

DIAMETER OF A TYPE-DEFINABLE LASCAR STRONG TYPE OVER A HYPERIMAGINARY

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ABSTRACT. Using the so called open analysis, Ludomir Newelski proved that type-definable Lascar strong types over real parameters have finite diameters in 2003. Later in 2008, Rodrigo Pelaez proved the same result in a more direct way, using the notion of c -free and weakly c -free extensions over a complete type, which were also introduced by Newelski. Recall that a hyperimaginary is an equivalence class of an $\emptyset$ -type-definable equivalence relation. We will extend the result of Pelaez to the context of hyperimaginaries, i.e. type-definable Lascar strong types over a hyperimaginary always have finite diameters.

Fix a complete theory T with a language $\mathcal{L}$ and a monster model $\mathcal{M}$ of T . Let E be an $\emptyset$ -type definable equivalence relation, a a real (possibly infinite) tuple, and $\mathbf{e} = a/E$ a hyperimaginary. Assume that there is a partial type $\pi(x)$ over ab such that $b_0 \models \pi(x)$ iff $b_0 \equiv_{\mathbf{e}}^L b$, i.e. the Lascar strong type of b over $\mathbf{e}$ is type-definable. We will show that there is a natural number $n_b < \omega$ such that for all b_0 having the same Lascar strong type of b over $\mathbf{e}$, the diameter (or Lascar distance) between b and b_0 over $\mathbf{e}$ is at most n_b , i.e. $d_{\mathbf{e}}(b, b_0) \leq n_b$.

The same result over an empty (or a set of real) parameter(s) is established in [4]. Many concepts and arguments will be from [4] but some critical modifications and new concepts will also be introduced to handle the situations over a hyperimaginary. We begin with the space of complete types over a hyperimaginary $\mathbf{e}$.

- Definition 1.** (1) Say a partial type $p(x)$ over a is a complete type over $\mathbf{e}$ if there is c such that for any $\varphi(x) \in \mathcal{L}(a)$, (if $f(c) \models \varphi(x)$ for all $f \in \text{Aut}_{\mathbf{e}}(\mathcal{M})$, then $\varphi \in p$). Denote this $p(x)$ by $\text{tp}_x(c/\mathbf{e})$, and call “the” complete type of c over $\mathbf{e}$ (with representation a). We may omit x and write $\text{tp}(c/\mathbf{e})$ if the variable is clear from the context.
- (2) $S_x(\mathbf{e}) = \{p(x) : p(x) \text{ is a complete type over } \mathbf{e}\}$ is the set of all complete types over a hyperimaginary $\mathbf{e}$ in the variable x . Likewise, we may omit x .
- (3) For $\varphi(x) \in \mathcal{L}(a)$, denote $[\varphi(x)]_{\mathbf{e}} = \{p(x) \in S_x(\mathbf{e}) : \varphi(x) \in p(x)\}$. Define likewise for a partial type $\Phi(x)$ over a .
- (4) For $\varphi(x) \in \mathcal{L}(a)$, denote $[\varphi(x)]_{\mathbf{e}}^{\text{con}} = \{p(x) \in S_x(\mathbf{e}) : p \text{ is consistent with } \varphi\}$. Define likewise for a partial type $\Phi(x)$ over a .

Remark 2. (1) The above definition of a complete type over a hyperimaginary can replace the old version for it : Recall that $\exists y(E(a, y) \wedge \text{tp}_{xy}(ca))$ would be a complete type of c over $\mathbf{e}$ in the old fashion, which is a subset of $p(x)$, and their solution sets coincide.

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- (2) We say $p(x) = \text{tp}(c/e) \in S(e)$ “the” complete type of c over e since it is unique : The parameters are restricted to a and $\text{tp}(c/e)$ is the maximal set of formulas $\varphi(x)$ over a such that $\{f(c) : f \in \text{Aut}_e(\mathcal{M})\} \subseteq \varphi(\mathcal{M})$.
- (3) $p(x) \in [\varphi(x)]_e$ iff $p(x)$ is inconsistent with $\neg\varphi(x)$ iff $p(x) \notin [\neg\varphi(x)]_e^{\text{con}}$, so $S_x(e) \setminus [\varphi(x)]_e$ is NOT the same as $[\neg\varphi(x)]_e$ in general.

Lemma 3. (1) $\{[\varphi(x)]_e : \varphi(x) \in \mathcal{L}(a)\}$ forms a basis of $S_x(e)$.

(2) $S_x(e)$ is Hausdorff and compact.

Proof. (1) $[x = x]_e = S_x(e)$ and $[\varphi(x)]_e \cap [\psi(x)]_e = [\varphi(x) \wedge \psi(x)]_e$.

- (2) If $p \neq q \in S_x(e)$, then $p \wedge q$ is inconsistent, so by compactness, there is $\varphi(x) \in p$ and $\psi(x) \in q$ such that $\varphi(x) \wedge \psi(x) \equiv \emptyset$. Then $p \in [\varphi]_e, q \in [\psi]_e$ and $[\varphi]_e \cap [\psi]_e = \emptyset$, proving that $S_x(e)$ is Hausdorff.

To show compactness, suppose there is an open cover $\{[\varphi_i(x)]_e : i \in I\}$ without any finite subcover. Pick any finite subset, say $\{\varphi_{i_0}, \dots, \varphi_{i_k}\}$. Then by supposition, there is $p(x) \in S_x(e)$ such that $p(x) \notin [\varphi_{i_0}]_e \cup \dots \cup [\varphi_{i_k}]_e$. Then by Remark 2(3), $p(x)$ is consistent with $\neg\varphi_{i_l}$ for each $l = 0, \dots, k$. So by compactness, there is $q(x) \in S_x(e)$ consistent with $\neg\varphi_i$ for each $i \in I$ (since $\exists x_{i \in I} (\bigwedge_{i \in I} \neg\varphi_i(x_i) \wedge \bigwedge_{i, j \in I} x_i \equiv_e x_j)$ is finitely consistent). Then by Remark 2(3) again, this $q(x) \notin \bigcup_{i \in I} [\varphi_i]_e = S_x(e)$, which is impossible. $\square$

Let $\pi(x)$ be a partial type over ab such that $b_0 \models \pi$ iff $b \equiv_e^L b_0$.

Definition 4. (1) $U \subseteq \mathcal{M}$ (not necessarily small) is called c-free over π if there is $n < \omega$ such that $f_0, \dots, f_{n-1} \in \text{Aut}_e(\mathcal{M})$ such that $\pi(\mathcal{M}) \subseteq \bigcup_{i < n} f_i(U)$ and $f_i(\pi(\mathcal{M})) = \pi(\mathcal{M})$ for every $i < n$.

(2) $\varphi(x) \in \mathcal{L}(\mathcal{M})$ is c-free over π if $\varphi(\mathcal{M})$ is c-free over π .

(3) A (partial) type $q(x)$ is c-free over π if for any $\varphi(x)$ such that $q(x) \vdash \varphi(x)$, $\varphi(x)$ is c-free over π .

Definition 5. $U \subseteq \mathcal{M}$ is weakly c-free over π if there is $V \subseteq \mathcal{M}$ such that V is not c-free over π but UV is c-free over π . Define for $\varphi(x) \in \mathcal{L}(\mathcal{M})$ and a partial type $q(x)$ to be weakly c-free over π in the same manner as above.

Remark 6. If $U \subseteq V$ and U is (weakly) c-free over π , then V is (weakly) c-free over π .

Definition 7. Define $F_x : S_x(a) \rightarrow S_x(e)$, $F_x(\text{tp}_x(c/a)) = \text{tp}_x(c/e)$.

Remark 8. F_x is possibly NOT continuous since $F_x^{-1}([\varphi(x)]_e)$ can be a proper subset of $[\varphi(x)] \subseteq S_x(a)$: Even if $p(x) \notin [\varphi(x)]_e$, it may happen that $F_x^{-1}(p) \cap [\varphi(x)] \neq \emptyset$ since there can be $c \models \varphi(x)$ such that there is some $f \in \text{Aut}_e(\mathcal{M})$, $f(c) \models \varphi(x)$.

Lemma 9. For any (partial) type $\Phi(x)$ over a , denote $[\Phi(x)] = \{p(x) \in S(a) : \Phi(x) \subseteq p(x)\}$. Then $F_x([\Phi(x)]) = \{q \in S(e) : q \text{ is consistent with } \Phi\}$ and F_x is a closed map.

Proof. Let $\Phi(x)$ be a partial type over a . Then

$$\begin{aligned}
 F_x([\Phi(x)]) &= F_x(\{p \in S(a) : p(\mathcal{M}) \subseteq \Phi(\mathcal{M})\}) \\
 &= F_x(\{p \in S(a) : \forall (\exists) c \models p, c \models \Phi\}) \\
 &= \{F_x(p) \in S(e) : \forall (\exists) c \models p, c \models \Phi\} \\
 &= \{q \in S(e) : \exists c \models q \text{ such that } c \models \Phi\} \\
 &= \{q \in S(e) : q \text{ is consistent with } \Phi\}.
 \end{aligned}$$

By compactness, $q \in S(e)$ is consistent with Φ iff q is consistent with every finite subset Φ_0 of Φ . Thus $F_x([\Phi(x)]) = \bigcap_{\substack{\Phi_0 \subseteq \Phi \\ \Phi_0 : \text{finite}}} [\bigwedge \Phi_0]_e^{\text{con}}$ where $\bigwedge \Phi_0$ is the conjunction of all formulas in Φ_0 , which is a single formula in $\mathcal{L}(a)$. Note that by Remark 2(3), each $[\bigwedge \Phi_0]_e^{\text{con}}$ is a basic closed set in $S(e)$, thus $F_x([\Phi(x)]) = \bigcap_{\substack{\Phi_0 \subseteq \Phi \\ \Phi_0 : \text{finite}}} [\bigwedge \Phi_0]_e^{\text{con}}$ is closed. $\square$

Definition 10. (1) $P = \{q(x, y) \in S_{xy}(e) : q(x, y) \cup \pi(x) \cup \pi(y) \text{ is consistent}\}$.
 (2) For $c \models \pi$, $P_{w,c} = \{q(x, y) \in P : q(c, y) \text{ is weakly } c\text{-free over } \pi\}$.

So far, no assumption on $\pi(x)$ is used, i.e. $\pi(x)$ would be any partial type having some parameters. Now we start exploiting the fact that $\pi(x)$ type-defines the Lascar strong type of b over e .

Lemma 11. (1) For any $p(x, y) = \text{tp}(b_1 b_2 / e) \in P$, $b_1 \equiv_e^L b_2$.
 (2) $P = [\pi(x) \cup \pi(y)]_e^{\text{con}}$ is closed in $S_{xy}(e)$.
 (3) Define $\Delta_{w,c}(x, y) = \{\varphi(x, y) \in \mathcal{L}(a) : \varphi(c, y) \text{ is not weakly } c\text{-free over } \pi\}$. Then $P_{w,c} = P \cap \bigcap_{\varphi \in \Delta_{w,c}} [\neg \varphi(x, y)]_e^{\text{con}}$. In particular, $P_{w,c}$ is closed in $S_{xy}(e)$.
 (4) There is a partial type $\Phi_{w,c}(x, y)$ over a such that $p \in P_{w,c}$ iff for all (some) $d \models p$, $d \models \Phi_{w,c}$. Thus $P_{w,c} = [\Phi_{w,c}(x, y)]_e^{\text{con}}$.

Proof. (1) Say $b_1^* b_2^* \models \text{tp}(b_1 b_2 / e) \wedge \pi(x) \wedge \pi(y)$. Then $b_1^* \equiv_e^L b_2^* \equiv_e^L b_2^*$ and $f(b_1^* b_2^*) = b_1 b_2$ for some $f \in \text{Aut}_e(\mathcal{M})$, hence $b_1 = f(b_1^*) \equiv_e^L f(b_2^*) = b_2$.
 (2) By Lemma 9, $P = [\pi(x) \cup \pi(y)]_e^{\text{con}} = F_{xy}([\pi(x) \cup \pi(y)])$ is closed.
 (3) Suppose there is $p(x, y) \in \bigcap_{\varphi \in \Delta_{w,c}} [\neg \varphi(x, y)]_e^{\text{con}}$ but $p(x, y) \notin P_{w,c}$. Then $p(c, y)$ is not weakly c -free over π , so there is $\psi(y) \in \mathcal{L}(\mathcal{M})$ such that $\psi(y)$ is not weakly c -free over π and $p(c, y) \vdash \psi(y)$. By compactness and Remark 6, there is $\varphi(c, y) \in p(c, y)$ not weakly c -free over π . But then $\varphi(x, y) \in \Delta_{w,c}(x, y)$, so $\neg \varphi(x, y)$ must be consistent with $p(x, y)$, a contradiction since $\varphi(x, y) \in p(x, y)$. Conversely, suppose there is $p(x, y) \in P_{w,c}$ but $p(x, y) \notin \bigcap_{\varphi \in \Delta_{w,c}} [\neg \varphi(x, y)]_e^{\text{con}}$. Then there is $\varphi(x, y) \in \Delta_{w,c}(x, y)$ such that $\neg \varphi(x, y)$ is inconsistent with $p(x, y)$, hence $p(x, y) \in [\varphi(x, y)]_e$ by Remark 2(3). But then $\varphi(c, y)$ is not weakly c -free over π and $p(c, y) \vdash \varphi(c, y)$, a contradiction to $p(x, y) \in P_{w,c}$. Now $P_{w,c}$ is clearly closed by Remark 2(3).
 (4) $\text{tp}(b_1 b_2 / e) \in P_{w,c}$ iff $\text{tp}(b_1 b_2 / e)$ is consistent with $\neg \psi(x, y)$ for any $\psi(x, y)$ such that $\psi(c, y)$ is not weakly c -free over π (by (3)) iff $b_1 b_2 \models \bigwedge_{\psi \in \Delta_{w,c}} \exists zw(xy \equiv_e zw \wedge \neg \psi(z, w))$, say $\Phi_{w,c}(x, y)$. Notice that this $\Phi_{w,c}(x, y)$ is e -invariant, so $\text{tp}(b_1 b_2 / e)$ is consistent with $\Phi_{w,c}(x, y)$ iff $\Phi_{w,c}(x, y) \subseteq \text{tp}(b_1 b_2 / e)$. $\square$

Proposition 12. Let $\emptyset \neq S \subseteq P_{w,c}(x, y)$ where $S = [\Psi(x, y)]_e$ for some (partial) type Ψ such that $[\Psi(x, y)]_e = [\Psi(x, y)]_e^{\text{con}}$. Then there are $c_1, \dots, c_k \models \pi$ such that for any $b_0 \models \pi$, there is $d \models \pi$ such that

- (1) $\text{tp}(b_0 d / e) \in S$.
- (2) $\text{tp}(c_i d / e) \in S$ for some $1 \leq i \leq k$.

Proof. We start with a claim.

Claim 1. $\Psi(c, y)$ is weakly c -free over π .

Proof of Claim 1. Suppose not, so there is $\delta'(c, y)$ such that $\delta'(c, y)$ is not weakly c -free over π and $\Psi(c, y) \vdash \delta'(c, y)$. Then by compactness and Remark 6, there is $\delta(c, y) \in$

$\bigwedge \Psi(c, y) = \{\psi_1 \wedge \cdots \wedge \psi_n : \psi_l \in \Psi(c, y), l \in \omega\}$, which is not weakly c-free over π . By Lemma 11(3), every element of $P_{w,c}$ must be consistent with $\neg\delta(x, y)$, i.e. $P_{w,c} \subseteq [\neg\delta(x, y)]_e^{con}$. But we have $[\Psi(x, y)]_e^{(con)} = S \subseteq P_{w,c}$ and $[\Psi(x, y)]_e^{(con)} \cap [\neg\delta(x, y)]_e^{con} = \emptyset$ (If not empty, say $p(x, y) \in S(e)$ witness it, then $p(\mathcal{M}) \subseteq \Psi(\mathcal{M})$ and $p(\mathcal{M}) \cap \neg\delta(\mathcal{M}) \neq \emptyset$, a contradiction), so every $p(x, y) \in P_{w,c}$ is not an element of S , a contradiction to $S \neq \emptyset$. $\square$

Using Claim 1, say $V \subseteq \mathcal{M}$ is not c-free over π but $\pi(\mathcal{M}) \subseteq \bigcup_{i=1}^k g_i(\Psi(c, \mathcal{M}) \cup V)$ where $g_i \in \text{Aut}_e(\mathcal{M})$ and $g_i(\pi(\mathcal{M})) = \pi(\mathcal{M})$ for each $1 \leq i \leq k$.

Claim 2. For each $b_0 \models \pi$, there is $d \in \Psi(b_0, \mathcal{M})$ such that $d \in \bigcup_{i=1}^k \Psi(g_i(c, \mathcal{M}))$.

Proof of Claim 2. Suppose not, so that there is $b_0 \models \pi$ such that for every $d \in \Psi(b_0, \mathcal{M})$, $d \notin \bigcup_{i=1}^k \Psi(g_i(c, \mathcal{M}))$. Notice that $d \models \pi : \text{tp}(b_0 d / e) \in P$ and $b_0 \models \pi$, so there is $b_0^* d^* \models \text{tp}(b_0 d / e) \wedge \pi(x) \wedge \pi(y)$. Then there is $f \in \text{Aut}_e(\mathcal{M})$ such that $f(b_0^* d^*) = b_0 d$, and this f also fixes $b / \equiv_e^L$ since $b \equiv_e^L b_0 \equiv_e^L b_0^*$. Thus $d = f(d^*) \models \pi$. So for all $d \in \Psi(b_0, \mathcal{M})$, $d \in \pi(\mathcal{M}) \subseteq \bigcup_{i=1}^k g_i(\Psi(c, \mathcal{M}) \cup V)$, hence $d \in \bigcup_{i=1}^k g_i(\Psi(c, \mathcal{M}) \cup V) \setminus \bigcup_{i=1}^k \Psi(g_i(c, \mathcal{M})) \subseteq \bigcup_{i=1}^k g_i((\Psi(c, \mathcal{M}) \cup V) \setminus \Psi(c, \mathcal{M}))$. Since b_0 and c realizes π , there is $g_{k+1} \in \text{Aut}_e(\mathcal{M})$ such that $g_{k+1}(c) = b_0$ and $g_{k+1}(\pi(\mathcal{M})) = \pi(\mathcal{M})$. Also, since $g_{k+1}(\Psi(c, \mathcal{M}) \cup V) = (g_{k+1}(\Psi(c, \mathcal{M}) \cup V) \setminus \Psi(b_0, \mathcal{M})) \cup \Psi(b_0, \mathcal{M})$ and $\Psi(b_0, \mathcal{M}) \subseteq \bigcup_{i=1}^k g_i((\Psi(c, \mathcal{M}) \cup V) \setminus \Psi(c, \mathcal{M}))$, $g_{k+1}(\Psi(c, \mathcal{M}) \cup V) \subseteq \bigcup_{i=1}^{k+1} (g_i(\Psi(c, \mathcal{M}) \cup V) \setminus \Psi(c, \mathcal{M}))$. Since $g_{k+1}(\Psi(c, \mathcal{M}) \cup V)$ is c-free over π , $\bigcup_{i=1}^{k+1} (g_i(\Psi(c, \mathcal{M}) \cup V) \setminus \Psi(c, \mathcal{M}))$ is also c-free over π , hence $(\Psi(c, \mathcal{M}) \cup V) \setminus \Psi(c, \mathcal{M}) \subseteq V$ is c-free over π , a contradiction. $\square$

Now by Claim 2, $g_1(c), \dots, g_k(c)$ satisfies : For any $b \models \pi$, there is $d \in \pi$ such that $\text{tp}(bd / e) \in S (= [\Psi(x, y)]_e^{(con)})$ and $\text{tp}(c_i d / e) \in S$ for some $1 \leq i \leq k$, completing the proof. $\square$

Let $\Gamma(x, y)$ be a partial type over a such that $b_0 b_1 \models \Gamma(x, y)$ iff there is an e -indiscernible sequence beginning with b_0, b_1 .

Definition 13. Say the diameter (or Lascar distance) between b_1 and b_2 is less than equal to n where $1 < n < \omega$, if

$$b_1 b_2 \models \exists y_1 \cdots y_{n-1} (\Gamma(x, y_1) \wedge \Gamma(y_1, y_2) \wedge \cdots \wedge \Gamma(y_{n-1}, y))$$

and denote it by $d_e(b_1, b_2) \leq n$. If $b_1 b_2 \models \Gamma(x, y)$, then say $d_e(b_1, b_2) \leq 1$.

Theorem 14. There is $n_b \in \omega$ such that if $b_0 \models \pi(x)$, then $d_e(b, b_0) \leq n_b$.

Proof. By Lemma 11(1), $P = \bigcup_{1 \leq n \in \omega} (P \cap [d_e(x, y) \leq n]_e)$, thus

$$\begin{aligned} P_{w,c} &= P \cap P_{w,c} = \bigcup_{1 \leq n \in \omega} (P \cap [d_e(x, y) \leq n]_e \cap P_{w,c}) \\ &= \bigcup_{1 \leq n \in \omega} (P_{w,c} \cap [d_e(x, y) \leq n]_e). \end{aligned}$$

Note that $[d_e(x, y) \leq n]_e = [d_e(x, y) \leq n]_e^{con}$ for any $1 \leq n < \omega$: ' $d_e(x, y) \leq n$ ' is e -invariant, so its solution set is a union of $\text{Aut}_e(\mathcal{M})$ -orbits. Thus by Lemma 11(4),

$$\begin{aligned} P_{w,c} \cap [d_e(x, y) \leq 1]_e^{(con)} &= [\Phi_{w,c}(x, y)]_e^{(con)} \cap [d_e(x, y) \leq 1]_e^{(con)} \\ &= [\Phi_{w,c}(x, y) \wedge (d_e(x, y) \leq 1)]_e^{(con)}. \end{aligned}$$

Now Proposition 12 can be applied with $S = [\Phi_{w,c}(x, y) \wedge (d_e(x, y) \leq 1)]_e^{(con)}$, so that there are $c_1, \dots, c_k \models \pi$ such that for any $b_0 \models \pi$, there are $d, d_0 \models \pi$ such that

- (1) $\text{tp}(bd/e), \text{tp}(b_0d_0/e) \in S$
- (2) $\text{tp}(c_{k_*}d/e), \text{tp}(c_{k_0}d_0/e) \in S$ for some $1 \leq k_*, k_0 \leq k$.

Let $M = \min\{n \in \omega : \{\text{tp}(c_i c_j/e) : 1 \leq i, j \leq k\} \subseteq [d_e(x, y) \leq n]_e\}$. Then

$$\begin{aligned} d_e(b, b_0) &\leq d_e(b, d) + d_e(d, c_{k_*}) + d_e(c_{k_*}, c_{k_0}) + d_e(c_{k_0}, d_0) + d_e(d_0, b_0) \\ &\leq 1 + 1 + M + 1 + 1 = 4 + M. \end{aligned}$$

□

Corollary 15. *Type-definable Lascar strong type over a hyperimaginary has finite diameter.*

REFERENCES

- [1] D. Lascar and A. Pillay, ‘Hyperimaginaries and automorphism groups’, *J. of Symbolic Logic* 66 (2001) 127-143.
- [2] B. Kim, ‘Simplicity Theory’, Oxford University Press (2014).
- [3] M. Ziegler, ‘Introduction to the Lascar group’, in *Tits buildings and the model theory of groups*, London Math. Lecture Note Series 291, Cambridge Univ. Press (2002) 279-298.
- [4] Rodrigo Pelaez Pelaez, ‘About The Lascar Group’, Dissertation, University of Barcelona (2008).
- [5] Ludomir Newelski, ‘The diameter of a Lascar strong type’, *Fundamenta Mathematicae* 176(2):157-170 (2003).

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