

On finite modified Nash V -determinacy of polynomial map-germs

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1 Introduction

We consider triviality of a family of zero-sets of real polynomial mappings. In [9], the author introduced the notion of modified Nash triviality as a natural and desirable one for it, and gave a modified Nash triviality theorem in the weighted homogeneous polynomial case. Following [9], T. Fukui, S. Koike and M. Shiota gave some fundamental results to construct local theory on modified Nash triviality. See [10] for the survey of this field. In this note, we discuss the problem of finite modified Nash V -determinacy. In particular, we give a characterization of the finite determinacy (Theorem I) and a sufficient condition for an r -jet $w \in J^r(2, 1)$ to be modified Nash V -sufficient (Theorem II).

2 Results

We first recall the definitions of a Nash manifold and a Nash mapping. A semialgebraic set of $\mathbf{R}^n$ is called a *Nash manifold* if it is a C^ω submanifold of $\mathbf{R}^n$. Let $M \subset \mathbf{R}^m$ and $N \subset \mathbf{R}^n$ be Nash manifolds. A C^ω mapping $f : M \rightarrow N$ is called a *Nash mapping* if the graph of f is semialgebraic in $\mathbf{R}^m \times \mathbf{R}^n$. See M. Shiota [16] for properties of Nash manifolds.

Let $M \subset \mathbf{R}^m$ be a Nash manifold possibly with boundary, and let $N_1, \dots, N_q$ be Nash submanifolds of M possibly with boundary, which together with $N_0 = \partial M$ are normal crossing. Assume that $\partial N_i \subset N_0$, $i = 1, \dots, q$. Then the following Nash Isotopy Lemma is given in [6]:

Theorem (2.1) ([6] Theorem I). *Let $\varpi : M \rightarrow \mathbf{R}^k$, $k > 0$, be a proper onto Nash submersion such that for every $0 \leq i_1 < \dots < i_s \leq q$, $\varpi|_{N_{i_1} \cap \dots \cap N_{i_s}} : N_{i_1} \cap \dots \cap N_{i_s} \rightarrow \mathbf{R}^k$ is a proper onto submersion. Then there exists a Nash diffeomorphism*

$$\phi = (\phi^*, \varpi) : (M; N_1, \dots, N_q) \rightarrow (M^*; N_1^*, \dots, N_q^*) \times \mathbf{R}^k$$

such that $\phi|_{M^} = \text{id}$, where Z^* denotes $Z \cap \varpi^{-1}(0)$ for a subset Z of M .*

Furthermore, if previously given are Nash diffeomorphisms $\phi_{i_j} : N_{i_j} \rightarrow N_{i_j}^ \times \mathbf{R}^k$, $0 \leq i_1 < \dots < i_a \leq q$, such that $\varpi \circ \phi_{i_j}^{-1}$ is the natural projection, and $\phi_{i_s} = \phi_{i_t}$ on $N_{i_s} \cap N_{i_t}$, then we can choose the above Nash diffeomorphism ϕ which satisfies $\phi|_{N_{i_j}} = \phi_{i_j}$, $j = 1, \dots, a$.*

Remark (2.2). In Theorem (2.1), we can replace $\mathbf{R}^k$ by one of the followings:

- (1) an open cuboid $\prod_{i=1}^k (a_i, b_i)$,
- (2) a closed cuboid $\prod_{i=1}^k [a_i, b_i]$,
- (3) a Nash manifold which is Nash diffeomorphic to an open simplex.

The above theorem is an effective tool to show modified Nash triviality for a family of zero-sets of polynomial (or Nash) mappings with isolated singularities. Therefore this is also useful to show finite modified Nash V -determinacy.

Let $\mathcal{N}(n, p)$ denote the set of Nash map-germs $: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, and let $\mathcal{A}(n, p)$ denote the set of analytic map-germs $: (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$.

Definition (2.3). We say that two map-germs $f_1, f_2 \in \mathcal{N}(n, p)$ are *modified Nash V -equivalent* (resp. *modified Nash equivalent*), if there are two Nash modifications $\pi_1 : M_1 \rightarrow \mathbf{R}^n$, $\pi_2 : M_2 \rightarrow \mathbf{R}^n$, and a Nash diffeomorphism $\Phi : (W_1, \pi_1^{-1}(0)) \rightarrow (W_2, \pi_2^{-1}(0))$ which induces a homeomorphism $\phi : (U_1, 0) \rightarrow (U_2, 0)$ such that $\phi((U_1, f_1^{-1}(0) \cap U_1)) = (U_2, f_2^{-1}(0) \cap U_2)$ (resp. $f_1 = f_2 \circ \phi$), where W_1, W_2 are semialgebraic neighborhoods of $\pi_1^{-1}(0)$ in M_1 and $\pi_2^{-1}(0)$ in M_2 , respectively, and U_1, U_2 are neighborhoods of 0 in $\mathbf{R}^n$.

Remark (2.4). Similarly, we can define the notions of modified C^k V -equivalence and modified C^k equivalence for elements of $\mathcal{A}(n, p)$.

Definition (2.5). We say that $f \in \mathcal{N}(n, p)$ is *finitely modified Nash V -determined*, if there is a positive integer r such that any $g \in \mathcal{N}(n, p)$ with $j^r g(0) = j^r f(0)$ is modified Nash V -equivalent to f .

We have a criterion for finite modified Nash V -determinacy of polynomial map-germs. We don't give the proof here.

Theorem I. For a polynomial map-germ $f : (\mathbf{R}^n, 0) \rightarrow (\mathbf{R}^p, 0)$, the following conditions are equivalent.

- (1) f is finitely modified Nash V -determined.
- (2) $f^{-1}(0) \cap S(f) \subset \{0\}$ as germs at $0 \in \mathbf{R}^n$.

Remark (2.6). Let $f \in \mathcal{A}(n, p)$. Originally, J. Bochnak and T. C. Kuo ([1]) proved that condition (2) in Theorem I is equivalent to finite V -determinacy of f . This was improved by M. Buchner and W. Kucharz ([4]). Precisely, they proved that condition (2) is equivalent to finite modified C^k V -determinacy of f where k is a positive integer.

Next we consider the problem of sufficiency of jets. We don't distinguish r -jets from their polynomial representatives of degree not exceeding r . Recently, T. Fukui has introduced some invariant for modified C^ω equivalence in the function case ([5]). Let N_∞ denote the set of positive integers and infinity, and let $\Lambda(\mathbf{R}^n, 0)$ denote the set of analytic arcs at $0 \in \mathbf{R}^n$, namely, the set of analytic maps $\lambda : [0, \epsilon) \rightarrow \mathbf{R}^n$ with $\lambda(0) = 0$, $\lambda(s) \neq 0$, $s > 0$.

Fukui's Invariant. For $f \in \mathcal{A}(n, 1)$, we define the following subset of N_∞ :

$$A(f) = \{O(f \circ \lambda) \mid \lambda \in \Lambda(\mathbf{R}^n, 0)\},$$

where $O(\phi)$ denotes the order of ϕ for $\phi \in \mathcal{A}(1, 1)$.

(2.7) If $f, g \in \mathcal{A}(n, 1)$ are modified C^ω equivalent, then $A(f) = A(g)$.

Example (2.8). Let $w = x^3 + 3xy^6 \in J^7(2, 1)$. By the Kuiper-Kuo theorem, w is C^0 -sufficient (see Theorem (2.11) and Remark (2.13) below). Let $f(x, y) = x^3 + 3xy^6 + y^8$. It is easy to see that $8 \notin A(w)$. On the other hand, $8 \in A(f)$. By (2.7), f is not modified C^ω -equivalent to w . Therefore w is not modified C^ω -sufficient.

Remark that in the above example, w satisfies the Kuiper-Kuo condition as a real 7-jet, but w does not do as a complex 7-jet. Let v be a complex r -jet, and let $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function such that $j^r h(0) = v$. Define $F : (\mathbb{C}^n \times \Omega, \{0\} \times \Omega) \rightarrow (\mathbb{C}, 0)$ by $F(z; t) = (1 - t)v(z) + th(z)$, where Ω is an open ball in $\mathbb{C}$ containing the interval $[0, 1]$. For $t \in \Omega$, let $f_t : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ denote the function defined by $f_t(z) = F(z; t)$, and set $V_t = f_t^{-1}(0)$. Suppose that v satisfies the Kuiper-Kuo condition. Then $F^{-1}(0)$ satisfies the Kuo's Ratio Test ([13]) along $\{0\} \times \Omega$. This condition is equivalent to the condition that $\mu^*(V_t)$ is constant (see J. Briançon-J. P. Speder [3], J. P. Henry-M. Merle [7], B. Teissier [17]). Therefore it follows from the main result in H. B. Laufer [15] that in the case of surface singularities, $F^{-1}(0)$ admits a *strong simultaneous resolution* over Ω in the sense of B. Teissier [18]. Then we have the following question:

Question (2.9). Let $w \in J^r(n, 1)$ be a real r -jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then, is w modified C^ω V -sufficient (or modified C^ω -sufficient)?

Let $w \in J^r(n, 1)$, and let $f \in \mathcal{A}(n, 1)$ (or $\mathcal{N}(n, 1)$) with $j^r f(0) = w$. After this, let $f_t : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ denote the function defined by $f_t(x) = (1 - t)w(x) + tf(x)$ for $t \in I = [0, 1]$. Concerning the above question, T. C. Kuo has proved

Theorem (2.10) ([14]). Let $w \in J^r(2, 1)$ be a real r -jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then w is modified C^ω sufficient in $A(2, 1)$. Furthermore, $\{f_t\}_{0 \leq t \leq 1}$ is modified C^ω trivial along I .

The author and T. C. Kuo have proved the following fact on Łojasiewicz inequalities.

Theorem (2.11). Let r be a positive integer. For $f \in \mathcal{A}(n, 1)$, the following conditions are equivalent.

(1) (The Kuiper-Kuo condition.) There are $c, \alpha > 0$ such that

$$|\text{grad } f(x)| \geq c|x|^{r-1} \text{ for } |x| < \alpha.$$

(2) (The Thom condition.) There are $K, A > 0$ such that

$$\sum_{i < j} \left| x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right|^2 + |f(x)|^2 \geq K|x|^{2r} \text{ for } |x| < A.$$

Remark (2.12). It is easy to see that Theorem (2.11) holds for a C^r function f .

Remark (2.13). Suppose that $w \in J^r(n, 1)$ satisfies the Kuiper-Kuo condition. Then w is C^0 -sufficient in C^r functions. This is well-known as the Kuiper-Kuo theorem ([11], [12]). The converse is also true (J. Bochnak and S. Łojasiewicz [2]). At the almost same time as Kuiper and Kuo, R. Thom showed if $w \in J^r(n, 1)$ satisfies the Thom condition, then w is C^0 -sufficient. The Kuiper-Kuo condition implies no coalescing of critical points of

$\{f_t\}_{0 \leq t \leq 1}$ in the sense of King ([8]) for any realization of w . On the other hand, the Thom condition implies that the Milnor radii of $\{f_t^{-1}(0)\}_{0 \leq t \leq 1}$ are uniformly positive. Therefore it seems that the Thom condition is stronger than the Kuiper-Kuo condition on the surface. But it follows from the above fact that Thom's result is equivalent to the Kuiper-Kuo theorem.

By using Theorems (2.1), (2.10) and (2.11), we can show

Theorem II. *Let $w \in J^r(2, 1)$ be a real r -jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then w is modified Nash V -sufficient in $\mathcal{N}(2, 1)$.*

Question (2.14). *Does Theorem II hold for general variables case?*

In the same way as Theorem II, we can reduce the above question to the following

Question (2.15). *Let $w \in J^r(n, 1)$ be a real r -jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then is w modified C^1 V -sufficient in $\mathcal{N}(n, 1)$? Furthermore, is $\{f_t^{-1}(0)\}_{0 \leq t \leq 1}$ modified C^1 V -trivial along I as set-germs, not as embedded varieties?*

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