

The first homology of IA_n with coefficients in spaces of Jacobi diagrams

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1 Introduction

The IA-automorphism group IA_n of the free group F_n of rank n is a normal subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of F_n . If $n = 2$, then IA_2 is equal to the inner automorphism group $\mathrm{Inn}(F_2)$ of F_2 and if $n \geq 3$, then $\mathrm{Inn}(F_n)$ is a proper subgroup of IA_n . The group IA_n is defined in a way similar to the Torelli group of a closed surface or a surface with one boundary component. That is, IA_n is the kernel of the canonical map from $\mathrm{Aut}(F_n)$ to the general linear group $\mathrm{GL}(n, \mathbb{Z})$ induced by the abelianization map of F_n . Therefore, we have a short exact sequence of groups

$$1 \rightarrow \mathrm{IA}_n \rightarrow \mathrm{Aut}(F_n) \rightarrow \mathrm{GL}(n, \mathbb{Z}) \rightarrow 1.$$

It follows from this short exact sequence that the rational (co)homology of IA_n admits an action of $\mathrm{GL}(n, \mathbb{Z})$. Since IA_n and the Torelli groups are defined in a similar way, some strategies of studying the (co)homology of the Torelli groups can be used to study the (co)homology of IA_n and vice versa.

Cohen–Pakianathan [3], Farb (unpublished) and Kawazumi [11] independently determined the first homology of IA_n by using the Johnson homomorphism for $\mathrm{Aut}(F_n)$, and we have

$$H_1(\mathrm{IA}_n, \mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_{\mathbb{Z}}, \bigwedge^2 H_{\mathbb{Z}}), \quad H_{\mathbb{Z}} = H_1(F_n, \mathbb{Z}).$$

In higher degrees, the rational (co)homology of IA_n has been studied by many authors [14, 15, 2, 4, 16, 7] and we have a conjectural structure of the whole rational cohomology of IA_n for sufficiently large n [5].

For any (right) $\mathrm{Aut}(F_n)$ -module M , the homology of IA_n with coefficients in M also admits a $\mathrm{GL}(n, \mathbb{Z})$ -module structure. To the best of our knowledge, however, the homology of IA_n with any non-trivial coefficients has not been computed.

The *Jacobi diagrams* are uni-trivalent graphs encoding the algebraic structures of Lie algebras and their representations. The space of Jacobi diagrams appears as the target space of the *Kontsevich invariant*, which is a universal finite type invariant for links and unifies all quantum invariants of links [13, 1]. Habiro and Massuyeau [6] extended the Kontsevich invariant to construct a functor from the category of bottom tangles in handlebodies to the category of Jacobi diagrams in handlebodies. By using the restriction

of this functor to the degree 0 part, the author constructed a functor A_d from the opposite category $\mathbf{F}^{\text{op}}$ of the category $\mathbf{F}$ of finitely generated free groups to the category of filtered vector spaces [8, 10]. By restricting the functor A_d to the automorphism group $\text{Aut}_{\mathbf{F}^{\text{op}}}(n) \cong \text{Aut}(F_n)^{\text{op}}$, we obtain a (right) action of $\text{Aut}(F_n)$ on the space $A_d(n)$.

In this report, we will exhibit our recent computation of the first homology of IA_n with non-trivial coefficients in the space $A_2(n)$ of Jacobi diagrams of degree 2.

2 Preliminaries

2.1 The space $A_d(n)$ of Jacobi diagrams

A *Jacobi diagram* is a uni-trivalent graph such that each trivalent vertex is equipped with a cyclic order.

Let $n \geq 0$. A *Jacobi diagram* on n -component oriented arcs X_n is a Jacobi diagram such that each connected component has at least one univalent vertex and univalent vertices are embedded into the interior of X_n . Two Jacobi diagrams on X_n are regarded as the same if there is a homeomorphism between them whose restriction to the arc components is isotopic to the identity map. The *degree* of a Jacobi diagram is defined as half the number of its vertices.

Let $A_d(n)$ denote the vector space over $\mathbb{Q}$ spanned by Jacobi diagrams of degree d on X_n modulo the STU relation:

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} - \begin{array}{c} \text{---} \quad \text{---} \\ | \quad | \\ \text{---} \end{array},$$

where the dotted lines represent a part of a Jacobi diagram and the solid lines represent a part of oriented arcs and where the trivalent vertex has the counter-clockwise order.

The vector space $A_d(n)$ admits a descending filtration of finite length

$$A_d(n) = A_{d,0}(n) \supset A_{d,1}(n) \supset A_{d,2}(n) \supset \cdots \supset A_{d,2d-2}(n) \supset A_{d,2d-1}(n) = 0,$$

where $A_{d,k}(n)$ is the subspace of $A_d(n)$ spanned by Jacobi diagrams of degree d with at least k trivalent vertices.

2.2 The action of $\text{Aut}(F_n)$ on $A_d(n)$

The author introduced an action of $\text{Aut}(F_n)$ on $A_d(n)$ in [8] and studied the $\text{Aut}(F_n)$ -module structure of $A_d(n)$ in [8, 10]. The action is defined by using the composition of morphisms of the category $\mathbf{A}$ of Jacobi diagrams in handlebodies, which was introduced in [6]. The objects of $\mathbf{A}$ are non-negative integers and the hom-space from m to n is the vector space spanned by (m, n) -Jacobi diagrams, which are Jacobi diagrams on X_n mapped into a handlebody of genus m in such a way that the endpoints of the arcs are uniformly distributed on the bottom line and the i -th component goes from the $2i$ -th point to the $(2i-1)$ -st point, where we count the endpoints from left to right. An automorphism $f \in \text{Aut}(F_n)$ is identified with an (n, n) -Jacobi diagram and the action of f on a Jacobi

diagram $u \in A_d(n)$ is defined as the composition of u and the corresponding diagram f in the category **A**. For example, the automorphism

$$f : F_3 \rightarrow F_3, \quad x_1 \mapsto x_1 x_2, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3$$

is identified with

$$f = \begin{array}{c} \text{Diagram with three vertical strands. The first strand has a loop that crosses over the second strand. The second strand has a loop that crosses over the third strand. The third strand is straight. Arrows point downwards on all strands.} \end{array} \in \mathbf{A}(3, 3),$$

and the action of f on the Jacobi diagram $u = \begin{array}{c} \text{Diagram with three strands meeting at a central vertex. Each strand has a loop. Arrows point outwards from the vertex.} \end{array}$ is given by

$$u \cdot f = \begin{array}{c} \text{Diagram with three strands meeting at a central vertex. The first strand has a loop that crosses over the second strand. The second strand has a loop that crosses over the third strand. The third strand is straight. Arrows point outwards from the vertex.} \end{array} = \begin{array}{c} \text{Diagram with three strands meeting at a central vertex. Each strand has a loop. Arrows point outwards from the vertex.} \end{array} - \frac{1}{2} \begin{array}{c} \text{Diagram with three strands meeting at a central vertex. Each strand has a loop. Arrows point outwards from the vertex.} \end{array}.$$

2.3 The $\text{Aut}(F_n)$ -module structure of $A_2(n)$

Here we recall the result in [8] on the $\text{Aut}(F_n)$ -module structure of $A_2(n)$.

Let $A'_2(n)$ and $A''_2(n)$ be the $\text{Aut}(F_n)$ -submodules of $A_2(n)$ generated by the elements P' and P'' , respectively, where

$$P' = \begin{array}{c} \text{Diagram with four strands meeting at a central vertex. Each strand has a loop. Arrows point outwards from the vertex.} \end{array} \in A_2(4), \quad \text{sym}_4 = \sum_{\sigma \in \mathfrak{S}_4} \sigma,$$

$$P'' = \begin{array}{c} \text{Diagram with four strands meeting at a central vertex. Each strand has a loop. Arrows point outwards from the vertex.} \end{array} \in A_2(4), \quad \text{alt}_2 = \sum_{\tau \in \mathfrak{S}_2} \text{sgn}(\tau) \tau.$$

For a pair (λ, μ) of partitions of non-negative integers, let $V_{\lambda, \mu}$ denote the irreducible algebraic $\text{GL}(n, \mathbb{Z})$ -representation corresponding to (λ, μ) .

Theorem 2.1 ([8, Proposition 7.5 and Theorem 7.9]). *For $n \geq 3$, we have*

$$A_2(n) = A'_2(n) \oplus A''_2(n),$$

where $A'_2(n) \cong V_{0,4}$ is a simple $\text{Aut}(F_n)$ -module, and where $A''_2(n)$ is an indecomposable $\text{Aut}(F_n)$ -module with the unique composition series

$$A''_2(n) \supsetneq A_{2,1}(n) \supsetneq A_{2,2}(n) \supsetneq 0,$$

whose composition factors are

$$A''_2(n)/A_{2,1}(n) \cong V_{0,2^2}, \quad A_{2,1}(n)/A_{2,2}(n) \cong V_{0,1^3}, \quad A_{2,2}(n) \cong V_{0,2}.$$

3 Main theorem

Here we will observe the computation of $H_1(\text{IA}_n, A_2(n))$.

By Theorem 2.1, we have

$$H_1(\text{IA}_n, A_2(n)) \cong H_1(\text{IA}_n, A'_2(n)) \oplus H_1(\text{IA}_n, A''_2(n)).$$

Since the action of $\text{Aut}(F_n)$ on $A'_2(n)$ factors through $\text{GL}(n, \mathbb{Z})$, which means that IA_n acts trivially on $A'_2(n)$, we have

$$H_1(\text{IA}_n, A'_2(n)) \cong H_1(\text{IA}_n, \mathbb{Q}) \otimes A'_2(n).$$

We can compute the irreducible decomposition of the tensor product of two algebraic $\text{GL}(n, \mathbb{Z})$ -representations by using the formula in [12], which is given by combining the Littlewood–Richardson rule. Then we have

$$\begin{aligned} H_1(\text{IA}_n, \mathbb{Q}) \otimes A'_2(n) &\cong (V_{0,1} \otimes V_{1^2,0}) \otimes V_{0,4} \\ &\cong \begin{cases} V_{1^2,41} \oplus V_{1^2,5} \oplus V_{1,4}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,3} & n \geq 4 \\ V_{1^2,5} \oplus V_{1,4}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,3} & n = 3. \end{cases} \end{aligned}$$

We also determined the irreducible decomposition of the first homology of IA_n with coefficients in $A''_2(n)$.

Theorem 3.1 ([9, Theorem 1.1]). *We have*

$$H_1(\text{IA}_n, A''_2(n)) \cong \begin{cases} V_{1^2,2^21} \oplus V_{1^2,32} \oplus V_{1,21^2} \oplus V_{1,2^2}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,21}^{\oplus 2} & n \geq 5 \\ V_{1^2,32} \oplus V_{1,21^2} \oplus V_{1,2^2}^{\oplus 2} \oplus V_{1,31} \oplus V_{0,21}^{\oplus 2} & n = 4 \\ V_{1,2^2} \oplus V_{1,31} \oplus V_{0,21}^{\oplus 2} & n = 3. \end{cases}$$

Outline of the proof. Use the long exact sequences of homology associated to the short exact sequences

$$0 \rightarrow A_{2,2}(n) \rightarrow A_{2,1}(n) \rightarrow A_{2,1}(n)/A_{2,2}(n) \rightarrow 0$$

and

$$0 \rightarrow A_{2,1}(n) \rightarrow A''_2(n) \rightarrow A''_2(n)/A_{2,1}(n) \rightarrow 0$$

and compute the image of the boundary homomorphisms. The case of $n = 3$ needs more computation than the other cases. $\square$

We also computed the twisted first homology of IO_n , which is the analogue of IA_n to the outer automorphism group of F_n .

We have

$$H_1(\text{IO}_n, A_2(n)) \cong H_1(\text{IO}_n, A'_2(n)) \oplus H_1(\text{IO}_n, A''_2(n)),$$

where

$$H_1(\text{IO}_n, A'_2(n)) \cong \begin{cases} V_{1^2,41} \oplus V_{1^2,5} \oplus V_{1,4} \oplus V_{1,31} & n \geq 4 \\ V_{1^2,5} \oplus V_{1,4} \oplus V_{1,31} & n = 3. \end{cases}$$

Theorem 3.2 ([9, Theorem 10.4]). *We have*

$$H_1(\mathrm{IO}_n, A_2''(n)) \cong \begin{cases} V_{1^2, 2^2 1} \oplus V_{1^2, 3 2} \oplus V_{1, 2 1^2} \oplus V_{1, 2^2} \oplus V_{1, 3 1} \oplus V_{0, 2 1} & n \geq 5 \\ V_{1^2, 3 2} \oplus V_{1, 2 1^2} \oplus V_{1, 2^2} \oplus V_{1, 3 1} \oplus V_{0, 2 1} & n = 4 \\ V_{1, 3 1} \oplus V_{0, 2 1} & n = 3. \end{cases}$$

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