big] dx = \operatorname{Re} \int_{B_{R,t}} \Phi f_\sigma(\tilde{x} \cdot \bar{\theta}) dx.$$

Since

$$\begin{aligned} & \frac{1}{r} \{ |\theta|^2 - |\tilde{x} \cdot \theta|^2 \} + \operatorname{Re} \frac{\partial_r k}{4k} |\theta|^2 + \left(\frac{\varphi'_0}{\varphi_0} - \operatorname{Re} \frac{\partial_r k}{2k} \right) \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} \\ &= \left(\frac{1}{r} - \frac{\varphi'_0}{\varphi_0} + \operatorname{Re} \frac{\partial_r k}{2k} \right) \{ |\theta|^2 - |\tilde{x} \cdot \theta|^2 \} + \frac{\varphi'_0}{2\varphi_0} |\theta|^2, \end{aligned} \quad (18)$$

it follows that

$$\begin{aligned} & \int_{S_t} \Phi \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} dS \geq \operatorname{Re} \int_{B_{R,t}} \Phi \left\{ \left\{ \frac{2\varphi'_0}{\varphi_0} - C\mu \right\} |\theta|^2 \right. \\ & \quad \left. - C_1 \mu |\theta|^2 - C_2 \mu |\sqrt{k}| |u| |\tilde{x} \cdot \theta| - |f| |\theta| \right\} dx \\ & \quad + \int_{B_{R,R+1}} \chi' \varphi_0 |\sqrt{k}|^{-1} \left\{ |\tilde{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right\} dx. \end{aligned}$$

Note the identity $\varphi'_0 = \mu_0 \varphi_0^2$. Then the Schwarz inequality and letting $t \rightarrow \infty$ show the desired assertion. $\square$

We need one more lemma, which is not obvious for exploding potentials.

Lemma 8 *For $\zeta \in \Gamma_\pm$ and $f \in L^2_{(\mu_0 \mathcal{H})^{-1}}(\mathbf{R}^n)$ let $u = R(\zeta)f$. Then u satisfies the radiation condition.*

Now, as is given in Eidus [8], the Theorem 2 is proved as follows.

Let $\{\zeta_j, f_j\} \subset \Gamma_\pm \times L^2_{\mu_1^{-1}}$ converges to $\{\zeta_0, f_0\}$ as $j \rightarrow \infty$. Since the other case is easier, we assume that $\zeta_0 = \lambda \pm i0$, $\lambda \in I$. Let $u_j = R(\zeta_j)f_k$.

(i) Each u_j satisfies the radiation conditions: *by Lemma 8.*

(ii) $\{u_k\}$ is pre-compact in $L^2_{\mu_0|\sqrt{k}|}$ if it is bounded in the same space, and every accumulation $u_0 \in L^2_{\mu_0|\sqrt{k}|}$ satisfies the radiation conditions: *by Rellich compactness criterion, Lemmas 6 (ii) and 7.*

(iii) The boundedness $\{u_j\}$ is proved by contradiction.

In fact, assume that there exists a subsequence, which we also write $\{u_j\}$, such that $\|u_j\|_{\mu_0|\sqrt{k}|} \rightarrow \infty$ as $j \rightarrow \infty$. Put $v_j = u_j / \|u_j\|_{\mu_2}$. Then as is explained above, $\{\zeta_j, v_j\}$ has a convergent subsequence, and if we denote the limit by $\{\lambda_0 \pm i0, v_0\}$, then it satisfies the eigenvalue problem (5) with $\lambda = \lambda_0$ and also

$$\|v_0\|_{\mu_0|\sqrt{k}|} = 1, \quad \|\partial_r v_0 + K_\pm v_0\|_{\varphi'_0|\sqrt{k}|^{-1}} < \infty, \quad (19)$$

where $K_{\pm} = K(x, \lambda_0 \pm i0)$. The second inequality implies

$$\liminf_{r \rightarrow \infty} \int_{S(r)} \sqrt{k}^{-1} |\partial_r v_0 + K_{\pm} v_0|^2 dS = 0$$

since $\varphi'_0(r) \notin L^1([R, \infty))$ for any $R > 0$ by Lemma 5. Comparing this with Theorem 1, we see that v_0 has a compact support in $x \in \mathbf{R}^n$. Hence, $v_0 \equiv 0$ by the unique continuation property for solutions to (5). But this contradicts to the first equation of (19).

(iv) If we apply Theorem 1 once more, then $\{u_j\}$ itself is shown to converge. $\square$

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