

On a mixed Hodge structure of an isolated singularity

京大 数理研 藤本 明

§0. Introduction.

0.1. In this note, a normal isolated singularity $\mathbb{X} = (X, p)$ is by definition an equivalence class of a germ of an analytic space that X determines at p , where X is a normal analytic space smooth outside a point p .

Let $\mathbb{X} = (X, p)$ be a normal isolated singularity. Let $K = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$. We define a K -module H_K^* by the formula

$$H_K^* = \varinjlim H^*(V-p, K),$$

where V runs through neighborhoods of p in X . This of course depends only on $\mathbb{X}$. We call H_K^* the cohomology group of $\mathbb{X}$ with coefficients in K .

Note that H_K^* could also be described as follows; assume that X is realized as an analytic subspace of a domain D of some $\mathbb{C}^N$. Let $K = X \cap S$ be the intersection of X

with a sufficiently small sphere S around p in $\mathbb{C}^N$. Then $H^* = H^*(K, K)$.

0.2 The concept of a mixed Hodge structure is introduced in [1].

Definition 0.2.1. A $\mathbb{Z}$ -module H is said to have a mixed Hodge structure, if the following conditions are satisfied; there exist i) a finite increasing filtration W on $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$, and ii) a finite decreasing filtration F on $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$ which satisfies the next properties; let $\bar{F}$ be the filtration on $H_{\mathbb{C}}$ conjugate to F , W_n denote also the ^{induced} filtration on $H_{\mathbb{C}}$, $H_n = \text{Gr}_{W_n}^{\mathbb{C}}(H_{\mathbb{C}}) = W_n(H_{\mathbb{C}})/W_{n-1}(H_{\mathbb{C}})$, and F_n (resp. $\bar{F}_n$) the filtration induced on H_n by F (resp. $\bar{F}$). Then we have a direct sum decomposition

$$H_n = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = F_n^p(H_n) \cap \bar{F}_n^q(H_n).$$

0.3 Then our result is.

Theorem. Let $X = (X, p)$ be a normal isolated singularity, and $H_{\mathbb{Z}}^*$ be a cohomology group of X with coefficients in $\mathbb{Z}$. Then $H_{\mathbb{Z}}^*$ has a natural mixed Hodge structure.

Complex

§ 1. cohomology at infinity

1.1. We denote by (X, A) a pair consisting of an (equidimensional)

complex manifold X and a divisor A with only normal crossings in X . Let (X_i, A_i) , $i=1,2$, be such pairs. Then a morphism $f: (X_1, A_1) \rightarrow (X_2, A_2)$ of (X_1, A_1) to (X_2, A_2) is a morphism $f: X_1 \rightarrow X_2$ with $f^{-1}(A_2) \subseteq A_1$.

Let $\mathcal{C}$ be the category whose objects are pairs (X, A) and whose morphisms are morphisms of pairs.

1.2. Hereafter in §1 and §2, we choose and fix a $(X, A) \in \mathcal{C}$. Then for any point $p \in A$, there exists a neighborhood $U \ni p$ in X with a local coordinate system $(z_1, \dots, z_n)$, $n = \dim X$, such that A is defined in U by the equation

$$(1) \quad z_1 \cdots z_r = 0,$$

for some r , $1 \leq r \leq n$.

Definition 1.3. The logarithmic de Rham complex $\mathcal{R}_X^*(A)$ is a complex of $\mathcal{O}_X$ -modules defined locally by

$$\mathcal{R}_X^0(A) := \mathcal{O}_X$$

$$\mathcal{R}_X^1(A) := \left\{ \sum_{i=1}^r a_i \frac{dz_i}{z_i} + \sum_{j=r+1}^n a_j dz_j \mid a_k \in \mathcal{O}_X, k=1, \dots, n \right\}$$

$$\mathcal{R}_X^p(A) := \wedge^p \mathcal{R}_X^1(A),$$

where the differential is the usual exterior differentiation.

The formation of $\mathcal{R}_X^*$ is a contravariant functor on $\mathcal{C}$.

Note that $\mathcal{R}_X^p(A)$ are locally free $\mathcal{O}_X$ -modules.

Dually we make the following

Definition 1.4. $\Sigma_x^* \langle A \rangle$ is a (locally free) complex of $\mathcal{O}_x$ -modules defined by

$$\Sigma_x^* \langle A \rangle := \text{Hom}(\mathcal{B}_x^{n-*} \langle A \rangle, \mathcal{B}_x^n),$$

differential being induced by that of $\mathcal{B}_x^* \langle A \rangle$.

We see easily the following facts:

(1.5.) $\Sigma_x^* \langle A \rangle$ is ^{naturally} the subcomplex of the usual Poincaré complex $\mathcal{B}_x^*$, generated locally as an $\mathcal{O}_x$ -algebra by the elements

$$(2) \quad z_{i_1} \cdots z_{i_r} dz_{i_{r+1}} \cdots dz_{i_r}, \quad \{i_1, \dots, i_r\} = \{1, \dots, n\}.$$

The formation of $\Sigma^* \langle \rangle$ is also a ^{contravariant} functor from the category $\mathcal{C}$, as is seen from (2).

1.6. The importance of $\mathcal{B}_x^* \langle A \rangle$ comes from the following lemma which is proved in [17] and [27].

Lemma 1.6.1. Let $U = X - A$ and $j: U \hookrightarrow X$ be the inclusion. Then the complex cohomology $H^*(U, \mathbb{C})$ of U can be calculated as a hypercohomology of the complex $\mathcal{B}_x^* \langle A \rangle$:

$$H^*(U, \mathbb{C}) = H^*(X, \mathcal{B}_x^* \langle A \rangle),$$

where the right side denotes the hypercohomology. See [1].

1.7. As for $\Sigma_x^* \langle A \rangle$, the next proposition holds.

Proposition 1.7. We have

$$H^*(U, \mathbb{C}) = H^*(X, \Sigma_x^* \langle A \rangle),$$

where $H_{\mathbb{Z}}^*$ denotes the cohomology with support in a closed set of X contained in U .

This immediately follows from the lemma below, since then $\Sigma_X^*(A)$ is a resolution of $\mathbb{C}_U$, $\mathbb{C}_U$ being the constant sheaf $\mathbb{C}$ on U extended by 0 to X .

Lemma 1.8. The complex $\Sigma_X^*(A)$ is exact.

Proof is attained quite analogously to that of ^(the) classical Dolbeault lemma for the complex $\mathcal{D}_X^*$, in view of (2).

Remark 1.9. From the exact sequence

$$0 \rightarrow \Sigma_X^*(A) \rightarrow \mathcal{D}_X^*(A) \rightarrow \mathcal{D}_X^*(A)/\Sigma_X^*(A) \rightarrow 0$$

we have the cohomology exact sequence

$$\dots \rightarrow H_{\mathbb{Z}}^0(U, \mathbb{C}) \rightarrow H^0(U, \mathbb{C}) \rightarrow H^0(X, \mathcal{D}_X^*(A)/\Sigma_X^*(A)) \rightarrow \dots$$

Then using five lemma we can get an isomorphism.

$$H_{\infty}^*(U, \mathbb{C}) \stackrel{\text{def}}{=} \lim_{\substack{\rightarrow \\ V, V \supset A}} H^*(V \setminus A, \mathbb{C}) = H^*(X, \mathcal{D}_X^*(A)/\Sigma_X^*(A)),$$

where in the middle term V runs through a h. h. d. of A in X .

§2 Mixed Hodge structure at infinity.

2.0. Let $A = \bigcup A_i$ be the decomposition of A into irreducible components A_i . We use the following notation. $A^{(p)} = \bigcup_{i_1 < \dots < i_p} A_{i_1} \cap \dots \cap A_{i_p}$, $U_{(p)} = A^{(p)} - A^{(p+1)}$, and $\tau_p: \tilde{A}^{(p)} \rightarrow X$ be the normalization composed with the inclusion $i_p: A^{(p)} \hookrightarrow X$.

Further for simplicity we assume that each irreducible component A_i is compact and nonsingular.

2.1. We set $K^* = K_x^*(A) = \mathcal{R}_x^*(A)/\Sigma_x^*(A)$. Note that this is naturally a complex of $\mathcal{O}_x$ -modules.

We define an increasing filtration W on $\mathcal{R}_x^*(A)$ by the formula:

$$\begin{aligned} W^s(\mathcal{R}_x^*(A)) &= \left\{ \sum_{i_1 < \dots < i_s} \frac{dz_{i_1}}{z_{i_1}} \wedge \dots \wedge \frac{dz_{i_s}}{z_{i_s}} \wedge \alpha_{i_1 \dots i_s}, \quad 1 \leq i_1 < \dots < i_s \leq r, \quad \alpha_{i_1 \dots i_s} \in \mathcal{R}_x^* \right\}, \text{ if } s \geq 0 \\ &= \left\{ \sum_{i_1 < \dots < i_t} z_{i_1} \dots z_{i_t} dz_{i_{t+1}} \wedge \dots \wedge dz_{i_r} \wedge \alpha_{i_1 \dots i_t}, \quad \{i_1, \dots, i_t\} \subseteq \{1, \dots, r\} \right. \\ &\quad \left. \text{and } \alpha_{i_1 \dots i_t} \in \mathcal{R}_x^* \right\} \quad \text{if } t = -s > 0. \end{aligned}$$

The induced filtration on K^* is still denoted by the same letter W . The formation of $W(K_x^*(A))$ also defines a functor. As for the associated gr. we have

$$(3) \begin{cases} G_r^s(K^*) \\ \cong T_{S*} \mathcal{R}_{\tilde{A}^{(w)}}^*[-s] & s \geq 0. \quad (\text{Poincaré residue [, 3.1.5].}) \\ \cong \Sigma_{\tilde{E}^{-1}(A^{(t+w)})}^* \langle \tilde{A}^{(w)} \rangle & t = -s > 0. \end{cases}$$

In fact, $W^s(\mathcal{R}_x^*)$ coincides with the kernel of the restriction map. $\kappa_s: \mathcal{R}_x^* \rightarrow \mathcal{R}_{A^{(k-s-1)}}^*$.

2.2. Hodge filtration F on K^* is defined also as that induced by the Hodge filtration (still denoted by F) on $\mathcal{R}_x^*(A)$. Here if we put $T_{(p)}^* = F^p(\mathcal{R}_x^*(A))$, then

$$K_{(p)}^s = 0 \quad s < p$$

$$K_{(p)}^s = \Omega_*^s(A) \quad s \geq p.$$

2.3. Now with the filters W and F defined, K^* becomes a doubly filtered complex (K^*, W, F) , functorial with respect to $(X, A) \in \mathcal{C}$. Then as usual we have various spectral sequences associated with this complex. In particular we consider the one arising from the filter W . By virtue of (3) in 2.1. we get

Lemma 2.3.1. The $E_1^{p,q}$ term is given by

$$\begin{aligned} E_1^{p,q} &= H^{p+q}(\tilde{A}^{(p)}, \mathbb{C}) \quad \text{if } p \geq 0. \\ &= H_c^{p+q}(U^{(p)}, \mathbb{C}), \quad H_c: \text{cohomology with compact supp.} \\ &\quad \text{if } p < 0. \end{aligned}$$

2.4. On each term $E_r^{p,q}$ of the spectral sequence F induces three kinds of filtrations, the first direct filtration F_d , the second direct filtration F_d^* and the recursive filtration F_r . The key point in the proof of Theorem 2.3. is

Lemma 2.4.1. (i) For every $r \geq 1$, the differential d_r of the above spectral sequence is strictly compatible with the recursive filtration on $E_r^{p,q}$.

(ii) On each term $E_r^{p,q}$, (r may be ∞), the 3 kinds of filtration coincide, and the filter F on $H_{\text{co}}^*(U, \mathbb{C})$ is compatible with the recursive filtration F_r on $E_{\infty}^{p,q}$.

Remark. 2.4.2. (a) (ii) is a consequence of (i). [1, Th. 1.3.16. Cor. 1.3.17].

$$d_r^{p,q} = 0 \quad \text{if } \begin{cases} p \geq 0 \\ r \geq 2 \end{cases} \quad [1, \text{Lemma 3.2.10}].$$

(b) Proof of (i) will be omitted. But as an explanation, we note the following facts. Put $\mathcal{Q}_X^* = \mathcal{R}_X^* / \Sigma_X^* \langle A \rangle$. (Note ^{that} this is different from that defined by Grothendieck. Grauert-Kneuper). Then since $W(\mathcal{R}_X^* \langle A \rangle) = \mathcal{R}_X^*$, we have an exact sequence of $\mathcal{Q}_X^*$ -modules.

$$(4) \quad 0 \rightarrow \mathcal{R}_X^* \rightarrow K_X^* \rightarrow \mathcal{R}_X^* \langle A \rangle / \mathcal{R}_X^* \rightarrow 0.$$

By virtue of Lemma 1.6.1 and of Lemma 1.8. we have isomorphisms.

$$(5) \quad \begin{cases} H^*(A, \mathcal{R}_X^*) \cong H^*(A, \mathbb{C}) \\ H^*(\mathcal{R}_X^* \langle A \rangle / \mathcal{R}_X^*, \mathbb{C}) \cong H_A^*(X, \mathbb{C}), \end{cases}$$

and the ^{of hypercohomology} sequence corresponding to (4) is nothing but the local ^{long exact} cohomology exact sequence.

$$(6) \rightarrow H^s(A, \mathbb{C}) \rightarrow H_{\infty}^s(V, \mathbb{C}) \rightarrow H_A^s(X, \mathbb{C}) \rightarrow \dots$$

On the other hand, (4) is ^{obviously} the sequence compatible with the f.lte W .

Hence on each term ^{of (6)} we have a filtration induced by F and it is natural to expect (6) is the sequence of mixed Hodge structures w.r. to. these W and F . As for H_A^s , this (= that H_A^s has mixed Hodge st.) is essentially contained in [1]. And for $H^s(A, \mathbb{C})$, the spectral seq. associated with W is nothing but the spectral sequence associated to the increasing sequence

of closed subspaces $A^{(p)}$ $p=0,1,\dots,r$ of A . The corresponding statement to (i) of Lemma 2.4.1. may be ^{then} proved by induction on r .
 (c). From ^{the} sequence (6), we ^(can) deduce easily that the filter W on each term of (6) arises from that on $H^*(\cdot, \mathbb{Q})$.

§3. Case of an isolated singularity.

3.1. Let $\tilde{X} \stackrel{=}{=} (X, P)$ be as in §2 and $f: \tilde{X} = (\tilde{X}, A) \rightarrow X$ be a resolution of $\tilde{X}$. Since $H_c^* = H_c^*(\tilde{X}) \cong H_u^*(U)$, $U = \tilde{X} - A$, by Lemma 2.4.1. and Remark 2.4.2 (c), we conclude that H_z^* has a mixed Hodge structure. Finally we have to show that this structure does not depend on the resolution chosen above. But this follows from [1, Théorème 1.2.10] by the same argument as in [1, 3.2.11. C].

References.

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- [2] Deligne, P., Equations différentielles à points singuliers réguliers, lecture note in Math. No.163. Springer. 1970