

Necessary condition for existence of conditional SIC-POVM

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1 Introduction

This paper is an announcement of our result and the detailed version will be submitted to somewhere. (See [2].)

POVMs (positive operator valued measure) on quantum systems are considered as a measurement in quantum physics. If POVMs have a good condition, then we can determine a state by results of a measurement. In this case, the POVM is called informationally complete and the process is called quantum state tomography.

First, we introduce SIC-POVMs (symmetric informationally complete POVM). SIC-POVMs is generated by vectors in $\mathbb{C}^n$ whose the absolute values of inner products of each vectors are same. Zauner conjectured that there exist such n^2 vectors in $\mathbb{C}^n$ for any n . But the existence is only proved when $n \leq 15$ and $n = 19, 24, 35, 48$ [5, 8].

Next, we introduce conditional SIC-POVMs. A state of a quantum system is a density matrix which has several parameters. When a few parameters are known, then SIC-POVM is not the best measurement to determine the state. Hence we need another POVM and it is a conditional SIC-POVM. Conditional SIC-POVMs are also generated by vectors in $\mathbb{C}^n$. But the existence of conditional SIC-POVMs depends on the system. We will discuss the details in Sect. 4.

2 Preliminaries

Definition 2.1 $\rho \in M_n(\mathbb{C})$ is called a density matrix (or state) if $\rho \geq 0$ and

$$\text{Tr}(\rho) = 1.$$

For any density matrix ρ , we can define a state $\hat{\rho}$ by

$$\hat{\rho}(X) = \text{Tr}(\rho X).$$

Conversely, any state is written by the above form. Therefore, there exists a one-to-one correspondence between density matrices and states.

Definition 2.2 A set of positive operators $\{P_i\}_{i=1}^k \subset M_n(\mathbb{C})$ is called a positive operator valued measure (POVM) if $P_i \geq 0$ ($1 \leq i \leq k$) and

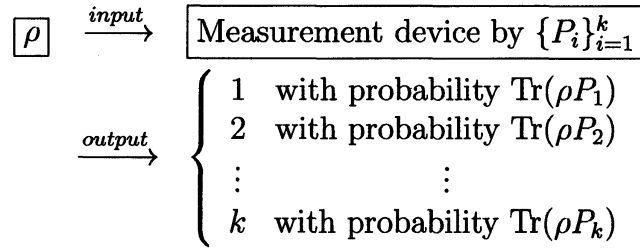
$$\sum_{i=1}^k P_i = I.$$

If P_i is a projection, then $\{P_i\}_{i=1}^k$ is called a projection valued measure (PVM).

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For a POVM $\{P_i\}_{i=1}^k$, we can consider a measurement device by using this POVM:



This means that we can know $\text{Tr}(\rho P_i)$ for all $(1 \leq i \leq k)$, if we have many copies of ρ . Therefore, if a POVM $\{P_i\}_{i=1}^k$ has a good condition, then we can determine the quantum state ρ by $\text{Tr}(\rho P_i)$. This is called **quantum state tomography**.

Definition 2.3 A POVM $\{P_i\}_{i=1}^k$ is called *informationally complete* if for density matrices $\rho \neq \sigma$, there exists P_i such that

$$\text{Tr}(\rho P_i) \neq \text{Tr}(\sigma P_i).$$

If a POVM $\{P_i\}_{i=1}^k$ is informationally complete, then we can determine the quantum state ρ by $\text{Tr}(\rho P_i)$. If ρ is a state in $M_n(\mathbb{C})$, then the following statement holds.

Theorem 2.4 A POVM $\{P_i\}_{i=1}^k \subset M_n(\mathbb{C})$ is informationally complete if and only if

$$\text{span}\{P_i\}_{i=1}^k = M_n(\mathbb{C}).$$

In particular, we can determine the quantum state ρ by $\text{Tr}(\rho P_i)$.

Example 2.5 For small $\varepsilon > 0$. Let

$$\begin{aligned}
 P_1 &= \frac{1}{2+2\varepsilon^2} \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon^2 \end{bmatrix}, P_2 = \frac{1}{2+2\varepsilon^2} \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon^2 \end{bmatrix}, \\
 P_3 &= \frac{1}{2+2\varepsilon^2} \begin{bmatrix} \varepsilon^2 & \varepsilon i \\ -\varepsilon i & 1 \end{bmatrix}, P_4 = \frac{1}{2+2\varepsilon^2} \begin{bmatrix} \varepsilon^2 & -i\varepsilon \\ i\varepsilon & 1 \end{bmatrix}.
 \end{aligned}$$

Then $\{P_i\}_{i=1}^4$ is an informationally complete POVM. For a state ρ in $M_n(\mathbb{C})$, by equations

$$\begin{aligned}
 (2+2\varepsilon^2)\text{Tr}(\rho P_1) &= \rho_{11} + \varepsilon\rho_{12} + \varepsilon\rho_{21} + \varepsilon^2\rho_{22} \\
 (2+2\varepsilon^2)\text{Tr}(\rho P_2) &= \rho_{11} - \varepsilon\rho_{12} - \varepsilon\rho_{21} + \varepsilon^2\rho_{22} \\
 (2+2\varepsilon^2)\text{Tr}(\rho P_3) &= \varepsilon^2\rho_{11} - i\varepsilon\rho_{12} + i\varepsilon\rho_{21} + \rho_{22} \\
 (2+2\varepsilon^2)\text{Tr}(\rho P_4) &= \varepsilon^2\rho_{11} + i\varepsilon\rho_{12} - i\varepsilon\rho_{21} + \rho_{22},
 \end{aligned}$$

we have

$$\begin{aligned} \rho = & \text{Tr}(\rho P_1) \begin{bmatrix} \frac{1}{1-\varepsilon^2} & \frac{1}{4\varepsilon(1+\varepsilon^2)} \\ \frac{1}{4\varepsilon(1+\varepsilon^2)} & \frac{-\varepsilon^2}{1-\varepsilon^2} \end{bmatrix} + \text{Tr}(\rho P_2) \begin{bmatrix} \frac{1}{1-\varepsilon^2} & \frac{-1}{4\varepsilon(1+\varepsilon^2)} \\ \frac{-1}{4\varepsilon(1+\varepsilon^2)} & \frac{-\varepsilon^2}{1-\varepsilon^2} \end{bmatrix} \\ & + \text{Tr}(\rho P_3) \begin{bmatrix} \frac{-\varepsilon^2}{1-\varepsilon^2} & \frac{i}{4\varepsilon(1+\varepsilon^2)} \\ \frac{-i}{4\varepsilon(1+\varepsilon^2)} & \frac{1}{1-\varepsilon^2} \end{bmatrix} + \text{Tr}(\rho P_4) \begin{bmatrix} \frac{-\varepsilon^2}{1-\varepsilon^2} & \frac{-i}{4\varepsilon(1+\varepsilon^2)} \\ \frac{i}{4\varepsilon(1+\varepsilon^2)} & \frac{1}{1-\varepsilon^2} \end{bmatrix}. \end{aligned}$$

Hence we can determine a state ρ . But this is not a good POVM to detect ρ . Since

$$\rho_{12} = \frac{1}{4\varepsilon(1+\varepsilon^2)} (\text{Tr}(P_1\rho) - \text{Tr}(P_2\rho) - i(\text{Tr}(P_3\rho) - \text{Tr}(P_4\rho))),$$

a small error causes a big difference.

If a POVM is informationally complete, then we can determine a state ρ by $\{\text{Tr}(\rho P_i)\}_{i=1}^k$. But by ℓ experiments, we can only obtain approximate values of $\{\text{Tr}(\rho P_i)\}_{i=1}^k$. Let the candidate generated by these approximate values be $\hat{\rho}$. A POVM is called optimal, if the expected value of

$$\|\rho - \hat{\rho}\|_2$$

is the minimum among all candidates generated by any POVM and ℓ experiments. If a POVM is optimal, then it satisfies the following condition.

Theorem 2.6 [4] *A POVM in $M_n(\mathbb{C})$ with rank one positive operators $\{\frac{n}{k}P_i\}_{i=1}^k$ are optimal POVM if and only if*

$$\sum_{i=1}^k \frac{n}{k} |P_i\rangle\langle P_i| = \frac{1}{n+1} (\text{id}_{M_n(\mathbb{C})} + |I\rangle\langle I|),$$

where $|P_i\rangle\langle P_i|$ is a superoperator $M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ with $A \mapsto \text{Tr}(AP_i)P_i$.

3 SIC-POVM

In this section, we introduce a SIC-POVM (symmetric informationally complete positive operator valued measure) which is an optimal POVM.

Definition 3.1 *A set of vectors $\{\xi_i\}_{i=1}^{n^2} \subset \mathbb{C}^n$ is called symmetric informationally complete POVM (SIC-POVM) if*

$$|\langle \xi_i, \xi_j \rangle| = \frac{1}{\sqrt{n+1}}.$$

A POVM generated by the above vectors

$$\left\{ \frac{1}{n} |\xi_i\rangle\langle\xi_i| \right\}_{i=1}^{n^2}$$

is also called a SIC-POVM, where $|x\rangle\langle y|z = \langle y, z\rangle x$ for all $x, y, z \in \mathbb{C}^n$.

A SIC-POVM is informationally complete. Indeed, if we assume

$$\sum_{i=1}^{n^2} a_i |\xi_i\rangle\langle\xi_i| = 0,$$

then for all $1 \leq j \leq n^2$ we have

$$0 = \text{Tr} \left(\sum_{i=1}^{n^2} a_i |\xi_i\rangle\langle\xi_i| \cdot |\xi_j\rangle\langle\xi_j| \right) = a_j + \frac{1}{n+1} \sum_{i \neq j} a_i.$$

So it is easy to see that $\{|\xi_i\rangle\langle\xi_i|\}_{i=1}^{n^2}$ is linearly independent. Moreover, for all $1 \leq j \leq n^2$,

$$\text{Tr} \left(\sum_{i=1}^{n^2} \frac{1}{n} |\xi_i\rangle\langle\xi_i| \cdot |\xi_j\rangle\langle\xi_j| \right) = \frac{1}{n} + \sum_{i \neq j} \frac{1}{n(n+1)} = 1.$$

Hence $\sum_{i=1}^{n^2} \frac{1}{n} |\xi_i\rangle\langle\xi_i| = I$.

Example 3.2 In $\mathbb{C}^2$,

$$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \xi_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \xi_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \lambda\sqrt{2} \end{bmatrix}, \xi_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \bar{\lambda}\sqrt{2} \end{bmatrix}$$

is a SIC-POVM, where $\lambda = e^{2\pi i/3} = \frac{-1 + \sqrt{3}i}{2}$.

In $\mathbb{C}^3$,

$$\begin{aligned} \xi_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \xi_2 = \frac{1}{2} \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix}, \xi_3 = \frac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{3} \\ 0 \end{bmatrix}, \\ \xi_4 &= \frac{1}{2} \begin{bmatrix} 1 \\ i \\ \sqrt{2} \end{bmatrix}, \xi_5 = \frac{1}{2} \begin{bmatrix} 1 \\ i \\ \sqrt{2}\lambda \end{bmatrix}, \xi_6 = \frac{1}{2} \begin{bmatrix} 1 \\ i \\ \sqrt{2}\bar{\lambda} \end{bmatrix}, \\ \xi_7 &= \frac{1}{2} \begin{bmatrix} 1 \\ -i \\ \sqrt{2} \end{bmatrix}, \xi_8 = \frac{1}{2} \begin{bmatrix} 1 \\ -i \\ \sqrt{2}\lambda \end{bmatrix}, \xi_9 = \frac{1}{2} \begin{bmatrix} 1 \\ -i \\ \sqrt{2}\bar{\lambda} \end{bmatrix}, \end{aligned}$$

is a SIC-POVM, where $\lambda = e^{2\pi i/3}$.

It is known that a SIC-POVM exists if $n \leq 15$ or $n = 19, 24, 35, 48$. Numerical solutions have been found when $n \leq 67$. But for other cases, the existence is an open problem. The following is conjectured by G. Zauner in 1999 [7].

Let

$$W = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda^{n-1} \end{bmatrix}, S = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix},$$

where $\lambda = \exp \frac{2\pi i}{n}$. $\{W^k S^\ell\}_{k,\ell=1}$ are called generalized Pauli matrices.

Definition 3.3 A unit vector ξ is called a fiducial vector if

$$\{W^k S^\ell \xi\}_{k,\ell=1}^n$$

is a SIC-POVM.

Conjecture (Zauner's conjecture [7]) A fiducial vector exists in $\mathbb{C}^n$ for all $n \geq 2$. In particular, a SIC-POVM exists in $\mathbb{C}^n$.

4 Conditional SIC-POVM

Recently, SIC-POVMs in arbitrary subspace of $M_n(\mathbb{C})$ are also considered. Let $M_n(\mathbb{C})$ be decomposed as

$$M_n(\mathbb{C}) = \mathbb{C}I \oplus \mathcal{A} \oplus \mathcal{B}$$

and $\rho \in M_n(\mathbb{C})$ be a density matrix. Let $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ be projections onto $\mathcal{A}$ and $\mathcal{B}$. Assume we know $P_{\mathcal{A}}\rho$. Then we want to know an optimal POVM which determines ρ . Since we already know $P_{\mathcal{A}}\rho$, SIC-POVM is not suitable.

Let $\dim \mathcal{A} = m$ then $\dim \mathcal{B} = n^2 - m - 1$ and let $N = n^2 - m$. For rank one informationally complete POVM $\{\frac{n}{k}P_i\}_{i=1}^k$, let

$$\mathcal{F} = \frac{n}{k}|P_i\rangle\langle P_i|.$$

Then the following theorem holds.

Theorem 4.1 [3] Rank one informationally complete POVM $\{\frac{n}{N}P_i\}_{i=1}^N$ is optimal if and only if

$$\mathcal{F} = |I\rangle\langle I| + \frac{n-1}{N-1}P_{\mathcal{B}}.$$

In this case,

$$\sum_{i=1}^k P_i = \frac{N}{n}I, \quad \text{Tr}(P_i P_j) = \frac{N-n}{n(N-1)}.$$

Such POVM is called a conditional SIC-POVM. Examples of conditional SIC-POVMs are following.

Example 4.2 If we do not have any information a priory about the state ($m = 0, N = n^2$), then

$$\text{Tr} P_i P_j = \frac{1}{n+1} \quad (i \neq j)$$

so the optimal POVM is the well-known SIC-POVM (if it exists).

Example 4.3 If we know the off-diagonal elements of the state, and we want to estimate the diagonal entries ($m = n^2 - n, N = n$), then from Theorem 4.1 it follows that the optimal POVM has the properties

$$\text{Tr} P_i P_j = 0 \quad (i \neq j), \quad \sum_{i=1}^n P_i = I, \quad \text{and} \quad P_i \text{ is diagonal.}$$

So the diagonal matrix units form an optimal POVM.

Example 4.4 If we know the diagonal elements of the state, and we want to estimate the off-diagonal entries ($m = n - 1, N = n^2 - n + 1$), then from Theorem 4.1 it follows that the optimal POVM has the properties

$$\text{Tr} P_i P_j = \frac{n-1}{n^2} \quad (i \neq j), \quad \sum_{i=1}^n P_i = \frac{n^2 - n + 1}{n} I$$

and P_i has a constant diagonal.

The existence is not clear generally, but if $n - 1$ is a prime power then it can be constructed. Details are written in [3].

Next, we present a necessary condition for existence of a conditional SIC-POVM.

Lemma 4.5 Let $\{P_i\}_{i=1}^N$ be a conditional SIC-POVM in $A \oplus C$ and let

$$Q_i = \sqrt{\frac{n(N-1)}{N(n-1)}} \left(P_i - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) I \right). \quad (1)$$

Then $\{Q_i\}_{i=1}^N$ is an orthonormal basis of $A \oplus C$.

Proof. For any $1 \leq i \leq N$, we have

$$\begin{aligned}
& \text{Tr} \left(\left(P_i - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) I \right)^2 \right) \\
&= \text{Tr} \left(P_i - \frac{2}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) P_i + \frac{1}{n^2} \left(1 + \sqrt{\frac{n-1}{N-1}} \right)^2 I \right) \\
&= 1 - \frac{2}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) + \frac{1}{n} \left(1 + 2\sqrt{\frac{n-1}{N-1}} + \frac{n-1}{N-1} \right) \\
&= 1 - \frac{1}{n} + \frac{n-1}{n(N-1)} \\
&= \frac{N(n-1)}{n(N-1)}.
\end{aligned}$$

Moreover, for any $1 \leq i < j \leq N$,

$$\begin{aligned}
& \text{Tr} \left(\left(P_i - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) I \right) \left(P_j - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) I \right) \right) \\
&= \frac{N-n}{n(N-1)} - \frac{2}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) + \frac{1}{n} \left(1 + 2\sqrt{\frac{n-1}{N-1}} + \frac{n-1}{N-1} \right) \\
&= \frac{N-n}{n(N-1)} - \frac{1}{n} + \frac{n-1}{n(N-1)} = 0.
\end{aligned}$$

These equations imply $\langle Q_i, Q_j \rangle = \text{Tr}(Q_i^* Q_j) = \delta_{ij}$ so that $\{Q_i\}_{i=1}^N$ is an orthonormal basis of $A \oplus C$. $\square$

Theorem 4.6 *If there exists a conditional SIC-POVM in $A \oplus C$, then for any $X \in B$ and any orthonormal basis $\{R_i\}_{i=1}^m$ of B ,*

$$\sum_{i=1}^m R_i^* X R_i = \frac{N-n}{n(n-1)} X.$$

Proof. Let $\{P_i\}_{i=1}^N$ be a conditional SIC-POVM in $A \oplus C$ and define $\{Q_i\}_{i=1}^N$ by (1). Then from the previous lemma, $\{Q_1, \dots, Q_N, R_1, \dots, R_m\}$ is an orthonormal basis of $M_n(\mathbb{C})$. It is well known that

$$\sum_{i=1}^N Q_i^* X Q_i + \sum_{i=1}^m R_i^* X R_i = \text{Tr}(X).$$

B is orthogonal to $A = \mathbb{C}I$ so that $\text{Tr}(X) = 0$. Hence we will calculate $\sum_{i=1}^N Q_i^* X Q_i$. Since P_i is a rank one projection, $P_i X P_i = t P_i$ for some $t \in \mathbb{C}$. But $\text{Tr}(P_i X P_i) =$

$\langle P_i, X \rangle = 0$ implies $t = 0$. Therefore $P_i X P_i = 0$. From the equation

$$\sum_{i=1}^N P_i = \frac{N}{n} I,$$

we have

$$\begin{aligned} & \frac{N(n-1)}{n(N-1)} \sum_{i=1}^N Q_i^* X Q_i \\ &= \sum_{i=1}^N \left(P_i - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) \right) X \left(P_i - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) \right) \\ &= \sum_{i=1}^N \left(P_i X P_i - \frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) (X P_i + P_i X) + \frac{1}{n^2} \left(1 + \sqrt{\frac{n-1}{N-1}} \right)^2 X \right) \\ &= \left(-\frac{1}{n} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) (X \sum_{i=1}^N P_i + \sum_{i=1}^N P_i X) + \frac{N}{n^2} \left(1 + \sqrt{\frac{n-1}{N-1}} \right)^2 X \right) \\ &= \left(-\frac{2N}{n^2} \left(1 + \sqrt{\frac{n-1}{N-1}} \right) + \frac{N}{n^2} \left(1 + 2\sqrt{\frac{n-1}{N-1}} + \frac{n-1}{N-1} \right) \right) X \\ &= \frac{N}{n^2} \left(-1 + \frac{n-1}{N-1} \right) X = \frac{N(n-N)}{n^2(N-1)} X. \end{aligned}$$

This implies the assertion. □

Example 4.7 Now we consider $M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. A density matrix

$$\rho = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

has reduced densities:

$$\rho_1 = \begin{bmatrix} a_{11} + a_{22} & a_{13} + a_{24} \\ a_{31} + a_{42} & a_{33} + a_{44} \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} a_{11} + a_{33} & a_{12} + a_{34} \\ a_{21} + a_{43} & a_{22} + a_{44} \end{bmatrix}.$$

The condition $\rho_1 = \rho_2$ implies

$$a_{22} = a_{33} \quad \text{and} \quad a_{13} + a_{24} = a_{12} + a_{34}.$$

Let

$$R_1 = \frac{1}{\sqrt{2}}(e_{22} - e_{33}), \quad R_2 = \frac{1}{2}(e_{12} - e_{13} - e_{24} + e_{34}), \quad R_3 = \frac{1}{2}(e_{21} - e_{31} - e_{42} + e_{43}),$$

and $B = \text{span}\{R_1, R_2, R_3\}$, then

$$\rho = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12}^* & b & a_{23} & c - a_{13} \\ a_{13}^* & a_{23}^* & b & c - a_{12} \\ a_{14}^* & c^* - a_{13}^* & c^* - a_{12}^* & a_{44} \end{bmatrix}$$

is orthogonal to B . But a conditional SIC-POVM for this ρ does not exist. Indeed, the equations

$$R_1^* R_1 R_1 = \frac{1}{2} R_1, \quad R_2^* R_1 R_2 = 0, \quad R_3^* R_1 R_3 = 0$$

imply $\sum_{i=1}^3 R_i^* R_1 R_i = \frac{1}{2} R_1$ and this is in contradict to the condition in Theorem 4.6.

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