

A New Parametric Method for Finding Efficient Solutions in Biobjective Shortest Route Problems

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§ 1. Preliminaries

Denote the variables representing two quantities we want to minimize in the biobjective programs by x and y . We call the coordinates plane having its orthogonal coordinates x and y the *original plane*, which is denoted by $\mathcal{P}$. The points of the plane $\mathcal{P}$ are written as $a = (x, y)$ and $b = (x', y')$, etc.. An order relation among the points of the plane $\mathcal{P}$ is given by the usual manner. Namely, for two points $a = (x, y)$ and $b = (x', y')$, we write

$$a \leq b \quad \text{iff } x \leq x' \text{ and } y \leq y'. \quad (1)$$

Let Ω be a non-empty finite subset of $\mathcal{P}$. We consider the optimization problem :

$$(P_0) \quad \text{Minimize } a \text{ subject to } a \in \Omega.$$

A point $a \in \Omega$ is said to be an *efficient solution* to the problem (P_0) , if there is no point $b \in \Omega$ such that $a \geq b$ and $a \neq b$.

For $t \in (0, 1)$, define a 2×2 matrix $G(t)$ by

$$G(t) = \begin{bmatrix} t & 1 \\ t & -1 \end{bmatrix}. \quad (2)$$

We call the matrix $G(t)$ a *transformation matrix*, and t is called a *transformation parameter*. Trivially, the matrix $G(t)$ is nonsingular for every $t \in (0, 1)$. Let

$$\begin{bmatrix} u \\ \alpha \end{bmatrix} = G(t) \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{P}. \quad (3)$$

The coordinates plane whose orthogonal coordinates are given by u and α of (4) is called the *transformed plane*, which is denoted by $\mathcal{H}$. The points of the plane $\mathcal{H}$ are written as $A = (u, \alpha)$ and $B = (v, \beta)$, etc..

Proposition 1. Let $a = (x, y)$ and $b = (x', y')$ be two points of $\mathcal{P}$. Let $0 < t \leq 1$ be arbitrary but fixed. Let $A = (u, \alpha)$ and $B = (v, \beta)$ be the points of $\mathcal{H}$ transformed from a and b by the transformation matrix $G(t)$,

respectively. Then the order relation $a \leq b$ is equivalently transformed to a relation on $\mathcal{H}$ as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} x' \\ y' \end{bmatrix} \Leftrightarrow |\alpha - \beta| \leq v - u. \quad (4)$$

§ 2. An order relation on the transformed plane

Definition 1. Let $0 \leq \lambda \leq 1$ be arbitrary. For two points $A = (u, \alpha)$ and $B = (v, \beta)$ of the transformed plane, we define an order relation $\leq^\lambda$ with the parameter λ by the following:

$$A \leq^\lambda B \stackrel{\text{def}}{\Leftrightarrow} \begin{cases} \text{(i)} & |\alpha - \beta| \leq v - u, \\ & \text{or} \\ \text{(ii)} & 0 < \lambda(\beta - \alpha) \leq |v - u| < \beta - \alpha, \\ & \text{or} \\ \text{(iii)} & 0 < v - u < \lambda|\alpha - \beta|, \\ & \text{or} \\ \text{(iv)} & u = v \text{ and } \alpha < \beta. \end{cases} \quad (5)$$

The four cases in the right-hand side of (5) are exclusive one another. Let a and b be the points transformed from A and B , respectively, by $G(t)^{-1}$.

Proposition 1, then, implies that the condition (i) in the right-hand side of (5) is equivalent to the relation $a \leq b$. This fact holds true regardless of the value of the transformation parameter t .

Proposition 2. For each $0 \leq \lambda \leq 1$, the order relation $\leq^\lambda$ is reflexive and asymmetric on $\mathcal{H}$.

Proposition 3. For any pair $A = (u, \alpha)$ and $B = (v, \beta)$ of points in the plane $\mathcal{H}$ and for any $0 \leq \lambda \leq 1$, either $A \leq^\lambda B$ or $B \leq^\lambda A$ necessarily holds.

Proposition 4. Let $0 \leq \lambda \leq 1$ be arbitrary, and let A and B be two points on $\mathcal{H}$. For every positive number μ and every point C on $\mathcal{H}$, then, it holds that

$$A \leq^\lambda B \Rightarrow \mu A \leq^\lambda \mu B \text{ and } A \pm C \leq^\lambda B \pm C.$$

§ 3. Descent sequence on the transformed plane

Proposition 5. Let $A = (u, \alpha)$ and $B = (v, \beta)$ be two points on $\mathcal{H}$ satisfying that

$$u < v \text{ and } \alpha < \beta. \quad (6)$$

Then A is smaller than B with respect to the order $\leq^\lambda$ for every $0 \leq \lambda \leq 1$.

Definition 2. Let $\{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$, where $m \geq 3$, be a finite sequence of points on $\mathcal{H}$. Then the sequence is said to be *descending to the right*, iff it holds that

$$\left. \begin{array}{l} u_i < u_{i+1}, \\ \alpha_i > \alpha_{i+1}, \end{array} \right\} \quad i = 1, 2, \dots, m-1. \quad (7)$$

For a sequence $\{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$ of points, if it holds that

$$\left. \begin{array}{l} u_i > u_{i+1}, \\ \alpha_i < \alpha_{i+1}, \end{array} \right\} \quad i = 1, 2, \dots, m-1, \quad (8)$$

then the sequence is, of course, descending to the right, by renumbering the index of points. But, in order to unify the numbering of points, when we speak of a sequence descending to the right, we suppose to imply the condition (7) but not (8).

We denote the gradient of the line segment connecting two points a and b on the original plane by γ_{ab} . Similarly, we denote the gradient of the line segment connecting two points A and B on the transformed plane by γ_{AB} .

Theorem 1. Let $\{a_i = (x_i, y_i) ; i = 1, 2, \dots, m\}$, where $m \geq 3$, be a sequence of points on $\mathcal{P}$ such that

$$\left. \begin{array}{l} x_i > x_{i+1}, \\ y_i < y_{i+1}, \end{array} \right\} \quad i = 1, 2, \dots, m-1. \quad (9)$$

Suppose that

$$\gamma_{a_{i-1}a_i} < \gamma_{a_i a_{i+1}}, \quad i = 2, 3, \dots, m-1. \quad (10)$$

Choose t such that $0 < t < \text{Min} \left\{ \left| \gamma_{a_{m-1}a_m} \right|, 1 \right\}$. Define (u_i, α_i) , $i = 1, 2, \dots, m$,

by

$$\begin{bmatrix} u_i \\ \alpha_i \end{bmatrix} = G(t) \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i = 1, 2, \dots, m, \quad (11)$$

and let $\mathcal{A} = \{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$. Then it holds that

- (i) The sequence $\mathcal{A}$ is descending to the right:
- (ii) $0 > \gamma_{A_{i-1}A_i} > \gamma_{A_i A_{i+1}}, \quad i = 2, 3, \dots, m-1.$

§ 4. Detection of efficient solutions

Definition 3. For two points $A = (u, \alpha)$ and $B = (v, \beta)$ on $\mathcal{H}$, if it holds that

$$|\alpha - \beta| > |u - v|, \quad (12)$$

then the points are said to be *mutually nondominant*. For a finite sequence $\{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$ of points on $\mathcal{H}$, if every two elements of the sequence are mutually nondominant, then it is said that the *sequence is mutually nondominant*.

Proposition 6. Let $A = (u, \alpha)$ and $B = (v, \beta)$ be points on $\mathcal{H}$. Let $t \in (0, 1)$ be arbitrary, and let $a = (x, y)$ and $b = (x', y')$ be the points transformed by $G(t)^{-1}$ from A and B , respectively. Then, A and B are mutually nondominant, if and only if, the relation

$$\left\{ \begin{array}{l} x < x' \text{ and } y > y', \\ \text{or} \\ x > x' \text{ and } y < y', \end{array} \right.$$

holds.

As we have stated in the preceding section, the order relation $\leq^\lambda$ is reflexive and asymmetric but not transitive on the whole plane $\mathcal{H}$. However, it can be shown that if we restrict ourselves to the family of sequences which are mutually nondominant and descending to the right, then the relation $\leq^\lambda$ is transitive on each of the sequences.

Theorem 2. Let $\mathcal{A} = \{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$ be mutually nondominant and descending to the right. Then the relation $\leq^\lambda$ is a total order relation on the sequence $\mathcal{A}$ for every $\lambda \in [0, 1]$.

Let $\mathbf{M}$ denote the set $\{1, 2, \dots, m\}$. Throughout the remainder of this section, it is assumed that $m \geq 3$.

Proposition 6. Let $\mathcal{A} = \{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$ be mutually nondominant and descending to the right. Suppose that the relations

$$\gamma_{A_{i-1}A_i} > \gamma_{A_iA_{i+1}}, \quad i = 2, 3, \dots, m-1. \quad (13)$$

hold. For each $k \in \mathbf{M}$, put

$$\lambda_{ki} = \frac{u_k - u_i}{\alpha_i - \alpha_k} \quad \text{for } i \in \mathbf{M} \setminus \{k\}. \quad (14)$$

Then we have

(i) for each $k \in \mathbf{M}$,

$$0 < \lambda_{ki} < 1 \quad \text{for } i \in \mathbf{M} \setminus \{k\}, \quad (15)$$

(ii) for each $k \in \mathbf{N}$,

$$\lambda_{ki} = \lambda_{ik} \quad \text{for } i \in \mathbf{M} \setminus \{k\}, \quad (16)$$

(iii) for each $k \in \mathbf{N}$,

$$\lambda_{ki} > \lambda_{k,i+1} \quad \text{for } i \in \mathbf{M} \setminus \{k-1, k, m\}, \quad (17)$$

$$(iv) \quad \lambda_{k-1,k} > \lambda_{k,k+1} \quad \text{for } k = 2, 3, \dots, m-1. \quad (18)$$

Now, let $\{a_i = (x_i, y_i) ; i = 1, 2, \dots, m\}$ be the whole set of efficient solutions to the problem (P_0) . Without loss of generality, we may assume that

$$\left. \begin{array}{l} x_i > x_{i+1}, \\ y_i < y_{i+1}, \end{array} \right\} \quad i = 1, 2, \dots, m-1. \quad (19)$$

It is well known that if the efficient solutions $\{a_i = (x_i, y_i) ; i = 1, 2, \dots, m\}$ satisfy the condition :

$$\gamma_{a_{i-1}a_i} > \gamma_{a_i a_{i+1}}, \quad i = 2, 3, \dots, m-1, \quad (20)$$

in addition to (37), then the solutions can be all detected by the usual scalarization method. In this paper, we consider another condition :

$$\gamma_{a_{i-1}a_i} < \gamma_{a_i a_{i+1}}, \quad i = 2, 3, \dots, m-1. \quad (21)$$

Theorem 3. Let $\{a_i = (x_i, y_i) ; i = 1, 2, \dots, m\}$ be the whole set of efficient solutions to the problem (P_0) . Suppose the conditions (19) and (21) to be satisfied. Choose t such that $0 < t < \text{Min} \left\{ \left| \gamma_{a_{m-1}a_m} \right|, 1 \right\}$, and define $\mathcal{A} = \{A_i = (u_i, \alpha_i) ; i = 1, 2, \dots, m\}$ by (11).

Then the sequence $\{\lambda_{ki}\}$ defined by (14) generates a partition of $[0, 1]$:

$$\begin{aligned} 1 &> \lambda_{12} > \lambda_{23} > \dots > \lambda_{k-1,k} > \lambda_{k,k+1} \\ &> \dots > \lambda_{m-2,m-1} > \lambda_{m-1,m} > 0, \end{aligned} \quad (22)$$

such that, with respect to the order criterion $\leq^\lambda$,

- (i) A_1 is the smallest one among $\mathcal{A}$ iff $1 \geq \lambda > \lambda_{12}$,
- (ii) for each k ($2 \leq k \leq m-1$), A_k is the smallest one among $\mathcal{A}$ iff $\lambda_{k-1,k} \geq \lambda > \lambda_{k,k+1}$,
- (iii) A_m is the smallest among $\mathcal{A}$ iff $\lambda_{m-1,m} \geq \lambda \geq 0$.

§ 5. Applications to biobjective shortest route problems

We consider a directed network (N, A, Γ) , where $N = \{1, 2, \dots, N\}$ is a finite set of nodes, A is a set of arcs whose elements are ordered pairs (i, j)

of distinct nodes, and $\Gamma = \{ \gamma_{ij} = (\gamma^1_{ij}, \gamma^2_{ij})^T \mid (i, j) \in A \}$: $\gamma_{ij} = (\gamma^1_{ij}, \gamma^2_{ij})^T$ denotes a biobjective distance attached to the directed arc (i, j) . Node 1 is assigned to a starting node, and node N to a terminal node.

Choose a transformation parameter t of an appropriate value, and transform the original data by the matrix $G(t)$. Put

$$T_{ij} = G(t) \gamma_{ij}, \quad \text{for } (i, j) \in A.$$

Let $A(i)$ denote the set of terminal nodes of all arcs emanating from node i .

Definition 4. For two points $A = (u, \alpha)$ and $B = (v, \beta)$ on $\mathcal{H}$, we define the relations $\prec$ by

$$A \prec B \quad \text{iff} \quad [|\alpha - \beta| \leq v - u \quad \text{and} \quad A \neq B].$$

Algorithm modified Dijkstra ;

begin

Choose $\lambda \in (0, 1)$ arbitrarily;

$S := N$;

$d(i) := +\infty$ for each node $i \in N \setminus \{1\}$;

$d(1) := 0$ and $\text{pred}(1) := 0$;

while $S \neq \emptyset$ **do**

begin

$D := \{ j \in S \mid (\exists i \in S) (d(i) \prec d(j)) \}$;

$M := S \setminus D$;

let $i_0 \in M$ be a node satisfying that $d(i_0) \leq^\lambda d(j)$ for $\forall j \in M$;

$S := S \setminus \{i_0\}$;

for each $j \in A(i_0)$ **do**

if $d(i_0) + T_{i_0 j} \leq^\lambda d(j)$ and $d(i_0) + T_{i_0 j} \neq d(j)$ **then**

$d(j) := d(i_0) + T_{i_0 j}$ and $\text{pred}(j) := i_0$;

end;

end.