

LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}E_8}$ AND AN ALGEBRA OF MIYAMOTO OF CENTRAL CHARGE $\frac{1}{2} + \frac{21}{22}$

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ABSTRACT. Motivated by a work of Miyamoto [17], we construct a vertex operator algebra U of central charge $\frac{1}{2} + \frac{21}{22}$ which has the full automorphism group isomorphic to the symmetry group S_3 . Actually, we show that the lattice vertex operator algebra $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to a tensor product of unitary Virasoro vertex operator algebras $\mathfrak{T} = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{52}{55}, 0) \otimes L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ and U is a certain coset subalgebra of $V_{\sqrt{2}E_8}$. We also show that U contains exactly 3 conformal vectors of central charge $1/2$ and the inner product between any two of them is $1/2^8$.

1. INTRODUCTION

This work is motivated by a recent article of Miyamoto [17]. In [17], Miyamoto studied a class of vertex operator algebra (VOA) generated by two rational conformal vectors e and f of central charge $1/2$. Among other things, he showed that if the inner product $\langle e, f \rangle$ is equal to $\frac{1}{2^8}$, then the vertex operator algebra U generated by e and f is of central charge $16/11$ and U contains a subalgebra isomorphic to $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$. Moreover, $\dim U_2 = 3$ and the full automorphism group of U is isomorphic to the symmetry group S_3 . In this paper, we shall construct explicitly a VOA

$$\begin{aligned} U \cong & L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 8) \\ & \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{7}{2}) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{45}{2}) \\ & \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{31}{16}) \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{175}{16}), \end{aligned}$$

in the lattice VOA $V_{\sqrt{2}E_8}$ and show that U satisfies all the properties mentioned in [17]. In fact, we shall show that the lattice VOA $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to a tensor product of the unitary Virasoro VOAs

$$\begin{aligned} \mathfrak{T} = & L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \\ & \otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{52}{55}, 0) \otimes L(\frac{21}{22}, 0) \otimes L(\frac{1}{2}, 0), \end{aligned}$$

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and obtain a complete decomposition of $V_{\sqrt{2}E_8}$ into a direct sum of irreducible $\mathfrak{I}$ -modules. The VOA U is actually a certain commutant (or coset) subalgebra associated with the above decomposition. We also notice that an automorphism of order 3 obtained from the abelian group $\sqrt{2}E_8/\sqrt{2}A_8$ induces a natural $\mathbb{Z}_3$ -action on U . This action together with the usual involution θ induced by -1 will form a group S_3 inside the automorphism group of U . In addition, we determine all conformal vectors of central charge $1/2$ inside U and show that the inner of any two of them is $1/2^8$ as mentioned by Miyamoto.

2. LATTICE VERTEX OPERATOR ALGEBRA $V_{\sqrt{2}E_8}$

2.1. The lattice $\sqrt{2}E_8$. Let $\alpha^0, \dots, \alpha^8$ be vectors in $\mathbb{R}^9$ such that $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$ for any $i, j = 0, \dots, 8$ and $L = \mathbb{Z}\alpha^0 \oplus \mathbb{Z}\alpha^1 \oplus \dots \oplus \mathbb{Z}\alpha^8$. Then L is isomorphic to the orthogonal sum of 9 copies of the root lattice A_1 . Let $\beta_i = -\alpha_{i-1} + \alpha_i$, $i = 1, \dots, 8$. Then $N = \text{span}_{\mathbb{Z}}\{\beta_1, \dots, \beta_8\}$ is isomorphic to the lattice $\sqrt{2}A_8$. Let

$$\gamma = \frac{1}{3}(2\alpha^0 + 2\alpha^1 + 2\alpha^2 - \alpha^3 - \alpha^4 - \alpha^5 - \alpha^6 - \alpha^7 - \alpha^8). \quad (2.1)$$

Then γ belongs to the dual lattice $N^* = \{x \in \mathbb{Q} \otimes_{\mathbb{Z}} N \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in N\}$ of N and the lattice K generated by γ and N is of rank 8. Moreover, we have

Lemma 2.1. $K \cong \sqrt{2}E_8$

Proof. First, we shall note that $\langle \gamma, \gamma \rangle = 4$ and $K = \langle \gamma, N \rangle = N \cup (\gamma + N) \cup (-\gamma + N)$. Moreover, $K/N \cong \mathbb{Z}_3$ as an abelian group.

Let $\theta_i = \frac{1}{\sqrt{2}}\beta_i = \frac{1}{\sqrt{2}}(-\alpha_{i-1} + \alpha_i)$ for $i = 1, \dots, 7$ and $\theta_8 = \frac{1}{\sqrt{2}}\gamma$. Then

$$\begin{aligned} \langle \theta_i, \theta_i \rangle &= 2 && \text{for } i = 1, \dots, 8, \\ \langle \theta_{i-1}, \theta_i \rangle &= -1 && \text{for } i = 2, \dots, 7, \\ \langle \theta_3, \theta_8 \rangle &= -1, && \text{and} \\ \langle \theta_i, \theta_j \rangle &= 0 && \text{for all other } 1 \leq i, j \leq 8. \end{aligned}$$

In other words, $\{\theta_1, \dots, \theta_8\}$ is a set of simple roots of the root lattice E_8 and hence

$$K \supset \text{span}_{\mathbb{Z}}\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \gamma\} \cong \sqrt{2}E_8.$$

Since $|K/N| = 3 = |\sqrt{2}E_8/\sqrt{2}A_8|$, $K = \text{span}_{\mathbb{Z}}\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \gamma\} \cong \sqrt{2}E_8$. □

Hence we also know that the vertex operator algebra

$$V_{\sqrt{2}E_8} \cong V_K = V_N \oplus V_{\gamma+N} \oplus V_{-\gamma+N}.$$

2.2. Conformal vectors in $V_{\sqrt{2}E_8}$. In this section, we shall study some conformal vectors in $V_{\sqrt{2}E_8}$. We shall show that the Virasoro element of the VOA $V_{\sqrt{2}E_8}$ can be decomposed into a sum of 10 mutually orthogonal conformal vectors $\tilde{\omega}^1, \dots, \tilde{\omega}^{10}$ and the central charge of $c(\tilde{\omega}^i)$ of $\tilde{\omega}^i$ are given by

$$c(\tilde{\omega}^i) = 1 - \frac{6}{(i+2)(i+3)} \quad \text{for } 1 \leq i \leq 8,$$

$$c(\tilde{\omega}^9) = \frac{1}{2}, \quad \text{and} \quad c(\tilde{\omega}^{10}) = \frac{21}{22}.$$

First, let us recall a construction of certain conformal vectors in $V_{\sqrt{2}A_l}$ from Dong et. al. [4]. Let Φ be the root system of A_l and Φ^+ and Φ^- the set of all positive roots and negative roots, respectively. Then

$$\Phi = \Phi^+ \cup \Phi^- = \Phi^+ \cup (-\Phi^+).$$

Consider a chain of root systems

$$\Phi = \Phi_l \supset \Phi_{l-1} \supset \dots \supset \Phi_1$$

such that Φ_i is a root system of type A_i . For any $i = 1, 2, \dots, l$, define

$$s^i = \frac{1}{2(i+3)} \sum_{\alpha \in \Phi_i^+} \left(\alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}) \right)$$

and

$$\omega = \frac{1}{2(l+1)} \sum_{\alpha \in \Phi_l^+} \alpha(-1)^2 \cdot 1.$$

It was shown by Dong et. al. [4] that the elements

$$\omega^1 = s^1, \quad \omega^i = s^i - s^{i-1}, \quad 2 \leq i \leq l, \quad \omega^{l+1} = \omega - s^l \quad (2.2)$$

are mutually orthogonal conformal vectors in $V_{\sqrt{2}A_l}$. The subalgebra $\text{Vir}(\omega^i)$ of the vertex operator algebra $V_{\sqrt{2}A_l}$ generated by ω^i is isomorphic to the Virasoro vertex operator algebra $L(c(\omega^i), 0)$ which is the irreducible highest weight module for the Virasoro algebra with central charge $c(\omega^i)$ and highest weight 0 and the central charge $c(\omega^i)$ of ω^i are given by

$$c(\omega^i) = 1 - \frac{6}{(i+2)(i+3)} \quad \text{for } 1 \leq i \leq l \quad \text{and} \quad c(\omega^{l+1}) = \frac{2l}{(l+3)}.$$

Since $\omega^1, \omega^2, \dots, \omega^{l+1}$ are mutually orthogonal, the subalgebra T of $V_{\sqrt{2}A_l}$ generated by these conformal vectors is a tensor product of $\text{Vir}(\omega^i)$'s, namely,

$$T = \text{Vir}(\omega^1) \otimes \dots \otimes \text{Vir}(\omega^{l+1})$$

$$\cong L(c(\omega^1), 0) \otimes \dots \otimes L(c(\omega^{l+1}), 0).$$

Moreover, $V_{\sqrt{2}A_l}$ is completely reducible as a T -module.

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For $l = 8$, there are 9 mutually orthogonal conformal vectors $\omega^1, \dots, \omega^9$ in $V_{\sqrt{2}A_8}$ and the central charge of $\omega^1, \dots, \omega^9$ are $\frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, \frac{25}{28}, \frac{11}{12}, \frac{14}{15}, \frac{52}{55}$ and $\frac{16}{11}$, respectively. In other words, $V_{\sqrt{2}A_8}$ contains a subalgebra isomorphic to

$$T = L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \otimes L(\frac{25}{28}, 0) \\ \otimes L(\frac{11}{12}, 0) \otimes L(\frac{14}{15}, 0) \otimes L(\frac{52}{55}, 0) \otimes L(\frac{16}{11}, 0)$$

The following lemma can be obtained by direct calculation.

Lemma 2.2. *Let γ be defined as in (2.1) and let*

$$a^1 = \sum_{\substack{\alpha \in (\gamma + \sqrt{2}A_8), \\ \langle \alpha, \alpha \rangle = 4}} e^\alpha \in V_{\gamma + \sqrt{2}A_8} \quad \text{and} \\ a^2 = \sum_{\substack{\alpha \in (-\gamma + \sqrt{2}A_8), \\ \langle \alpha, \alpha \rangle = 4}} e^\alpha \in V_{-\gamma + \sqrt{2}A_8}.$$

Then a^1 and a^2 are both highest weight vectors of weight $(0, 0, 0, 0, 0, 0, 0, 2)$ with respect to the action of T .

Lemma 2.3. *Let $u = a^1 + a^2 = \sum_{\substack{\alpha \in (\gamma + \sqrt{2}A_8), \\ \langle \alpha, \alpha \rangle = 4}} (e^\alpha + e^{-\alpha})$. Then*

$$\tilde{\omega}^9 = \frac{11}{32}\omega^9 + \frac{1}{32}u \quad \text{and} \quad \tilde{\omega}^{10} = \frac{21}{32}\omega^9 - \frac{1}{32}u$$

are mutually orthogonal conformal vectors of central charge $1/2$ and $21/22$, respectively. Moreover, they are orthogonal to $\omega^1, \dots, \omega^8$.

Proof. First, we shall note that for any α, β with square norm 4,

$$(e^\alpha)_1 e^\beta = \begin{cases} e^{\alpha+\beta} & \text{if } \langle \alpha, \beta \rangle = -2 \\ \alpha(-1)^2 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

and

$$\langle e^\alpha, e^\beta \rangle = (e^\alpha)_3 e^\beta = \begin{cases} 1 & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases} \quad (2.4)$$

Then by direct computation, we have

$$u_1 u = 2(231\omega^9 + 10u), \quad \omega_1^9 \omega^9 = 2\omega^9 \quad \text{and} \quad \omega_1^9 u = 2u.$$

Now, it is easy to verify that both $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are conformal vectors.

Since $\sqrt{2}A_8$ has exactly 72 vectors of square norm 4 and $\gamma + \sqrt{2}A_8$ and $-\gamma + \sqrt{2}A_8$ each has 84 vectors of square norm 4, we also have

$$\langle \omega^9, \omega^9 \rangle = \frac{8}{11}, \quad \langle \omega^9, u \rangle = 0, \quad \text{and} \quad \langle u, u \rangle = 168. \quad (2.5)$$

Therefore,

$$\langle \tilde{\omega}^9, \tilde{\omega}^9 \rangle = \frac{1}{4}, \quad \langle \tilde{\omega}^9, \tilde{\omega}^{10} \rangle = 0, \quad \text{and} \quad \langle \tilde{\omega}^{10}, \tilde{\omega}^{10} \rangle = \frac{21}{44}$$

and hence $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are mutually orthogonal conformal vectors of central charge $1/2$ and $21/22$. By the definition, it is also clear that $\tilde{\omega}^9$ and $\tilde{\omega}^{10}$ are orthogonal to $\{\omega^1, \dots, \omega^8\}$ as ω^9 and u are orthogonal to $\{\omega^1, \dots, \omega^8\}$. $\square$

As a corollary, we have

Corollary 2.4. *The lattice VOA $V_{\sqrt{2}E_8}$ contains a subalgebra isomorphic to*

$$\begin{aligned} \mathfrak{T} = & L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right) \otimes L\left(\frac{25}{28}, 0\right) \\ & \otimes L\left(\frac{11}{12}, 0\right) \otimes L\left(\frac{14}{15}, 0\right) \otimes L\left(\frac{52}{55}, 0\right) \otimes L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 0\right), \end{aligned}$$

Proof. Let $\tilde{\omega}^i = \omega^i$ for $i = 1, 2, \dots, 8$. Then $\{\tilde{\omega}^1, \dots, \tilde{\omega}^{10}\}$ is a set of mutually orthogonal conformal vectors of central charge $\frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}, \frac{25}{28}, \frac{11}{12}, \frac{14}{15}, \frac{52}{55}, \frac{1}{2}$ and $\frac{21}{22}$, respectively. Hence, the subalgebra generated by $\{\tilde{\omega}^1, \dots, \tilde{\omega}^{10}\}$ is isomorphic to $\mathfrak{T}$. $\square$

Remark 2.5. Note that the vector $v = a^1 - a^2$ is a highest weight vector of weight $(0, 0, 0, 0, 0, 0, 0, 1/16, 31/16)$ with respect to $\mathfrak{T}$.

2.3. Decomposition of $V_{\sqrt{2}E_8}$ as $\mathfrak{T}$ -submodules. Next, we shall study the decomposition of $V_{\sqrt{2}E_8}$ as a direct sum of $\mathfrak{T}$ -modules. First, let us recall the following theorem from [13].

Theorem 2.6. *The lattice VOA $V_{\sqrt{2}A_8}$ can be decomposed as*

$$V_{\sqrt{2}A_8} \cong V_N \cong \bigoplus_{\substack{0 \leq k_j \leq j+1, \\ j=0, \dots, 8, \\ k_j \equiv 0 \pmod{2}}} L(c_1, h_{k_0+1, k_1+1}^1) \otimes \dots \otimes L(c_l, h_{k_7+1, k_8}^8) \otimes W(k_8), \quad (2.6)$$

where $W(0)$ is a simple VOA, known as parafermion algebra or W -algebra, of central charge $16/11$ and $W(k), k = 0, 2, 4, 6, 8$, are irreducible $W(0)$ -modules.

Since $V_{\gamma+\sqrt{2}A_8}$ and $V_{-\gamma+\sqrt{2}A_8}$ are irreducible $V_{\sqrt{2}A_8}$ -modules and both of them contain highest weight vectors of weight $(0, 0, 0, 0, 0, 0, 0, 0, 2)$ with respect to T , we also have

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$$V_{\gamma+\sqrt{2}A_8} \cong \bigoplus_{\substack{0 \leq k_j \leq j+1, \\ j=0, \dots, 8, \\ k_j \equiv 0 \pmod{2}}} L(c_1, h_{k_0+1, k_1+1}^1) \otimes \cdots L(c_l, h_{k_7+1, k_8}^8) \otimes P(k_8), \quad (2.7)$$

$$V_{-\gamma+\sqrt{2}A_8} \cong \bigoplus_{\substack{0 \leq k_j \leq j+1, \\ j=0, \dots, 8, \\ k_j \equiv 0 \pmod{2}}} L(c_1, h_{k_0+1, k_1+1}^1) \otimes \cdots L(c_l, h_{k_7+1, k_8}^8) \otimes Q(k_8), \quad (2.8)$$

where $P(k_l)$ and $Q(k_l)$ are irreducible $W(0)$ -modules whose structure are yet to be determined.

Now, let $U = U(0) = \{V \in V_{\sqrt{2}E_8} \mid (\tilde{w}^i)_1 v = 0 \text{ for } i = 1, 2, \dots, 8\}$. Then, U is a VOA of central charge $16/11$ and by combining Corollary 2.4 and (2.6 – 2.8), we have

Theorem 2.7. *The lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as*

$$V_{\sqrt{2}E_8} \cong \bigoplus_{\substack{0 \leq k_j \leq j+1, \\ j=0, \dots, 8, \\ k_j \equiv 0 \pmod{2}}} L(c_1, h_{k_0+1, k_1+1}^1) \otimes \cdots L(c_l, h_{k_7+1, k_8+1}^l) \otimes U(k_8), \quad (2.9)$$

where $U(k) = W(k) + P(k) + Q(k)$, $k = 0, 2, 4, 6, 8$, are $U(0)$ – modules.

Remark 2.8. Let σ be an automorphism of $V_{\sqrt{2}E_8}$ defined by

$$\sigma(u) = e^{\frac{2\pi i}{3}\langle \gamma, \beta \rangle} \quad \text{for any } u \in M(1) \otimes e^\beta \subset V_{\sqrt{2}E_8}.$$

and let θ be an automorphism of $V_{\sqrt{2}E_8}$ induces by the isometry $\beta \rightarrow -\beta$ of $\sqrt{2}E_8$. Then the subgroup generated by σ and θ is isomorphic to S_3 . Moreover, σ and θ induce some nontrivial automorphisms of order 3 and order 2 on the subVOA $U(0)$ respectively. In fact, they induce automorphisms of order 3 and order 2 on the submodules $U(k)$, $k = 0, 2, 4, 6, 8$, also. By abuse of notation, we shall still denote them by σ and θ .

Note also that the automorphism σ is in fact induced from the order 3 symmetry among the 3 cosets of $\sqrt{2}A_8$ in $\sqrt{2}E_8$.

Next let us determine the structure of $U(0)$. Since $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ is rational and contained in $U(0)$, $U(0)$ and $U(k)$, $k = 2, 4, 6, 8$, are direct sum of irreducible $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ -modules. On the other hand,

$$\begin{aligned} &L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0), \quad L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 8), \quad L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{7}{2}), \\ &L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{45}{2}), \quad L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{31}{16}), \quad \text{and} \quad L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{175}{16}), \end{aligned}$$

are the only irreducible modules of $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ which have integral weights.

Hence,

$$\begin{aligned} U(0) = & A_1 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) \oplus A_2 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 8) \\ & \oplus A_3 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{7}{2}) \oplus A_4 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{45}{2}) \\ & \oplus A_5 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{31}{16}) \oplus A_6 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{175}{16}), \end{aligned}$$

where $A_1, \dots, A_6$ are the multiplicities of the irreducible summands.

Similarly, we also have

$$\begin{aligned} U(2) = & B_1 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{13}{11}) \oplus B_2 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{35}{11}) \\ & \oplus B_3 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{15}{22}) \oplus B_4 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{301}{22}) \\ & \oplus B_5 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{21}{176}) \oplus B_6 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{901}{176}), \end{aligned}$$

$$\begin{aligned} U(4) = & C_1 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{50}{11}) \oplus C_2 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{6}{11}) \\ & \oplus C_3 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{1}{22}) \oplus C_4 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{155}{22}) \\ & \oplus C_5 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{85}{176}) \oplus C_6 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{261}{176}), \end{aligned}$$

$$\begin{aligned} U(6) = & D_1 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{111}{11}) \oplus D_2 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{1}{11}) \\ & \oplus D_3 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{35}{22}) \oplus D_4 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{57}{22}) \\ & \oplus D_5 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{533}{176}) \oplus D_6 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{5}{176}), \end{aligned}$$

and

$$\begin{aligned} U(8) = & E_1 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{196}{11}) \oplus E_2 L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, \frac{20}{11}) \\ & \oplus E_3 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{117}{22}) \oplus E_4 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{7}{22}) \\ & \oplus E_5 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{1365}{176}) \oplus E_6 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{133}{176}), \end{aligned}$$

for some suitable B_i, C_i, D_i and E_i . Note that the weights of $U(2), U(4), U(6)$, and $U(8)$ are $2/11 + \mathbb{Z}, 6/11 + \mathbb{Z}, 1/11 + \mathbb{Z}$, and $9/11 + \mathbb{Z}$, respectively.

Now by comparing the characters of the left and the right hand sides of (2.9), we find that all A_i 's, B_i 's, C_i 's, D_i 's, and E_i 's are equal to 1.

Hence we have

$$\begin{aligned} U(0) \cong & L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 0\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 8\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{2}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{45}{2}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{31}{16}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{175}{16}\right), \end{aligned}$$

$$\begin{aligned} U(2) \cong & L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{13}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{35}{11}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{15}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{301}{22}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{21}{176}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{901}{176}\right), \end{aligned}$$

$$\begin{aligned} U(4) \cong & L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{50}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{6}{11}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{1}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{155}{22}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{85}{176}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{261}{176}\right), \end{aligned}$$

$$\begin{aligned} U(6) \cong & L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{111}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{1}{11}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{35}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{57}{22}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{533}{176}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{5}{176}\right), \end{aligned}$$

and

$$\begin{aligned} U(8) \cong & L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{196}{11}\right) \oplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, \frac{20}{11}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{117}{22}\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{22}\right) \\ & \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{1365}{176}\right) \oplus L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{133}{176}\right), \end{aligned}$$

Theorem 2.9. *U is a simple VOA and $U(k)$ for $k = 0, 2, 4, 6, 8$ are irreducible U -modules.*

Proof. Since

$$\begin{aligned} U(0) = & L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0) + L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{45}{2}) \\ & + L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 8) + L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{7}{2}) \\ & + L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{31}{16}) + L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{175}{16}) \end{aligned}$$

as an $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$ -module, by the fusion rules, U is clearly simple.

Now, by the fusion rules and the decomposition, it is also clear that $U(k)$ for $k = 0, 2, 4, 6, 8$, are irreducible as U -modules. $\square$

3. CONFORMAL VECTORS IN U

In this section, we shall compute all the conformal vectors in U . First, we shall note that $\dim U_2 = 3$ and $\{\tilde{\omega} = \omega^9, u, v\}$ forms a basis of U_2 .

Theorem 3.1. *There are exactly 7 conformal vectors in U , namely, the Virasoro element $\tilde{\omega}$ of U , 3 conformal vectors of central charge $1/2$ and 3 conformal vectors of central charge $21/22$.*

Proof. First we shall note that U_2 is spanned by $\{\tilde{\omega}, u, v\}$. Let $x = a\tilde{\omega} + bu + cv$ be a conformal vector in U_2 . Then $x_1x = 2x$. Since $\tilde{\omega}_1\tilde{\omega} = 2\tilde{\omega}$, $\tilde{\omega}_1u = 2u$, $\tilde{\omega}_1v = 2v$, $u_1u = 2(231\tilde{\omega} + 10u)$, $u_1v = -20v$, and $v_1v = 2(-231\tilde{\omega} + 10u)$, by direct computation, we know that

$$\begin{aligned} a^2 + 231b^2 - 231c^2 &= a, \\ 2ab + 10b^2 + 10c^2 &= b, \quad \text{and} \\ 2ac - 20bc &= c. \end{aligned} \tag{3.1}$$

Solving the above equations, we obtain 7 non-trivial solutions, namely,

$$\begin{aligned} \{a = 1, b = 0, c = 0\}, \\ \{a = \frac{11}{32}, b = \frac{1}{32}, c = 0\}, \quad \{a = \frac{21}{32}, b = \frac{-1}{32}, c = 0\}, \\ \{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{\sqrt{-3}}{64}\}, \quad \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{\sqrt{-3}}{64}\}, \\ \{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{-\sqrt{-3}}{64}\}, \quad \{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{-\sqrt{-3}}{64}\}. \end{aligned}$$

When $\{a = 1, b = 0, c = 0\}$, $x = \tilde{\omega}$ is the Virasoro element of U .

When $\{a = \frac{11}{32}, b = \frac{1}{32}, c = 0\}$, $\{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{\sqrt{-3}}{64}\}$, or $\{a = \frac{11}{32}, b = \frac{-1}{64}, c = \frac{-\sqrt{-3}}{64}\}$, $\langle x, x \rangle = 1/4$ and x is a conformal vector of central charge $1/2$.

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When $\{a = \frac{21}{32}, b = \frac{-1}{32}, c = 0\}$, $\{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{\sqrt{-3}}{64}\}$, or $\{a = \frac{21}{32}, b = \frac{1}{64}, c = \frac{-\sqrt{-3}}{64}\}$, $\langle x, x \rangle = 21/44$ and x is a conformal vector of central charge $21/22$. $\square$

Lemma 3.2. *Let $e^1 = \frac{11}{32}w^9 + \frac{1}{32}u$, $e^2 = \frac{11}{32}w^9 - \frac{1}{64}u + \frac{\sqrt{-3}}{64}v$, and $e^3 = \frac{11}{32}w^9 - \frac{1}{64}u - \frac{\sqrt{-3}}{64}v$ be the three rational conformal vectors of central charge $\frac{1}{2}$ in U . Then $\langle e^i, e^j \rangle = \frac{1}{2^8}$ if $i \neq j$.*

Proof. By (2.4), it is easy to show that

$$\langle \omega^9, \omega^9 \rangle = \frac{8}{11}, \quad \langle u, u \rangle = 168, \quad \langle v, v \rangle = -168,$$

and

$$\langle \omega^9, u \rangle = \langle \omega^9, v \rangle = \langle u, v \rangle = 0.$$

Thus, we have

$$\langle e^i, e^j \rangle = \begin{cases} 1/2^8 & \text{if } i \neq j, \\ 1/4 & \text{if } i = j, \end{cases}$$

as desired. $\square$

Theorem 3.3. *Let U_2 be the Griess algebra of U . Then $\text{Aut } U_2 \cong S_3$.*

Proof. Let g be an element of $\text{Aut } U_2$. Then it will induce a permutation on the three conformal vectors e^1, e^2 and e^3 . Since U_2 is generated by e^1, e^2 and e^3 , $\text{Aut } U_2$ must itself be a permutation subgroup on $\{e^1, e^2, e^3\}$. On the other hand, by our construction, $\text{Aut } U_2$ already contains elements of order 3 and order 2, namely σ and θ . Thus $\text{Aut } U_2 \cong S_3$. $\square$

Theorem 3.4. *The full automorphism group of U is isomorphic to S_3 .*

Proof. Let $g \in \text{Aut } U$ and let G be the subgroup of $\text{Aut } U$ generated by σ and θ . Since

$$\text{Aut } U_2 = \{h|_{U_2} \mid h \in G\},$$

there exists an $h \in G$ such that $gh^{-1}|_{U_2} = \text{id}_{U_2}$. In particular, $\rho = gh^{-1}$ will fix the conformal vectors $\tilde{\omega}^9, \tilde{\omega}^{10}$ and thus fixes the subVOA $L(1/2, 0) \otimes L(21/22, 0)$. Hence ρ will map highest weight vectors to highest weight vectors of the same type. Moreover in U highest weight vectors are unique (up to scalar multiple) and ρ preserves their inner product. Hence ρ must fix U . Thus $g = h \in G$ and $\text{Aut } U = G \cong S_3$. $\square$

Remark 3.5. Recall from Miyamoto [14] that for each conformal vector e of central charge $1/2$, one can define an automorphism τ_e by

$$\tau_e = \begin{cases} 1 & \text{on the summands isomorphic to } L(1/2, 0) \text{ or } L(1/2, 1/2), \\ -1 & \text{on the summands isomorphic to } L(1/2, 1/16). \end{cases}$$

In the VOA U , τ_{e^1} actually corresponds the permutation $e^2 \leftrightarrow e^3$ and τ_{e^2} corresponds to $e^1 \leftrightarrow e^3$. On the other hand, the order 3 automorphism σ corresponds to the cyclic permutation $e^1 \rightarrow e^2 \rightarrow e^3 \rightarrow e^1$. Hence we have

$$\sigma = \tau_{e^2}\tau_{e^1}.$$

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