

RESEARCH ARTICLE

Coupled finite element analysis of the dynamics of poroelastic media considering the relative fluid acceleration

Jiawei Xu^{*1} | Ryosuke Uzuoka² | Kyohei Ueda²

¹School of Highway, Chang'an University, Xi'an, Shaanxi 710064, China

²Disaster Prevention Research Institute, Kyoto University, Gokasho, Uji, Kyoto 611-0011, Japan

Correspondence

*Jiawei Xu, School of Highway, Chang'an University, Xi'an, Shaanxi 710064, China. Email: xu_jw1023@163.com

Present Address

Middle Section of South 2nd Ring Road, Xi'an, Shaanxi 710064, China

Abstract

This paper mainly discusses the dynamics of poroelastic media using the finite element analysis based on the $u-v-p$ full formulation, where u , v , and p denote the solid displacement, relative fluid velocity with respect to solid velocity, and pore fluid pressure. It incorporates the effect of relative fluid acceleration with respect to solid on soil dynamic response. The $u-v-p$ formulation is first verified through the comparison with the analytical solution. After that, the response of one-dimensional saturated soil column and two-dimensional partially saturated soil layer subjected to vertical loading with different combinations of soil permeability and loading frequency are investigated using the analyses with the $u-v-p$ and $u-p$ formulations, based on which the effect of relative pore fluid motion is discussed and the differences in the soil response between two kinds of analysis approaches are examined. Results reveal that the rapid fluid flow has a pronounced effect on soil dynamics and the soil with a greater degree of saturation is more sensitive to the increase of loading frequency and soil permeability. The soil dynamic response can basically be divided into two main categories that represent insignificant and significant flow motion depending on loading frequency and soil permeability. Furthermore, despite vertical loading, horizontal soil response can also be affected by the large relative pore fluid flow when loading frequency and soil permeability are large. In addition, the simplified formulation can still be applicable to predict dynamics of poroelastic media in cases with high loading frequency and low soil permeability.

KEYWORDS:

Coupled analysis; poroelastic media; dynamics; full formulation; relative fluid acceleration

1 | INTRODUCTION

Modeling the dynamics of porous media is of great importance in geotechnical engineering problems. Since porous media generally consists of solid, water, and/or air, it falls into the category of multiphase material, the dynamic behavior of which can be described using macroscopic continuum mechanics and porous media theory¹. The governing equations in the porous media theory describe the mass and linear momentum balance equations of porous media, thus becoming a robust tool to determine the motion of solid and fluid as well as their interaction. With the adoption of the appropriate analysis approach, the coupled deformation and flow problem can be solved, the result of which is then applicable to predict the hydromechanical response of soil in geotechnical engineering. In the study of porous media dynamics, Biot first proposed equations for saturated porous media

under quasi-static load² and extended it to dynamics^{3,4,5}. Later on, different formulations were developed for the dynamics and statics of porous media and were named after the primary variables needed to be solved in the analysis, and they can generally be classified into full and simplified formulations. The full formulation represents the whole set of Biot equations (full Biot), while the simplified formulation is derived based on some assumptions. The primary variables are usually solid displacement $\mathbf{u}$, fluid displacement $\mathbf{U}$, average fluid velocity $\mathbf{w}$ or relative fluid velocity $\mathbf{v}$, average relative fluid displacement $\bar{\mathbf{u}}$ ($\mathbf{w} = n\mathbf{v} = n(\dot{\mathbf{U}} - \dot{\mathbf{u}})$ and $\bar{\mathbf{u}} = n(\mathbf{U} - \mathbf{u})$, where n is the porosity), and fluid pressure p . The commonly applied three-variable full formulations are $\mathbf{u}-\mathbf{w}-p$ formulation^{3,4,5,6,7,8,9,10,11,12}, $\mathbf{u}-\mathbf{U}-p$ formulation^{13,14,15}, and $\mathbf{u}-\bar{\mathbf{u}}-p$ formulation^{19,20}. Furthermore, when fluids are assumed to be compressible, the pore pressure variable p can be determined explicitly; as for incompressible fluids, it can also be determined using the penalty parameter. Two-variable formulations can also be found written as $\mathbf{u}-\mathbf{U}$ formulation^{13,14,23,24,25,26,27}, $\mathbf{u}-\mathbf{w}$ formulation^{28,29}, $\mathbf{u}-\bar{\mathbf{u}}$ formulation^{14,16,17,18}, and $\dot{\mathbf{u}}-\mathbf{w}$ ^{21,22}. Nevertheless, when the relative fluid velocity is rather small compared with the solid velocity, the relative fluid acceleration can be neglected, then the generally non-zero relative fluid velocity and the relative fluid displacement can be eliminated from the governing equations, making the solid displacement $\mathbf{u}$ and fluid pressure p primary variables and thus leading to the simplified $\mathbf{u}-p$ formulation^{6,14,19,16,31,32,33,34,35,36,37,38,39,40,41,42,43,44}.

Zienkiewicz et al.⁶ investigated the applicability of the full Biot theory, $\mathbf{u}-p$ formulation, and quasi-static consolidation formulation based on the one-dimensional analytical study of linearly elastic saturated soil under periodic surface loading. Three zones of applicability of various assumptions were also proposed to estimate which formulation was appropriate when the ranges of loading frequency and soil permeability were available. Gajo et al.¹⁴ evaluate the accuracy and performance of the $\mathbf{u}-\mathbf{U}-p$ formulation in studying the propagation of harmonic waves in fluid-saturated linear isotropic porous media, which was also compared with those of $\mathbf{u}-\mathbf{U}$ and $\mathbf{u}-\bar{\mathbf{u}}$ formulations. It was found that the $\mathbf{u}-\mathbf{U}-p$ formulation was capable of resolving the problem with incompressible pore fluid and solid grain and it became better than other two formulations as the compressibility of pore fluid and solid grain increased. Jeremic et al.¹⁵ employed the $\mathbf{u}-\mathbf{U}-p$ formulation to describe the dynamics of fully saturated porous materials, where the viscous damping is appropriately modeled and simulated by taking into account the interaction between solid and fluid. With the use of the formulation, liquefaction of saturated level and sloping grounds were investigated, involving both loose and dense sand conditions. Ulker and Rahman¹⁹ examined the fully dynamic, partly dynamic, and quasi-static formulations using the analytical approaches with the $\mathbf{u}-\bar{\mathbf{u}}-p$ formulation and concluded that the regions of validity for three different formulations mainly dependent on three dimensionless parameters relating to soil permeability, loading frequency, and spatial variation of loading. The range of applicability of these formulations were also suggested in the study. Navas et al.¹⁸ investigated the applicability of the proposed meshfree numerical scheme with the $\mathbf{u}-\bar{\mathbf{u}}$ formulation to the dynamic behavior of one-dimensional soil in three zones⁶ featuring different soil permeability and loading frequencies, finding that the larger permeability and frequency led to more unstable and higher peaks of pore water pressure. Toyoda and Noda¹¹ performed the one-dimensional consolidation analysis of a linearly elastic saturated soil model with a wide range of soil permeability coefficients to evaluate the performance of the $\mathbf{u}-\mathbf{w}-p$ formulation, based on which characteristics such as the S-shaped settlement–time relation, non-harmonic oscillation in the settlement/pressure–time relation in highly permeable soil, and inconsistency in the vertical load and immediate pressure were identified in the study.

For soil dynamics problems with a common frequency range encountered in the geotechnical earthquake engineering, the analysis based on the $\mathbf{u}-p$ formulation is generally thought capable of guaranteeing sufficient accuracy^{6,30} and it is usually adopted for the merits of fairly reasonable assumptions in usual cases; however, since the applicability of different formulations is dependent on both the loading frequency and soil permeability, it still remains unclear for two-dimensional problem whether the simplified formulation is applicable when both high-permeability soil and large-frequency loading are involved. In fact, the full formulation is preferable to the simplified version when dealing with high-frequency and high-permeability problem since it describes the fluid flow in more detail. The high-frequency problem can be found in blast loading that has a typically shorter excitation period, e.g. 0.005 s–2 s⁴⁶. Since loading frequency can affect the strength and moduli of soil, which will in turn have an impact on the behavior of soil under different loading conditions, it is necessary to investigate the soil response considering the effect of high frequency. On the other hand, the high-permeability problem can be encountered when dealing with sandy soil, which is usually classified to be highly permeable and drained, so the effect of permeability should be examined in the dynamic analysis. Another case where high-permeability may arise is the dynamic centrifuge modeling of soil response. The permeability of saturated soil sample is usually reduced by a factor N (N is the ratio of the centrifugal acceleration to the Earth's gravity) to satisfy the scaling law for the permeability coefficient. This is usually achieved by mixing a fluid N times more viscous than water into soil so that dynamic time will match the diffusion time. However, when water is used as the pore fluid in the soil sample^{47,48}, the scaling law is thus not perfectly satisfied and the flow is regarded to be N times faster. In this situation, the fast flow may not be neglected especially for a large N value such as 50 or even greater, hence the impact of fast fluid flow

on soil dynamics should be properly evaluated when the numerical analysis is carried out to reproduce or validate against the physical modeling outcome.

The literature review also indicates that the main work on soil dynamics considering the effects of soil permeability or loading frequency is usually based on the one-dimensional or simple two-dimensional soil model. Whether the simplified soil model is applicable to complex problem in geotechnical engineering still needs to be investigated into. In addition, the dynamics of unsaturated soil that is commonly encountered in the geotechnical problems has not been widely studied yet, so the analysis approach for the dynamics of unsaturated porous media is still an important subject requires further attention.

Table 1 Comparison of typical formulations for porous media dynamics

Formulation	Comment	Definition of primary variable
$u-U-p$	It represents the whole set of Biot equations ⁶ ; it can treat pore fluid as incompressible ¹⁵ ; it is applicable to any range of frequency ⁶ ; it is difficult to track the large deformation of fluid.	u : solid displacement; U : pore fluid displacement;
$u-U$	It is the reduced format of $u-w-p$ formulation when pore fluid or solid is compressible ¹⁴ ; it may induce volumetric locking when large difference in volumetric compressibility between pore fluid and solid exists ¹⁵ ; it is difficult to track the large deformation of fluid.	w : average relative velocity of pore fluid with respect to solid velocity, i.e., Darcy velocity; v : relative velocity of pore fluid with respect to solid velocity, i.e., seepage velocity;
$u-w-p$	It represents the whole set of Biot equations ¹⁰ ; it enables continuous w even on the discontinuous porosity surface ¹⁰ ; it has discontinuous v at nodes ¹⁰ .	$w = nv$;
$u-w$	It is the reduced format of $u-w-p$ formulation when pore fluid or solid is compressible ¹⁴ ; it enables the continuous w even on the discontinuous porosity surface; it has discontinuous v at nodes.	n : porosity; p : pore fluid pressure.
$u-v-p$	It is simple since v can be treated directly in the balance equations ^{40,41} ; v can be continuous at nodes ^{40,41} .	
$u-p$	It simplifies porous media dynamics without considering the relative acceleration between pore fluid and solid ^{6,14,38,41} ; it requires special numerical treatment to prevent volumetric locking ^{7,20,38,41,45} .	

Among the abovementioned formulations, the commonly used ones listed in Table 1 are worth extra attention. The $u-p$ formulation captures the movement of soil skeleton and the change in pore pressure and it has a simpler expression compared to other types of formulations. This formulation neglects the relative acceleration of pore fluid with respect to solid^{6,14,38,41}. In the case of incompressible pore fluid and very low permeability, the formulation requires special treatment of the approximation function for pore fluid to prevent volumetric locking^{7,20,38,41,45}. The majority of the currently available implementations are based on this formulation as it can produce reasonably reliable results in most coupled problems where the permeability and loading frequency are insignificant. The $u-U-p$ formulation represents the whole set of Biot equations⁶ and uses an additional unknown variable of pore fluid pressure to stabilize the solution of the coupled system¹⁵. The pore fluid pressure is dependent on pore fluid displacement as it can be calculated once the volumetric compressibility of pore fluid is known. This formulation can treat pore fluid as incompressible and simplifies the description of the physical model for fluid permeation¹⁵; it is applicable to any range of frequency as it does not make additional assumptions on pore fluid motion imposed by the $u-p$ formulation⁶. The $u-U$ formulation, which is reduced from the $u-U-p$ formulation, can be obtained after eliminating p from the continuity equation when the pore fluid or solid is compressible¹⁴. The $u-U$ formulation tracks the movement of both soil skeleton and pore fluid and it is complete in the sense of basic variable; however, it may experience volumetric locking in situations featuring large difference in volumetric compressibility between pore fluid and solid skeleton¹⁵ and the direct treatment of fluid displacement U can cause numerical difficulty since it is difficult to track the large deformation of fluid. The $u-w-p$ formulation also represents the whole set of Biot equations¹⁰, which enables the achievement of continuous Darcy velocity w even on the discontinuous porosity surface¹⁰. Its reduced format—the $u-w$ formulation—can also be derived by eliminating p from the continuity equation when pore fluid or solid is compressible¹⁴. Similar to the $u-w-p$ formulation, the $u-p$ formulation has continuous w even on

the discontinuous porosity surface but the seepage velocity $\mathbf{v}$ at nodes is discontinuous. The main difference between $\mathbf{u}-\mathbf{w}-p$ formulation and $\mathbf{u}-\mathbf{v}-p$ formulation is that the former enables the continuous $\mathbf{w}$ even on the discontinuous porosity surface¹⁰. However, since porosity n is defined at integration points, $\mathbf{w}$ can be calculated at integration points to ensure the continuous $\mathbf{v}$ at nodes^{40,41}. Furthermore, the $\mathbf{u}-\mathbf{v}-p$ formulation is simple since $\mathbf{v}$ can be treated directly in the balance equations^{40,41}. In comparison with other formulations, the $\mathbf{u}-\mathbf{v}-p$ formulation is still less applied to the analysis of porous media dynamics at present. As an effective tool in predicting the overall behavior of porous media, the analysis based on $\mathbf{u}-\mathbf{v}-p$ formulation needs thorough examination and its applicability merits detailed investigation.

In this study, the $\mathbf{u}-\mathbf{v}-p$ formulation, which does not neglect the relative acceleration of fluid with respect to solid, is proposed to express the governing equations for soil dynamics. The effects of the relative fluid acceleration are discussed where the poroelastic dynamics is the focus. The objective is to examine the effects of soil permeability and loading frequency on soil dynamics especially two-dimensional elastic soil dynamics and investigate the applicability of the $\mathbf{u}-\mathbf{v}-p$ and $\mathbf{u}-p$ formulations to the corresponding analysis.

2 | THEORETICAL FRAMEWORK

2.1 | Governing equations

Porous media is modeled by the three-phase mixture consisting solid, water, and air. The governing equations for the porous media can be written in five major formulations, namely, the conservation of momentum for the whole porous media, the conservation of momentum for the water phase, the conservation of momentum for the air phase, the conservation of mass for the water phase, and the conservation of mass for the air phase. The detailed description of related equations and the derivation of the governing equations are supplemented in Appendix A.

2.1.1 | $\mathbf{u}-\mathbf{v}-p$ formulation

In the $\mathbf{u}-\mathbf{v}-p$ formulation, it is assumed that the soil particle is incompressible, temperature is constant, and no mass exchange exists between the solid particle and fluid.

The momentum balance equation of the mixture is written as:

$$\rho \mathbf{a}^s + \rho^w \left[\frac{D^s \mathbf{v}^{ws}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{ws})\} \mathbf{v}^{ws} \right] + \rho^a \left[\frac{D^s \mathbf{v}^{as}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{as})\} \mathbf{v}^{as} \right] = \text{div} \boldsymbol{\sigma} + \rho \mathbf{b} \quad (1)$$

where ρ , ρ^w , and ρ^a are the density of the mixture, pore water, and pore air; $\mathbf{a}^s$ is the solid acceleration; $\mathbf{v}^s$, $\mathbf{v}^{ws}$, and $\mathbf{v}^{as}$ are the solid velocity and relative velocities of pore water and pore air with respect to solid; $\boldsymbol{\sigma}$ is the total stress tensor; $\mathbf{b}$ is the body force.

The momentum balance equation of the pore water is written as:

$$\rho^{wR} \left[\mathbf{a}^s + \frac{D^s \mathbf{v}^{ws}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{ws})\} \mathbf{v}^{ws} \right] = -\text{grad} p^w + \rho^{wR} \mathbf{b} - n s^w \rho^{wR} g (k^w)^{-1} \mathbf{v}^{ws} \quad (2)$$

where ρ^{wR} is the intrinsic density of pore water, which has the relationship with the density of pore water ρ^w defined in Equation A5 in Appendix A; p^w is the pore water pressure; n is the porosity; s^w is the degree of water saturation; k^w is the permeability coefficient of pore water; g is the gravitational acceleration.

The momentum balance equation of the pore air is written as:

$$\rho^{aR} \left[\mathbf{a}^s + \frac{D^s \mathbf{v}^{as}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{as})\} \mathbf{v}^{as} \right] = -\text{grad} p^a + \rho^{aR} \mathbf{b} - n s^a \rho^{aR} g (k^a)^{-1} \mathbf{v}^{as} \quad (3)$$

where ρ^{aR} is the intrinsic density of pore air, which has the relationship with the density of pore water ρ^w defined in Equation A6 in Appendix A; p^a is the pore air pressure; s^a is the degree of air saturation; k^a is the permeability coefficient of pore air.

The mass balance equation of the pore water is written as:

$$\left(\frac{n s^w \rho^{wR}}{K^w} - c n \rho^{wR} \right) \frac{D^s p^w}{Dt} + c n \rho^{wR} \frac{D^s p^a}{Dt} + s^w \rho^{wR} \text{div} \mathbf{v}^s + \text{div} (n s^w \rho^{wR} \mathbf{v}^{ws}) = 0 \quad (4)$$

where K^w is the bulk modulus of pore water; c is the specific moisture capacity of the soil-water characteristics curve (SWCC), i.e., the derivative of the water saturation with respect to the suction, which may be obtained with the aid of SWCC.

The mass balance equation of the pore air is written as:

$$\left(\frac{ns^a \rho^{aR}}{K^a} - cn\rho^{aR} \right) \frac{D^s p^a}{Dt} + cn\rho^{aR} \frac{D^s p^a}{Dt} + s^a \rho^{aR} \text{div } \mathbf{v}^s + \text{div} (ns^a \rho^{aR} \mathbf{v}^{as}) = 0 \quad (5)$$

where K^a is the bulk modulus of pore air.

Equations 1–5 constitute the governing equations of the $\mathbf{u}$ – $\mathbf{v}$ – p formulation with primary variables of $\mathbf{u}^s$, $\mathbf{v}^{ws}$, $\mathbf{v}^{as}$, p^w , and p^a .

2.1.2 | $\mathbf{u}$ – p formulation

In the $\mathbf{u}$ – p formulation, it is further assumed that the relative accelerations of pore fluid (pore water and pore air) with respect to solid and the convection terms are neglected, in addition to the assumptions made in the $\mathbf{u}$ – $\mathbf{v}$ – p formulation.

When the time derivatives of the relative velocity of pore fluid with respect to solid, as well as the convection terms, in Equations 1, 2, and 3 are neglected, the momentum balance equations for the mixture, pore water, and pore air in the $\mathbf{u}$ – p formulation are written as:

$$\rho \mathbf{a}^s = \text{div } \boldsymbol{\sigma} + \rho \mathbf{b} \quad (6)$$

Substitution of the momentum balance equations of the pore water and air in Equations 2 and 3 with the omission of the time derivatives of the relative velocity of pore fluid with respect to solid as well as the convection terms into the mass balance equations of pore fluids results in

$$\left(\frac{ns^w \rho^{wR}}{K^w} - cn\rho^{wR} \right) \frac{D^s p^w}{Dt} + cn\rho^{wR} \frac{D^s p^w}{Dt} + s^w \rho^{wR} \text{div } \mathbf{v}^s + \text{div} \left(\frac{k^w}{g} (-\text{grad } p^w + p^{wR} \mathbf{b} - \rho^{wR} \mathbf{a}^s) \right) = 0 \quad (7)$$

$$\left(\frac{ns^a \rho^{aR}}{K^a} - cn\rho^{aR} \right) \frac{D^s p^a}{Dt} + cn\rho^{aR} \frac{D^s p^a}{Dt} + s^a \rho^{aR} \text{div } \mathbf{v}^s + \text{div} \left(\frac{k^a}{g} (-\text{grad } p^a + p^{aR} \mathbf{b} - \rho^{aR} \mathbf{a}^s) \right) = 0 \quad (8)$$

Equations 6–8 constitute the governing equations of the $\mathbf{u}$ – p formulation with primary variables of $\mathbf{u}^s$, p^w , and p^a .

2.2 | Soil–water characteristic curve

Soil–water characteristic curve (SWCC) describes the relationship between the effective degree of water saturation s_e^w and suction p^c and can be mathematically written using the VG model⁴⁹:

$$s_e^w = \frac{s_s^w - s_r^w}{s_s^w - s_r^w} = \left\{ 1 + (\alpha p^c)^n \right\}^{-m} \quad (9)$$

where s_s^w and s_r^w are the maximum and minimum degrees of water saturation. α , n , and m are the material parameters; suction p^c is equal to $p^a - p^w$ and there is

$$\frac{D^s s^w}{Dt} = c \frac{D^s p^c}{Dt} = \frac{\partial s^w}{\partial p^c} \frac{D^s p^c}{Dt} \quad (10)$$

where c is the specific moisture capacity of SWCC and varies with the suction or degree of saturation, and it can be obtained by differentiating Equation 9 with respect to p^c .

2.3 | Permeability coefficient function

Permeability coefficient function (PCF) describes the relationship of the permeability coefficient k^w and the effective degree of water saturation s_e^w , which takes the following form in the VG model:

$$k^w = k_s^w (s_e^w)^\zeta \left[1 - \left\{ 1 - (s_e^w)^{\frac{1}{m}} \right\}^m \right]^2 \quad (11)$$

$$k^a = k_s^a (1 - s_e^w)^\eta \left\{ 1 - (s_e^w)^{\frac{1}{m}} \right\}^{2m} \quad (12)$$

where k_s^w and k_s^a are the saturated water and air permeability coefficients; ζ and η are the constant material parameters.

2.4 | Effective stress

The Cauchy effective stress of solid skeleton is assumed as an additive decomposition of the form³⁸

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma}'_{\text{inv}} + \boldsymbol{\sigma}'_{\text{vis}} \quad (13)$$

where $\boldsymbol{\sigma}'_{\text{inv}}$ and $\boldsymbol{\sigma}'_{\text{vis}}$ are the inviscid and viscous parts of $\boldsymbol{\sigma}'$ respectively. The inviscid part of the Cauchy effective stress can be obtained by the hyperelastic constitutive equation (neo-Hookean model⁵⁰) as

$$\boldsymbol{\sigma}'_{\text{inv}} = \frac{\mu}{J^s}(\mathbf{b}^s - \mathbf{I}) + \frac{\lambda}{J^s}(\ln J^s)\mathbf{I} \quad (14)$$

where λ and μ are the Lamé's constants of neo-Hookean model. The Jacobian J^s is defined as $\det(\mathbf{F}^s)$ with the deformation gradient $\mathbf{F}^s$ of solid skeleton. The left Cauchy-Green tensor $\mathbf{b}^s$ of solid skeleton is defined as $\mathbf{F}^s(\mathbf{F}^s)^T$. The viscous part of the Cauchy effective stress for the solid skeleton can be obtained by Kelvin constitutive equation⁵¹ as

$$\boldsymbol{\sigma}'_{\text{vis}} = \alpha_{\text{vis}} \mathbf{c} : \mathbf{d}^s \quad (15)$$

where α_{vis} is the material parameter reflecting the viscous damping characteristics of the solid skeleton and $\mathbf{d}^s$ is the deformation rate tensor of solid skeleton. The fourth-order spatial elasticity tensor $\mathbf{c}$ in the current configuration can be obtained as

$$\mathbf{c} = \frac{\lambda}{J^s} \mathbf{I} \otimes \mathbf{I} + \frac{2}{J^s}(\mu - \lambda \ln J^s) \mathbf{I} \quad (16)$$

where $\mathbf{I}$ is the fourth-order identity tensor.

The elastic model is used in all the analyses in this study.

3 | FINITE ELEMENT FORMULATION

The numerical stability issues relating to the $\mathbf{u}\text{--}\mathbf{v}\text{--}p$ formulation can be attributed to improper use of time integration scheme and spatial discretization method. Also, in problems where the fluid or solid is nearly incompressible, the system of partial differential equations can become ill-conditioned, leading to numerical locking. Moreover, many porous media problems involve nonlinear material behavior, which can introduce additional stability challenges. In addition, spatial oscillation may be a problem when using the $\mathbf{u}\text{--}\mathbf{v}\text{--}p$ formulation, where the insufficient spatial discretization such as large mesh size can cause the boundary with discontinuous porosity to show oscillation in results.

The finite element formulation of time-dependent problems involves two steps, i.e., spatial discretization and temporal discretization. The spatial discretization is first conducted, including the establishment of the weak forms and approximation of primary variables. The standard Galerkin method is used to obtain the weak forms of the governing equations by multiplying them with weight functions and integrating over the domain. The primary variables are approximated using the isoparametric mixed finite element, with 8 nodes for displacement and 4 nodes for pressure. In this way, the inf-sup or LBB condition is satisfied.

For the temporal discretization, the Newmark method is used to advance transient problems in time. To handle the nonlinearities of the governing equations, linearization of the weak form is performed with construction of directional derivative and iteration using the Newton-Raphson iterative method at each time step. The linearization process is conducted first in the reference configuration due to the constant initial elemental volume during linearization, then the linearized weak forms of governing equations is transformed by a push forward operation to the current configuration.

By linearizing the governing equations at each iteration using the Newton-Raphson method, the nonlinear problem is transformed into a sequence of linearized equations where the solution is updated until convergence is achieved. Within each time step, a sequence of linear problems is solved to find the configuration satisfying the residual equation.

3.1 | Initial and boundary conditions

The momentum and mass balance equations are solved in a boundary value problem. The boundary condition for the The initial conditions for the five primary variables are described using the following conditions

$$\mathbf{u}^s = \mathbf{u}_{s0}^s; \mathbf{v}^{ws} = \mathbf{v}_{s0}^{ws}; \mathbf{v}^{as} = \mathbf{v}_{s0}^{as}; p^w = p_{s0}^w; p^a = p_{s0}^a \quad (17)$$

where the subscript s0 indicates the reference configuration of solid phase.

The boundary conditions for the five primary variables $\mathbf{u}^s$, $\mathbf{v}^{ws}$, $\mathbf{v}^{as}$, p^w , and p^a are given by

$$\mathbf{u}^s = \bar{\mathbf{u}}^s \text{ on } \partial B_u^s \quad (18)$$

$$\boldsymbol{\sigma} \mathbf{n} = (\boldsymbol{\sigma}' - p^a \mathbf{I} + s^w p^c \mathbf{I}) \mathbf{n} = \bar{\mathbf{t}} \text{ on } \partial B_t^s \quad (19)$$

$$\mathbf{v}^{ws} = \bar{\mathbf{v}}^{ws} \text{ on } \partial B_{uw}^s \quad (20)$$

$$\boldsymbol{\sigma}^w \mathbf{n} = -p^w \mathbf{I} \mathbf{n} = \bar{\mathbf{t}}^w \text{ on } \partial B_{tw}^s \quad (21)$$

$$\mathbf{v}^{as} = \bar{\mathbf{v}}^{as} \text{ on } \partial B_{ua}^s \quad (22)$$

$$\boldsymbol{\sigma}^a \mathbf{n} = -p^a \mathbf{I} \mathbf{n} = \bar{\mathbf{t}}^a \text{ on } \partial B_{ta}^s \quad (23)$$

$$p^w = \bar{p}^w \text{ on } \partial B_{wp}^s \quad (24)$$

$$n s^w \rho^{wR} \mathbf{v}^{ws} \cdot \mathbf{n} = \mathbf{q}^w \cdot \mathbf{n} = \bar{\mathbf{q}}^w \text{ on } \partial B_{wq}^s \quad (25)$$

$$p^a = \bar{p}^a \text{ on } \partial B_{ap}^s \quad (26)$$

$$n s^a \rho^{aR} \mathbf{v}^{as} \cdot \mathbf{n} = \mathbf{q}^a \cdot \mathbf{n} = \bar{\mathbf{q}}^a \text{ on } \partial B_{aq}^s \quad (27)$$

where $\bar{\mathbf{u}}^s$, $\bar{\mathbf{v}}^{ws}$, and $\bar{\mathbf{v}}^{as}$ are the prescribed vectors of solid displacement, relative pore water velocity, and relative pore air velocity at the boundaries ∂B_u^s , ∂B_{uw}^s , and ∂B_{ua}^s ; $\bar{p}^w$ and $\bar{p}^a$ are the prescribed vectors of pore water pressure and pore air pressure at the boundaries ∂B_{wp}^s and ∂B_{ap}^s ; $\bar{\mathbf{t}}$, $\bar{\mathbf{t}}^w$, and $\bar{\mathbf{t}}^a$ are the prescribed vectors of traction acting on the mixture, water phase, and air phase at the boundaries ∂B_t^s , ∂B_{tw}^s , and ∂B_{ta}^s ; $\bar{\mathbf{q}}^w$ and $\bar{\mathbf{q}}^a$ are the prescribed mass flux on a unit area along the unit normal vector $\mathbf{n}$ at the boundaries ∂B_{wq}^s and ∂B_{aq}^s . All boundary segments have relationships as follows:

$$\begin{aligned} \partial B^s &= \partial B_u^s \cup \partial B_t^s = \partial B_{uw}^s \cup \partial B_{tw}^s = \partial B_{ua}^s \cup \partial B_{ta}^s = \partial B_{wp}^s \cup \partial B_{wq}^s = \partial B_{ap}^s \cup \partial B_{aq}^s \\ \partial B_u^s \cap \partial B_t^s &= \partial B_{uw}^s \cap \partial B_{tw}^s = \partial B_{ua}^s \cap \partial B_{ta}^s = \partial B_{wp}^s \cap \partial B_{wq}^s = \partial B_{ap}^s \cap \partial B_{aq}^s = 0 \end{aligned} \quad (28)$$

3.2 | Weak forms

3.2.1 | $u-v-p$ formulation

The strong forms (differential equations) of the governing equations 1–5 in the $u-v-p$ formulation are transformed to the weak forms (integral equations) using the weighted residuals.

The weak form of the momentum balance equation of the mixture is derived by multiplying Equations 1 and 19 by a test function $\mathbf{w}^s$ that is zero on the prescribed displacement boundary ∂B_u^s and adding them together after integrating over the total domain:

$$\begin{aligned} f^s &= \int_{B^s} \rho \mathbf{w}^s \cdot \mathbf{a}^s dv + \int_{B^s} \text{grad } \mathbf{w}^s : \boldsymbol{\sigma}' dv - \int_{B^s} p^f \text{div } \mathbf{w}^s dv - \int_{B^s} \rho \mathbf{w}^s \cdot \mathbf{b} dv \\ &\quad - \int_{\partial B_t^s} \mathbf{w}^s \cdot \bar{\mathbf{t}} da + \int_{B^s} \rho^w \mathbf{w}^s \cdot \left[\frac{D^s \mathbf{v}^{ws}}{Dt} + \{ \text{grad } (\mathbf{v}^s + \mathbf{v}^{ws}) \} \mathbf{v}^{ws} \right] dv \\ &\quad + \int_{B^s} \rho^a \mathbf{w}^s \cdot \left[\frac{D^s \mathbf{v}^{as}}{Dt} + \{ \text{grad } (\mathbf{v}^s + \mathbf{v}^{as}) \} \mathbf{v}^{as} \right] dv = 0 \end{aligned} \quad (29)$$

The weak form of the momentum balance equation of the pore water is derived by multiplying Equations 2 and 21 by a test function $\mathbf{w}^{ws}$ that is zero on the prescribed displacement boundary ∂B_{uw}^s and adding them together after integrating over the

total domain:

$$\begin{aligned}
 f^{\text{ws}} = & \int_{B^s} \rho^{\text{wR}} \mathbf{w}^{\text{ws}} \cdot \left[\mathbf{a}^s + \frac{D^s \mathbf{v}^{\text{ws}}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{\text{ws}})\} \mathbf{v}^{\text{ws}} \right] dv \\
 & - \int_{B^s} \mathbf{w}^{\text{ws}} \cdot \left\{ \rho^{\text{wR}} \mathbf{b} - n s^{\text{w}} \rho^{\text{wR}} g (k^{\text{w}})^{-1} \mathbf{v}^{\text{ws}} \right\} dv \\
 & - \int_{\partial B_{\text{tw}}^s} \mathbf{w}^{\text{ws}} \cdot \bar{\mathbf{t}}^{\text{w}} da - \int_{B^s} p^{\text{w}} \text{div} \mathbf{w}^{\text{ws}} dv = 0
 \end{aligned} \tag{30}$$

The weak form of the momentum balance equation of the pore air mixture is derived by multiplying Equations 3 and 23 by a test function $\mathbf{w}^{\text{as}}$ that is zero on the prescribed displacement boundary ∂B_{ua}^s and adding them together after integrating over the total domain:

$$\begin{aligned}
 f^{\text{as}} = & \int_{B^s} \rho^{\text{aR}} \mathbf{w}^{\text{as}} \cdot \left[\mathbf{a}^s + \frac{D^s \mathbf{v}^{\text{as}}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{\text{as}})\} \mathbf{v}^{\text{as}} \right] dv \\
 & - \int_{B^s} \mathbf{w}^{\text{as}} \cdot \left\{ \rho^{\text{aR}} \mathbf{b} - n s^{\text{a}} \rho^{\text{aR}} g (k^{\text{a}})^{-1} \mathbf{v}^{\text{as}} \right\} dv \\
 & - \int_{\partial B_{\text{ta}}^s} \mathbf{w}^{\text{as}} \cdot \bar{\mathbf{t}}^{\text{a}} da - \int_{B^s} p^{\text{a}} \text{div} \mathbf{w}^{\text{as}} dv = 0
 \end{aligned} \tag{31}$$

The weak form of the mass balance equation of the pore water is derived by multiplying Equations 4 and 25 by a test function w^{w} that is zero on the prescribed pressure boundary ∂B_{wp}^s and adding them together after integrating over the total domain:

$$\begin{aligned}
 f^{\text{w}} = & \int_{B^s} w^{\text{w}} \left[\left(\frac{n s^{\text{w}} \rho^{\text{wR}}}{K^{\text{w}}} - c n \rho^{\text{wR}} \right) \frac{D^s p^{\text{w}}}{Dt} + c n \rho^{\text{wR}} \frac{D^s p^{\text{a}}}{Dt} + s^{\text{w}} \rho^{\text{wR}} \text{div} \mathbf{v}^s \right] dv \\
 & + \int_{\partial B_{\text{wq}}^s} w^{\text{w}} \bar{q}^{\text{w}} da - \int_{B^s} \text{grad} w^{\text{w}} \cdot (n s^{\text{w}} \rho^{\text{wR}} \mathbf{v}^{\text{ws}}) dv = 0
 \end{aligned} \tag{32}$$

The weak form of the mass balance equation of the pore air is derived by multiplying Equations 5 and 27 by a test function w^{a} that is zero on the prescribed pressure boundary ∂B_{ap}^s and adding them together after integrating over the total domain:

$$\begin{aligned}
 f^{\text{a}} = & \int_{B^s} w^{\text{a}} \left[\left(\frac{n s^{\text{a}} \rho^{\text{aR}}}{K^{\text{a}}} - c n \rho^{\text{aR}} \right) \frac{D^s p^{\text{a}}}{Dt} + c n \rho^{\text{aR}} \frac{D^s p^{\text{w}}}{Dt} + s^{\text{a}} \rho^{\text{aR}} \text{div} \mathbf{v}^s \right] dv \\
 & + \int_{\partial B_{\text{aq}}^s} w^{\text{a}} \bar{q}^{\text{a}} da - \int_{B^s} \text{grad} w^{\text{a}} \cdot (n s^{\text{a}} \rho^{\text{aR}} \mathbf{v}^{\text{as}}) dv = 0
 \end{aligned} \tag{33}$$

3.2.2 | u - p formulation

The strong forms (differential equations) of the governing equations 6–8 in the u - p formulation are transformed to the weak forms (integral equations) using the weighted residuals.

The weak form of the momentum balance equation of the mixture is derived by multiplying Equations 6 and 19 by a test function $\mathbf{w}^s$ that is zero on the prescribed displacement boundary ∂B_{u}^s and adding them together after integrating over the total domain:

$$s^s = \int_{B^s} \rho \mathbf{w}^s \cdot \mathbf{a}^s dv + \int_{B^s} \text{grad} \mathbf{w}^s : \boldsymbol{\sigma}' dv - \int_{B^s} \bar{p} \text{div} \mathbf{w}^s dv - \int_{B^s} \rho \mathbf{w}^s \cdot \mathbf{b} dv - \int_{\partial B_{\text{t}}^s} \mathbf{w}^s \cdot \bar{\mathbf{t}} da = 0 \tag{34}$$

The weak form of the mass and momentum balance equation of the pore water is derived by multiplying Equations 7 and 25 by a test function w^{w} that is zero on the prescribed displacement boundary ∂B_{wp}^s and adding them together after integrating over

the total domain:

$$s^w = \int_{B^s} w^w \left(\frac{n s^w \rho^{wR}}{K^w} - c n \rho^{wR} \right) \frac{D^s p^w}{Dt} dv + \int_{B^s} w^w c n \rho^{wR} \frac{D^s p^a}{Dt} dv + \int_{B^s} w^w s^w \rho^{wR} \mathbf{v}^s dv - \int_{B^s} \text{grad } w^w \cdot \left\{ \frac{k^w}{g} (-\text{grad } p^w + \rho^{wR} \mathbf{b} - \rho^{wR} \mathbf{a}^s) \right\} dv + \int_{\partial B_{wq}^s} w^w \bar{q}^w da = 0 \quad (35)$$

The weak form of the mass and momentum balance equation of the pore air is derived by multiplying Equations 8 and 27 by a test function w^a that is zero on the prescribed displacement boundary ∂B_{ap}^s and adding them together after integrating over the total domain:

$$s^a = \int_{B^s} w^a \left(\frac{n s^a \rho^{aR}}{K^a} - c n \rho^{aR} \right) \frac{D^s p^a}{Dt} dv + \int_{B^s} w^a c n \rho^{aR} \frac{D^s p^w}{Dt} dv + \int_{B^s} w^a (1 - s^w) \rho^{aR} \mathbf{v}^s dv - \int_{B^s} \text{gard } w^a \cdot \left\{ \frac{k^a}{g} (-\text{gard } p^a + \rho^{aR} \mathbf{b} - \rho^{aR} \mathbf{a}^s) \right\} dv + \int_{\partial B_{aq}^s} w^a \bar{q}^a da = 0 \quad (36)$$

3.3 | Spatial discretization

In numerical models based on coupled formulations for porous media, the inf-sup or LBB condition appears in the form of an incompressibility constraint on the spatial discretization of the fluid mass balance. If the inf-sup condition is violated, non-negligible spurious oscillations will occur in the numerical fluid pressure results. In the context of the finite element method, different models based on various strategies have developed to satisfy the inf-sup condition, ranging from the least squares Galerkin technique to the stabilization approaches to the use of element generating unequal order interpolation functions.

In this study, the Q8P4 mixed element in Figure 1 is adopted to satisfy the inf-sup condition. The Q8P4 mixed element uses an 8-node quadrilateral isoparametric element with 4 corner nodes and 4 midside nodes for displacement, velocity, and acceleration (Q8) and a 4-node quadrilateral isoparametric element with 4 midside nodes for pressure (P4). The key to satisfying the inf-sup condition with this element pair lies in the choice of shape functions and the way they are integrated over the elements. The pressure gradient in any element is appropriately bounded by the displacement gradient, where the interpolation used for pressure is at one order lower than that used for the displacement field, i.e. unequal order interpolation.

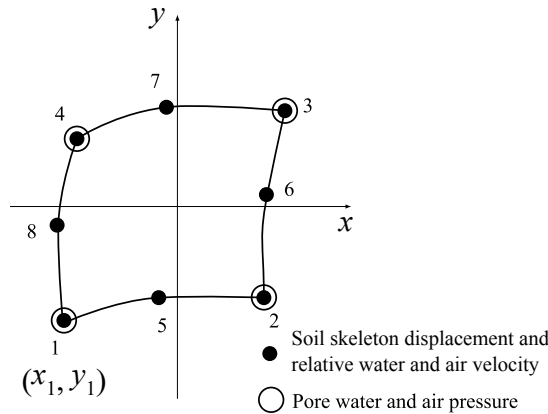


Figure 1 Q8P4 element

The Q8 element provides a higher order of interpolation for displacement, which is beneficial for capturing the displacement field accurately. The P4 element, being linear, ensures that the pressure field is continuous and well-behaved across element interfaces. The combination of the Q8 element for displacement and the P4 element for pressure ensures that the pressure gradient is appropriately represented and bounded by the displacement gradient, thereby satisfying the inf-sup condition.

The displacement, velocity, and acceleration of solid, as well as relative velocities of pore water and air over the element domain are interpolated in the following manners:

$$\mathbf{u}^s = \sum_{n=1}^8 N_n^d \mathbf{u}_n^s, \quad \mathbf{v}^s = \sum_{n=1}^8 N_n^d \mathbf{v}_n^s, \quad \mathbf{a}^s = \sum_{n=1}^8 N_n^d \mathbf{a}_n^s, \quad \mathbf{v}^{ws} = \sum_{n=1}^8 N_n^d \mathbf{v}_n^{ws}, \quad \mathbf{v}^{as} = \sum_{n=1}^8 N_n^d \mathbf{v}_n^{as} \quad (37)$$

where N_n^d is the 8-node shape function for displacement and its derivatives; $\mathbf{u}_n^s$, $\mathbf{v}_n^s$, and $\mathbf{a}_n^s$ are the nodal displacement, velocity, and acceleration of solid; $\mathbf{v}_n^{ws}$ and $\mathbf{v}_n^{as}$ are the nodal relative velocities of pore water and air with respect to solid.

The pressures of pore water and air over the element domain are interpolated in the following manners:

$$p^w = \sum_{n=1}^4 N_n^p p_n^w, \quad p^a = \sum_{n=1}^4 N_n^p p_n^a \quad (38)$$

where N_n^p is the 4-node shape function for pressure; p_n^w and p_n^a are the nodal pressures of pore water and air.

3.4 | Temporal discretization

Given the time-dependent nature of the governing equations, the Newmark method is introduction for temporal discretization. The displacement and velocity of solid at time $t + \Delta t$ are linked with those known at time t according to the following expressions:

$$\mathbf{u}^s = \mathbf{u}_t^s + \Delta t \mathbf{v}_t^s + \frac{1}{2} \Delta t^2 (1 - 2\beta) \mathbf{a}_t^s + \beta \Delta t^2 \mathbf{a}^s \quad (39)$$

$$\mathbf{v}^s = \mathbf{v}_t^s + (1 - \gamma) \Delta t \mathbf{a}_t^s + \gamma \Delta t \mathbf{a}^s \quad (40)$$

The relative velocities of pore water and air at time $t + \Delta t$ are linked with those known at time t according to the following expressions:

$$\mathbf{v}^{ws} = \mathbf{v}_t^{ws} + (1 - \gamma) \Delta t \dot{\mathbf{v}}_t^{ws} + \gamma \Delta t \dot{\mathbf{v}}^{ws} \quad (41)$$

$$\mathbf{v}^{as} = \mathbf{v}_t^{as} + (1 - \gamma) \Delta t \dot{\mathbf{v}}_t^{as} + \gamma \Delta t \dot{\mathbf{v}}^{as} \quad (42)$$

where γ and β are parameters that determine the stability and accuracy characteristics of the method; the subscript t indicates the previous time step and Δt is the time increment, and variables with the subscript t are used to calculate the variables in the next time step $t + \Delta t$; $\dot{\mathbf{v}}^{ws}$ and $\dot{\mathbf{v}}^{as}$ are the material time derivatives of relative velocities of pore water and air with respect to solid.

The pressures of pore water and air at time $t + \Delta t$ are linked with those known at time t according to the following expressions:

$$p^w = p_t^w + \Delta t \dot{p}_t^w + \frac{1}{2} \Delta t^2 (1 - 2\beta) \ddot{p}_t^w + \beta \Delta t^2 \ddot{p}^w \quad (43)$$

$$\dot{p}^w = \dot{p}_t^w + (1 - \gamma) \Delta t \ddot{p}_t^w + \gamma \Delta t \ddot{p}^w \quad (44)$$

$$p^a = p_t^a + \Delta t \dot{p}_t^a + \frac{1}{2} \Delta t^2 (1 - 2\beta) \ddot{p}_t^a + \beta \Delta t^2 \ddot{p}^a \quad (45)$$

$$\dot{p}^a = \dot{p}_t^a + (1 - \gamma) \Delta t \ddot{p}_t^a + \gamma \Delta t \ddot{p}^a \quad (46)$$

where $\dot{p}^w$ and $\dot{p}^a$ are the material time derivatives of pressures of pore water and air with respect to solid; $\ddot{p}^w$ and $\ddot{p}^a$ are the second material time derivatives of pressures of pore water and air with respect to solid.

3.5 | Linearization

The directional derivatives of the left-hand side terms of the weak forms in Section 3.2 are constructed and the Newton-Raphson method is used for linearization. Herein, the linearization of the three-field $\mathbf{u}-\mathbf{v}-p$ formulation is carried out. As for the two-field $\mathbf{u}-p$ formulation, the operation is similar.

The weak form of governing equations in the $\mathbf{u}-\mathbf{v}-p$ formulation to be linearized can be written as

$$\begin{aligned} f^s(\mathbf{a}^s, \dot{\mathbf{v}}^{ws}, \dot{\mathbf{v}}^{as}, \ddot{p}^w, \ddot{p}^a) &= 0 \\ f^{ws}(\mathbf{a}^s, \dot{\mathbf{v}}^{ws}, \dot{\mathbf{v}}^{as}, \ddot{p}^w, \ddot{p}^a) &= 0 \\ f^{as}(\mathbf{a}^s, \dot{\mathbf{v}}^{ws}, \dot{\mathbf{v}}^{as}, \ddot{p}^w, \ddot{p}^a) &= 0 \\ f^w(\mathbf{a}^s, \dot{\mathbf{v}}^{ws}, \dot{\mathbf{v}}^{as}, \ddot{p}^w, \ddot{p}^a) &= 0 \\ f^a(\mathbf{a}^s, \dot{\mathbf{v}}^{ws}, \dot{\mathbf{v}}^{as}, \ddot{p}^w, \ddot{p}^a) &= 0 \end{aligned} \quad (47)$$

The linearized formulations, after taking directional derivatives of Equation 47, take the following forms:

$$\begin{aligned}
 Df^s [\Delta \mathbf{a}^s] + Df^s [\Delta \dot{\mathbf{v}}^{ws}] + Df^s [\Delta \dot{\mathbf{v}}^{as}] + Df^s [\Delta \ddot{\mathbf{p}}^w] + Df^s [\Delta \ddot{\mathbf{p}}^a] &= -f_k^s \\
 Df^{ws} [\Delta \mathbf{a}^s] + Df^{ws} [\Delta \dot{\mathbf{v}}^{ws}] + Df^{ws} [\Delta \dot{\mathbf{v}}^{as}] + Df^{ws} [\Delta \ddot{\mathbf{p}}^w] + Df^{ws} [\Delta \ddot{\mathbf{p}}^a] &= -f_k^{ws} \\
 Df^{as} [\Delta \mathbf{a}^s] + Df^{as} [\Delta \dot{\mathbf{v}}^{ws}] + Df^{as} [\Delta \dot{\mathbf{v}}^{as}] + Df^{as} [\Delta \ddot{\mathbf{p}}^w] + Df^{as} [\Delta \ddot{\mathbf{p}}^a] &= -f_k^{as} \\
 Df^w [\Delta \mathbf{a}^s] + Df^w [\Delta \dot{\mathbf{v}}^{ws}] + Df^w [\Delta \dot{\mathbf{v}}^{as}] + Df^w [\Delta \ddot{\mathbf{p}}^w] + Df^w [\Delta \ddot{\mathbf{p}}^a] &= -f_k^w \\
 Df^a [\Delta \mathbf{a}^s] + Df^a [\Delta \dot{\mathbf{v}}^{ws}] + Df^a [\Delta \dot{\mathbf{v}}^{as}] + Df^a [\Delta \ddot{\mathbf{p}}^w] + Df^a [\Delta \ddot{\mathbf{p}}^a] &= -f_k^a
 \end{aligned} \tag{48}$$

where $\Delta \mathbf{a}^s$, $\Delta \dot{\mathbf{v}}^{ws}$, $\Delta \dot{\mathbf{v}}^{as}$, $\Delta \ddot{\mathbf{p}}^w$, and $\Delta \ddot{\mathbf{p}}^a$ are the variations at the current iterative step $k+1$; f_k^s , f_k^{ws} , f_k^{as} , f_k^w , and f_k^a at the right-hand side are the residuals at the previous iterative step k .

The total solution is obtained from the incremental relationships with expressions of

$$\begin{aligned}
 \mathbf{a}_{k+1}^s &= \mathbf{a}_k^s + \Delta \mathbf{a}^s \\
 \dot{\mathbf{v}}_{k+1}^{ws} &= \dot{\mathbf{v}}_k^{ws} + \Delta \dot{\mathbf{v}}^{ws} \\
 \dot{\mathbf{v}}_{k+1}^{as} &= \dot{\mathbf{v}}_k^{as} + \Delta \dot{\mathbf{v}}^{as} \\
 \ddot{\mathbf{p}}_{k+1}^w &= \ddot{\mathbf{p}}_k^w + \Delta \ddot{\mathbf{p}}^w \\
 \ddot{\mathbf{p}}_{k+1}^a &= \ddot{\mathbf{p}}_k^a + \Delta \ddot{\mathbf{p}}^a
 \end{aligned} \tag{49}$$

In this manner, a sequence of linear problems (with index k) is solved within each time step to find the configuration satisfying the residual equations.

4 | VERIFICATION

4.1 | Consolidation of soil column under constant load

The consolidation of a saturated soil column 1 m in width and 10 m in height under a constant surface load of 10 kPa, as illustrated in Figure 2, is used for verification. The soil column is divided into 100 elements and each element has a height of 0.1 m and a width of 1 m. There are 503 nodes in total. The solid displacement and relative water velocity on the side boundaries are fixed in the horizontal direction but remain free in the vertical direction. Two end points at the bottom are fixed both horizontally and vertically and the middle point is fixed only vertically. The top surface of the soil column is the drainage boundary while the bottom surface is impermeable boundary. The $\mathbf{u}-\mathbf{v}-p$ analysis approach and the analytical method are used to calculate the settlement during consolidation. The main material properties of soil are listed in Table 2. The coefficient in Newmark integration method is $\gamma = 1.0$. The convergence tolerance of the governing equations is taken as 1.0×10^{-8} . The initial excess pore water pressure does not change with depth, so the analytical solution to the degree of consolidation under the one-side drainage based on Terzaghi's one-dimensional consolidation theory⁵² is summarized as follows:

$$\begin{aligned}
 U_s &= 1 - \sum_{i=0}^{\infty} \frac{2}{M} (\sin MZ) \exp(-M^2 T_v) \\
 M &= \frac{\pi(2i+1)}{2}, \quad Z = \frac{z}{H}, \quad T_v = \frac{c_v t}{H^2}
 \end{aligned} \tag{50}$$

where U_s is the degree of consolidation at small strain, M is the intermediate parameter related to the positive integer i that varies from 0 to ∞ , Z is the normalized soil depth, z is the soil depth, H is the height of the soil column that equals 10 m, T_v is the small-strain dimensionless time factor, and c_v is the coefficient of consolidation, which is taken as $0.04 \text{ m}^2/\text{s}$.

The finite-strain solution to the normalized void ratio proposed by Morris⁵³ is

$$\begin{aligned}
 E(Z, T) &= E(0, T) + \frac{4(1 - E(0, T))}{\pi} \sum_{j=1,3,5}^{\infty} \frac{1}{n} \sin\left(\frac{j\pi Z}{2}\right) \exp\left(\frac{-j^2 \pi^2 T}{4}\right) \\
 E(Z, T) &= \frac{e(z, t)}{e(0, 0)}, \quad Z = \frac{z}{H}, \quad T = \frac{g' t}{H^2}, \quad g' = -\frac{k}{\gamma_w (1 + e)} \frac{\partial \sigma'}{\partial e}
 \end{aligned} \tag{51}$$

where $E(Z, T)$ is the normalized void ratio, $E(0, T)$ is the normalized void ratio at the soil surface, $e(Z, T)$ is the void ratio, $e(0, 0)$ is the initial void ratio, T is the finite-strain dimensionless time factor, j is the positive odd integer with values from 1 to

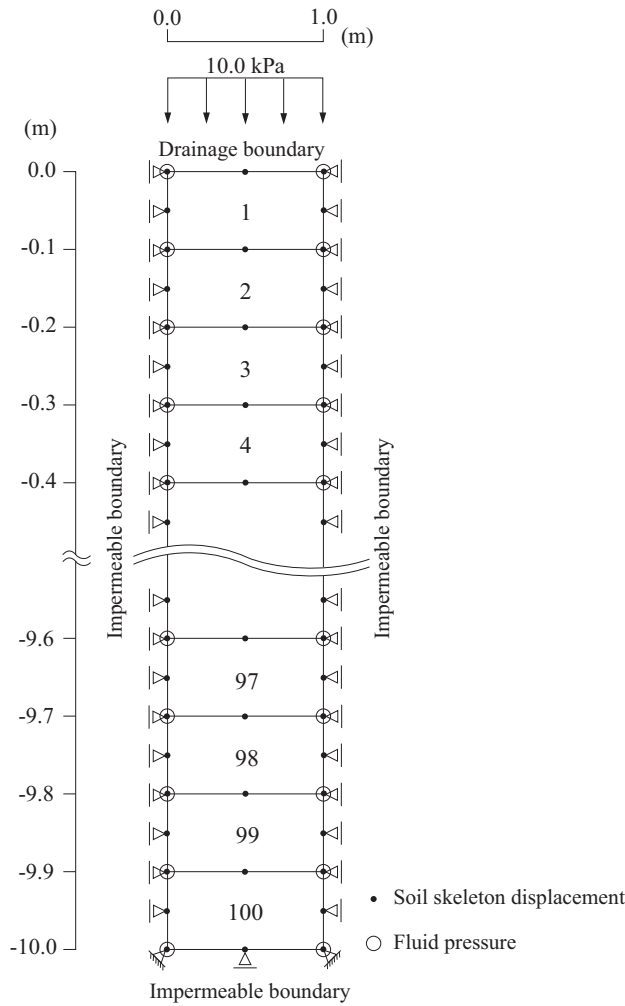


Figure 2 Soil column under static load

Table 2 Parameters for consolidation analysis

Parameter	Value
Void ratio, e_0	1.0
Permeability coefficient, k_s^w	1.0×10^{-2} m/s
Bulk modulus of pore water, K^w	1.0×10^{10} kPa
Lamé's constant, λ	10.0 kPa
Lamé's constant, μ	10.0 kPa

1 ∞ , g' is the finite-strain coefficient of consolidation and is assumed to be constant in the analytical solution, σ' is the effective
2 stress, k is the saturated permeability coefficient, and γ_w is the unit weight of water.
3 The analysis results are presented in Figure 3 with the predicted surface settlement using the analytical solution⁵³ and $u-v-p$
4 analysis based on small- and finite-strain theories. In small deformation theory, it is assumed that the first derivatives of dis-
5 placements are so small that the squares and products of these derivatives are negligible compared to the linear terms and are
6 thus omitted. The Cauchy stress and Lagrangian strain at finite deformation are adopted. The results show that the two ap-
7 proaches adopted in the quasi-static consolidation analysis produce the surface settlement of soil that match well with each
8 other. The small-strain analysis produces a slightly larger soil settlement than the finite-strain analysis due to the lack of updated
9 configuration.

4.2 | Consolidation of soil column under dynamic load

11 The consolidation analysis of soil column under dynamic loading in Figure 4 is presented here. The soil column has a height of
12 100 m and a width of 1 m, and it is divided into 5000 elements 0.02 m in height and 1 m in width. There are 25003 nodes in
13 total. The boundary conditions are similar to those of the soil column in Figure 2. Step and sine loading both have a magnitude
14 of 1 kPa and last for 0.2 s, and the sine loading has a period of 0.1 s.

15 Simon et al.⁵⁴ derived the analytical solution to the response of one-dimensional saturated elastic porous columns using Biot's
16 two-phase mixture theory⁴ based on the assumption of an infinite elastic column. When the soil column is subjected to loading

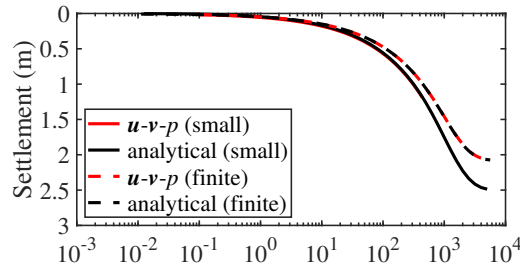


Figure 3 Surface settlement during consolidation using different methods

on its surface, the solid displacement u and the relative displacement of pore water with respect to solid w are determined as

$$w(\xi, \tau) = \frac{d}{(1-d\beta)\sqrt{a}} \int_0^\tau f(\tau - \bar{\tau}) e^{-b\bar{\tau}/2a} I_0 \left[\frac{b\sqrt{\bar{\tau}^2 - a\xi^2}}{2a} \right] 1(\bar{\tau} - \xi\sqrt{a}) d\bar{\tau}$$

$$u(\xi, \tau) = - \int_0^\tau f(\tau - \bar{\tau}) 1(\bar{\tau} - \xi) d\bar{\tau} - \beta w(\xi, \tau) \quad (52)$$

where ξ is the spatial coordinate, τ is non-dimensional time, $f(\tau)$ is a load function, $1(\tau)$ is a unit step function, I_0 is the modified zero-order Bessel function, and d is the material parameter by Biot⁴.

Parameters τ , β , a , and b are defined as

$$\tau = \frac{t\gamma^f}{\rho k}, \quad \beta = \frac{\rho^{fR}}{\rho}, \quad a = \frac{\gamma - \beta^2}{\kappa - \beta^2}, \quad b = \frac{1}{\kappa - \beta^2}$$

$$\gamma = \frac{\beta}{n}, \quad \kappa = \frac{Q}{\lambda + 2\mu + d^2 Q}, \quad \frac{1}{Q} = \frac{n}{K^f} + \frac{d-n}{K^s} \quad (53)$$

where t is time, γ^f is the unit weight of pore fluid, k is the permeability coefficient, ρ^{fR} is the real density of fluid, ρ is the density of total mixture, n is the porosity, λ and μ are the Lamé constants, K^f is the bulk modulus of fluid and K^s is the bulk modulus of soil particle.

The material parameters used in the analysis are provided in Table 3. The $u-v-p$ analysis based on the small strain theory is used to calculate the solid displacement and relative pore fluid displacement. The coefficients in Newmark integration method are $\gamma = 0.6$ and $\beta = 0.3025$. The convergence tolerance of the governing equations is 1.0×10^{-8} . Results from the Finite element analysis and the analytical solution proposed by Simon et al.⁵⁴ are reported in Figure 5. The predictions are very similar in two cases, so the $u-v-p$ analysis proves to have a good performance in predicting the consolidation-induced displacement during dynamic loading.

5 | DYNAMIC RESPONSE OF ONE-DIMENSIONAL SATURATED SOIL COLUMN

To test the applicability of the analyses based on $u-v-p$ and $u-p$ formulations in predicting the dynamic response of one-dimensional soil column in various scenarios, i.e., different ranges of soil permeability and loading frequencies, simulations with the small strain measure are carried out. The soil column model 10 m in height and 0.5 m in width is subject to dynamic load on the top surface, as plotted in Figure 6. The soil column is divided into 200 elements, and each element has the same thickness of 0.05 m. The material properties of the soil column are listed in Table 5. The soil is fully saturated and the top surface is the only permeability boundary. The solid displacement and relative velocity of pore water with respect of solid velocity are fixed horizontally on both sides of the model, whereas on the bottom of the model they are fixed horizontally and vertically. The vertical dynamic load is in sinusoidal waveform and has a magnitude of 100 kPa/m. According to Zienkiewicz et al.⁶, two dimensionless parameters Π_1 and Π_2 used to divide the zones of applicability of different formulations are written as

$$\Pi_1 = \frac{2k}{\pi\beta g} \frac{f_0^2}{f}, \quad \Pi_2 = \pi^2 \left(\frac{f}{f_0} \right)^2 \quad (54)$$

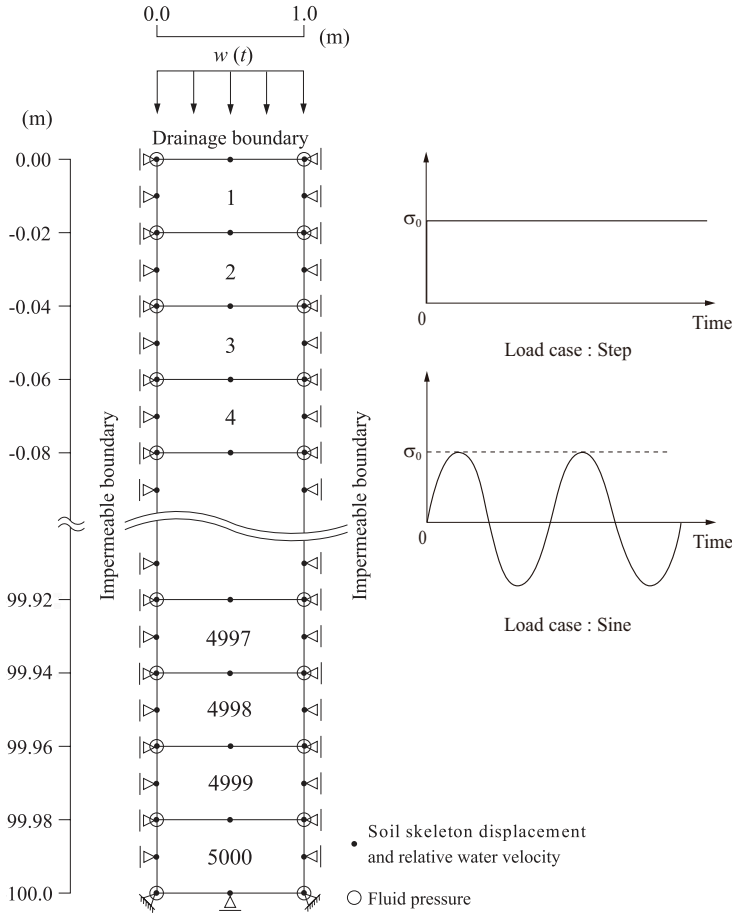


Table 3 Parameters⁵⁴ for dynamic consolidation analysis

Parameter	Value
Initial porosity, n	0.333
Density of mixture, ρ	306 kg/m ³
Density of pore fluid, ρ^{fR}	297.7 kg/m ³
Unit weight of pore fluid, γ^{f}	2.917 kN/m ³
Permeability coefficient, k	0.01425 m/s
Lamé's constant, λ	833.3 kPa
Lamé's constant, μ	1250.0 kPa
Bulk modulus of soil skeleton, K^{s}	∞
Bulk modulus of pore fluid, K^{f}	39999.0 kPa

Figure 4 Soil column under dynamic load

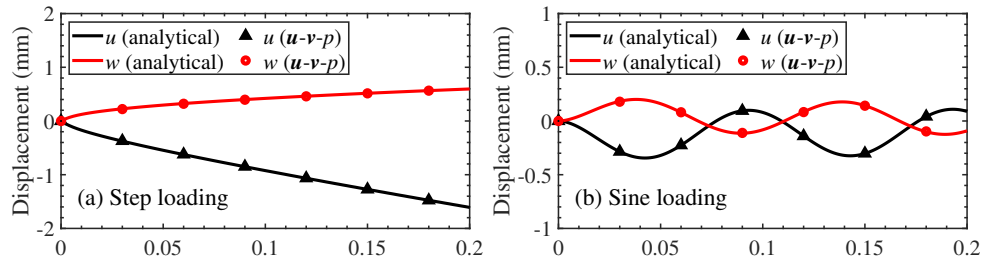


Figure 5 Displacement during consolidation under step and sine loading

- 1 where k is the permeability of the soil; f is the frequency of the loading wave; f_0 is the natural frequency of the soil column;
- 2 β is the ratio of fluid density to bulk density that is taken as 0.333; g is the gravitational acceleration, i.e., 9.81 m/s². Detailed
- 3 explanation of main variables can be found in Zienkiewicz et al.⁶
- 4 A total of 25 cases with different combinations of Π_1 and Π_2 calculated using Equation 54, as listed in Table 4, are arranged
- 5 with the purpose of investigating the difference between $u-v-p$ and $u-p$ analyses in simulating the dynamic response of soil to
- 6 longitudinal loading. Π_1 and Π_2 in these cases are within the range 0.01 and 100.
- 7 The finite deformation measure is adopted in the analysis. The coefficients in Newmark integration method are $\gamma = 0.6$ and
- 8 $\beta = 0.3025$. The convergence tolerance of the governing equations is 1.0×10^{-10} . The total number of steps in the dynamic
- 9 analysis is 400 in all cases. The time increment in the dynamic analysis in the case of dynamic loading with the frequency of
- 10 1 Hz is 0.016 s and the time increment in the analysis in the case of dynamic loading with the higher frequency is reduced

- 1 according to the ratio between the frequency and 1 Hz, for example, for dynamic loading 50 Hz in frequency, the time increment
 2 in the analysis is 0.00032 s. The magnitude and number of cycles of dynamic loading are the same in all cases.

Table 4 Different pairs of soil permeability and load period

Case	Π_1	Π_2	Frequency f (Hz)	Permeability k (m/s)
Case1	100	100	160	3.27×10^1
Case2	10	100	160	3.27
Case3	1	100	160	3.27×10^{-1}
Case4	0.1	100	160	3.27×10^{-2}
Case5	0.01	100	160	3.27×10^{-3}
Case6	100	10	50	1.03×10^1
Case7	10	10	50	1.03
Case8	1	10	50	1.03×10^{-1}
Case9	0.1	10	50	1.03×10^{-2}
Case10	0.01	10	50	1.03×10^{-3}
Case11	100	1	16	3.27
Case12	10	1	16	3.27×10^{-1}
Case13	1	1	16	3.27×10^{-2}
Case14	0.1	1	16	3.27×10^{-3}
Case15	0.01	1	16	3.27×10^{-4}
Case16	100	0.1	5	1.03
Case17	10	0.1	5	1.03×10^{-1}
Case18	1	0.1	5	1.03×10^{-2}
Case19	0.1	0.1	5	1.03×10^{-3}
Case20	0.01	0.1	5	1.03×10^{-4}
Case21	100	0.01	1.6	3.27×10^{-1}
Case22	10	0.01	1.6	3.27×10^{-2}
Case23	1	0.01	1.6	3.27×10^{-3}
Case24	0.1	0.01	1.6	3.27×10^{-4}
Case25	0.01	0.01	1.6	3.27×10^{-5}

3 Displacement and pore water pressure of the central node on the ground surface and the central node in the middle (5 m
 4 in height) of the soil column model in Case 1 ($\Pi_1 = 100$ and $\Pi_2 = 100$), Case 14 ($\Pi_1 = 0.1$ and $\Pi_2 = 1$), and Case 25
 5 ($\Pi_1 = 0.01$ and $\Pi_2 = 0.01$) are plotted in Figure 7. Case 25 in Zone I and Case 14 in Zone II characterize the soil behavior with
 6 slow and moderate speeds respectively, where the relative acceleration of pore water with respect to solid is negligible, thus
 7 the two different analyses do not differ in the predicted displacement and pore water pressure. However, when the soil behavior
 8 demonstrates fast phenomena, which is represented by Zone III, the two different analyses generate clear distinction in their
 9 prediction. Regarding the solid displacement, when the soil behavior shifts from undrained (i.e., Cases 14 and 25 in Figure 7b
 10 and c) to drained (i.e., Cases 1 in Figure 7a), the solid displacement predicted by $u-v-p$ and $u-p$ analyses become different,
 11 which is mainly shown by the magnitude of the time history curve. The similar phenomenon is also revealed by the pore water
 12 pressure response in Figure 7d, e and f. Apart from that, the pore water pressure in the $u-v-p$ analysis lags behind that in the
 13 $u-p$ analysis when it comes to the drained soil behavior in Figure 7d.

14 The distinction of simulation outcomes between two different kinds of analyses are further assessed using the EARTH (Error
 15 Assessment of Response Time Histories) metric proposed by Sarin et al.⁵⁵, in which the differences in magnitude, slope, and
 16 phase (which are called magnitude error, slope error, and phase error) between two time histories can be quantitatively evaluated.
 17 The basics are briefly introduced. Firstly, the time history of a simple harmonic signal is described by

$$y(t_i) = Y \cos(\omega t_i + \phi) \quad (55)$$

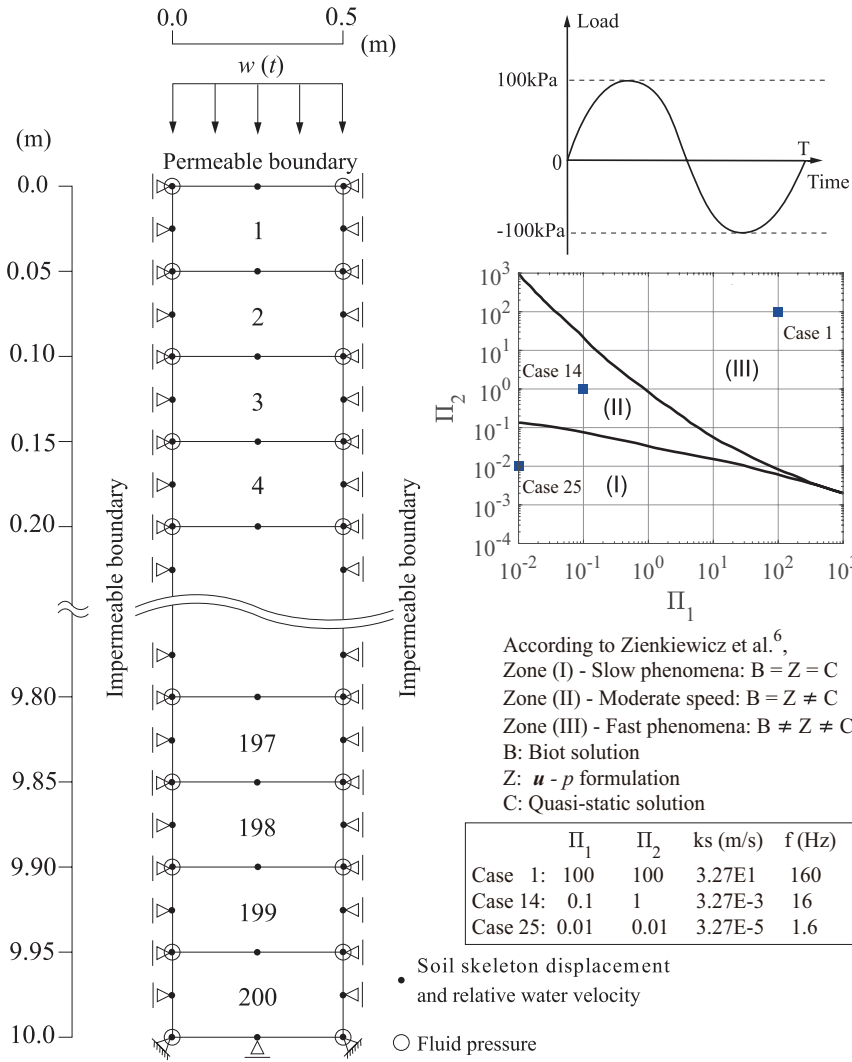


Table 5 Parameters⁶ for dynamic analysis of one-dimensional saturated soil column

Parameter	Value
Initial porosity, n_{0s}	0.333
Density of solid, ρ^{sR}	4000 kg/m ³
Density of fluid, ρ^{fR}	1000 kg/m ³
Bulk modulus of fluid, K^f	1000 MPa
Lamé's constant, λ	3570 kPa
Lamé's constant, μ	2380 kPa

Figure 6 One-dimensional soil column under dynamic load with three different conditions

- where Y is the magnitude, ω is the frequency, ϕ is the phase, and t_i is the value of time at time index i . The DTW (Dynamic Time Warping) method was adopted in EARTH so that the errors of magnitude, frequency, and phase can be assessed independently.
- The magnitude error is calculated using

$$\text{Error}_{\text{magnitude}} = \frac{\|A^{[ts+w]} - B^{[ts+w]}\|}{\|B^{[ts+w]}\|} \quad (56)$$

- where A and B denote the two time histories to be compared; $ts + w$ is the phase-shifted DTW modified time history with a cost function of

$$d[i, j] = \left\{ (a_i^{ts} - b_j^{ts})^2 + (t_i - t_j)^2 \right\} \left| \left(\frac{dA^{ts}}{dt} \right)_{t=t_i} - \left(\frac{dB^{ts}}{dt} \right)_{t=t_j} \right| \quad (57)$$

- where the superscript ts demotes the time-shifted time history; a_i^{ts} and b_j^{ts} are the data in time histories A and B ; subscripts i and j denote the locations of data in two time histories.

- The slope error is calculated based on the derivative of the time history using

$$\text{Error}_{\text{slope}} = \frac{\|A^{[ts+d+w]} - B^{[ts+d+w]}\|}{\|B^{[ts+d+w]}\|} \quad (58)$$

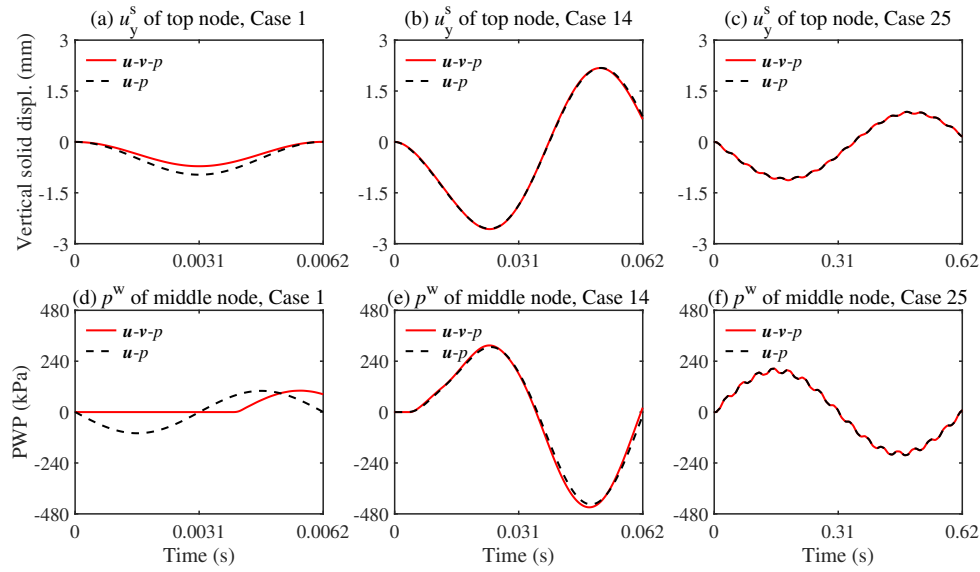


Figure 7 Vertical solid displacement and pore water pressure of nodes on the top and in the middle of the soil column in Cases 1, 14, and 25

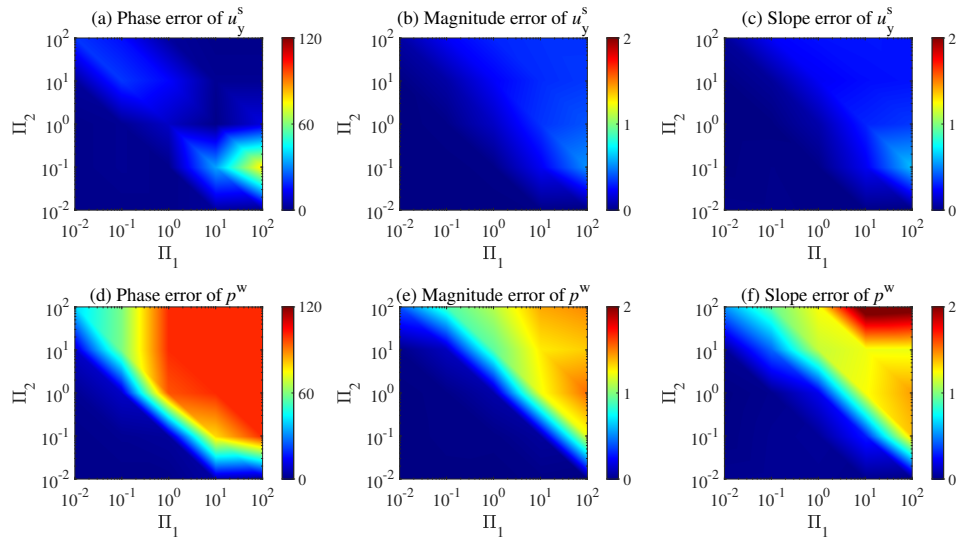


Figure 8 Phase error, magnitude error, and slope error of solid displacement and pore water pressure of nodes on the top and in the middle of the soil column

- 1 where the superscript $ts + d + w$ denotes that the time history is processed by the sequence of time shifting for the global phase
- 2 effect, derivative computation, and then DTW.
- 3 The phase error is calculated using

$$\text{Error}_{\text{phase}} = e^{\left(\frac{n_* - c}{r}\right)} \quad (59)$$

- 4 where n_* is the number of shifted time steps; c is the critical time shift value below which the phase error can be regarded
- 5 negligible and r is the rate of phase error increase above the critical value c .
- 6 With the application of the EARTH metric, the magnitude error, slope error, and phase error of the simulated results such
- 7 as nodal pore water pressure and displacement during loading obtained from the $u-p$ and $u-v-p$ analyses can be quantitatively
- 8 compared. Figure 8 contains the discrepancies of the time histories of solid displacement and pore water pressure between two

kinds of analyses in terms of magnitude error, slope error, and phase error when the soil column is subjected to vertical periodic loading under different combinations of Π_1 and Π_2 .

Regarding the situation featuring low permeability but extremely fast dynamic phenomena within a very short time span such as fracture forerunning in Ni et al.⁵⁶ and Khoei et al.⁵⁷ and supershear earthquake ruptures in Pampillon et al.⁵⁸, the changes in volumetric strain will lead to pressure changes without significant fluid-solid relative displacement, suggesting that the porous flow can be treated as undrained. In these situations, the solid and fluid phases work together as a multiphase medium through which stress waves propagate. It is characteristics of large Π_2 large but small Π_1 , so the soil dynamic response may fall into the upper left-hand corner of the $\Pi_1 - \Pi_2$ space, which is close to Zone II in Figure 6, then the effect of relative fluid flow becomes insignificant. Therefore, the $u-p$ formulation can still be used with good accuracy to predict the dynamics of porous media.

It is worth noting that in Case 14 with a frequency of 16 Hz, which is a quite high frequency for common geotechnical problems, two kinds of analyses do not vary from each other in the prediction. If the frequency continues to grow to 160 Hz while the permeability keeps constant, which causes Π_2 to increase and Π_1 to reduce, the variation between two kinds of analyses still does not show much difference because the representative (Π_1, Π_2) point is still likely to approach Zone II in Figure 6. It is only when both permeability and frequency increase to a certain level that the large discrepancy starts to take place, such as Case 1 where the permeability and frequency are both considerably increased, so Π_1 and Π_2 become large enough to give rise to the significant difference between two kinds of analyses.

6 | DYNAMIC RESPONSE OF TWO-DIMENSIONAL PARTIALLY SATURATED SOIL LAYER

The dynamic response of partial saturated soil layer subjected to vertical sinusoidal loading on its surface is investigated. The finite model is detailed in Figure 9. This response of soil layer to dynamic loading are also taken as an example for discussion in previous studies^{38,41}. Four cases shown in Figure 9 with different combination of loading frequency and soil permeability coefficient are investigated. The soil layer is 10 m in depth and 10 m in width with the ground water table located 5 m under the surface, and it is divided into 100 square eight-node elements with 341 nodes in total. The material properties are given in Table 6. Vertical sinusoidal loading is imposed on the top surface of the soil layer on the left hand side covering space 1 m in width, and it is expressed as $w(t) = 3 - 3 \cos(2\pi ft)$, where f is the frequency of the load.

The right and left boundaries of the soil layer are supported by vertical rollers, the bottom boundary is supported by horizontal rollers, and the upper boundary is free. As for pore air, the left and right boundaries are impermeable while air drainage is allowed at the upper boundary. As for pore water, the hydrostatic pressure is utilized, so the linear pressure distribution is used initially in the selfweight analysis, whereas the upper and bottom boundaries are impermeable in the dynamic analysis. The horizontal relative velocities of pore water and air on two vertical boundaries are zero, and the vertical relative velocities of pore water and air on the bottom horizontal boundary are also zero.

The small deformation analysis is adopted in the analysis. The coefficients in Newmark integration method are $\gamma = 0.6$ and $\beta = 0.3025$. The convergence tolerance of the governing equations is 1.0×10^{-10} . The total number of steps in the dynamic analysis is 400 in all cases. The time increment in the dynamic analysis in the case of dynamic loading with the frequency of 1 Hz is 0.016 s and the time increment in the analysis in the case of dynamic loading with the higher frequency is reduced according to the ratio between the frequency and 1 Hz, for example, for dynamic loading 50 Hz in frequency, the time increment in the analysis is 0.00032 s. The magnitude and number of cycles of dynamic loading are the same in all cases.

Table 6 Parameters⁴¹ used for dynamic analysis of soil layer under sinusoidal loading

Parameter	Value
Initial porosity, n_{0s}	0.5
Densities of solid particle, water, and air, ρ^{sR} , ρ^{wR} , and ρ^{aR}	2.6×10^3 , 1.0×10^3 , and 1.2 (unit: kg/m ³)
Bulk module of water, K^w	2200 MPa
Lamé's constants, λ and μ	6 MPa and 4 MPa
Minimum and maximum saturation, s_r^w and s_s^w	0.10 and 0.99
SWCC parameters of LG model, a_{lg} , b_{lg} , c_{lg} , ξ_{lg} , and η_{lg}	0.6, -6, 0.1, 3.0, and 1.5

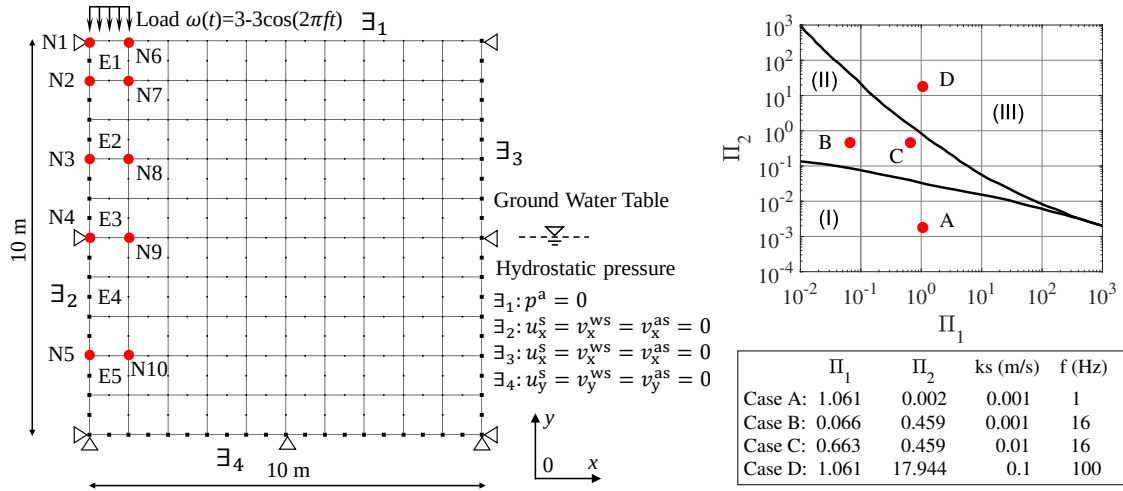


Figure 9 Soil layer under vertical sinusoidal loading with four different conditions

Figures 10 and 11 report the development of pore water and air pressures of soil layer subjected to vertical sinusoidal loading by means of analyses based on three-field $u-v-p$ and two-field $u-p$ formulations. The results are from Cases A, B, C, and D with different combinations of Π_1 and Π_2 that represent different soil behavior divided by Zienkiewicz et al⁶. The initial values of node N1, N2, and N3 becomes increasingly greater due to the rising degree of water saturation as the depth of soil grows. When soil permeability is relatively small, such as Case A, the change of pore water pressure during loading is marked since the soil is not well drained and the excess pore water pressure tends to build quickly and displays considerable peaks. As soil permeability grows, the buildup of pore water pressure becomes less prominent in Cases B, C, and D. Additionally, the differences in pore fluid pressures between two analyses become significant when soil behavior transits from Zone I to Zone II and Zone III. As the soil becomes more saturated, the two analyses exhibit more deviation in prediction, especially the results of Case D recorded in Figure 10d, h, and l. For the development of pore air pressure, the large difference between two analyses is also found in node N4 in Case D, which is near the water table.

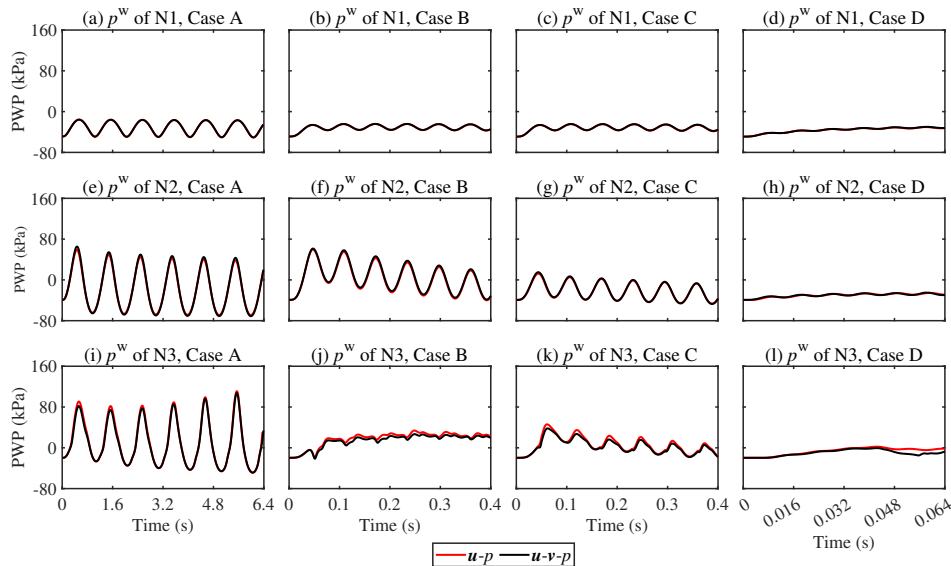


Figure 10 Nodal pore water pressures of soil layer during dynamic loading

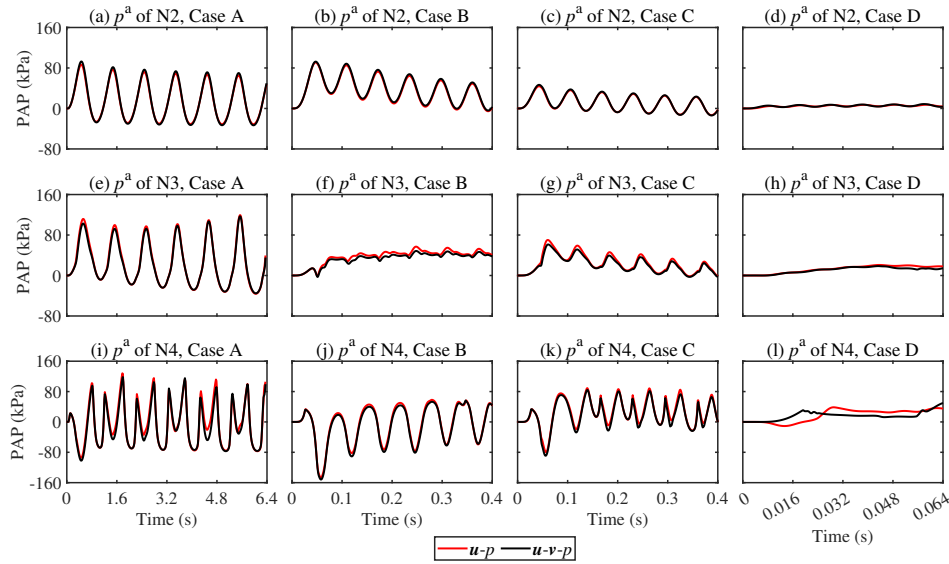


Figure 11 Nodal pore air pressures of soil layer during dynamic loading

Figure 12a–h compares the distributions of pore water pressures of soil layer during dynamic loading in Cases A, B, C and D. In Case A where both the loading frequency and soil permeability are low, the distinction in the predicted pore fluid pressures between two analyses is subtle. Under loading with the same frequency, when the soil permeability is greater (Case C), the pore water pressures dissipate much faster than the that in the lower permeability situation (Case B), and the differences between two analyses become noticeable but not considerable in the saturated area below the loading application zone especially the area below the ground water table. Furthermore, the discrepancy in the prediction manifest itself to the greater degree as the soil becomes more drained, particularly in Case D. Additionally, the development of pore water pressure mostly occurs in the areas near the left-hand boundary because of the direct influence stemming from dynamic loading, and the greatest difference in pore water pressure is found in the deeper part of the soil layer, i.e., the saturated zone in the bottom left corner area in Case D.

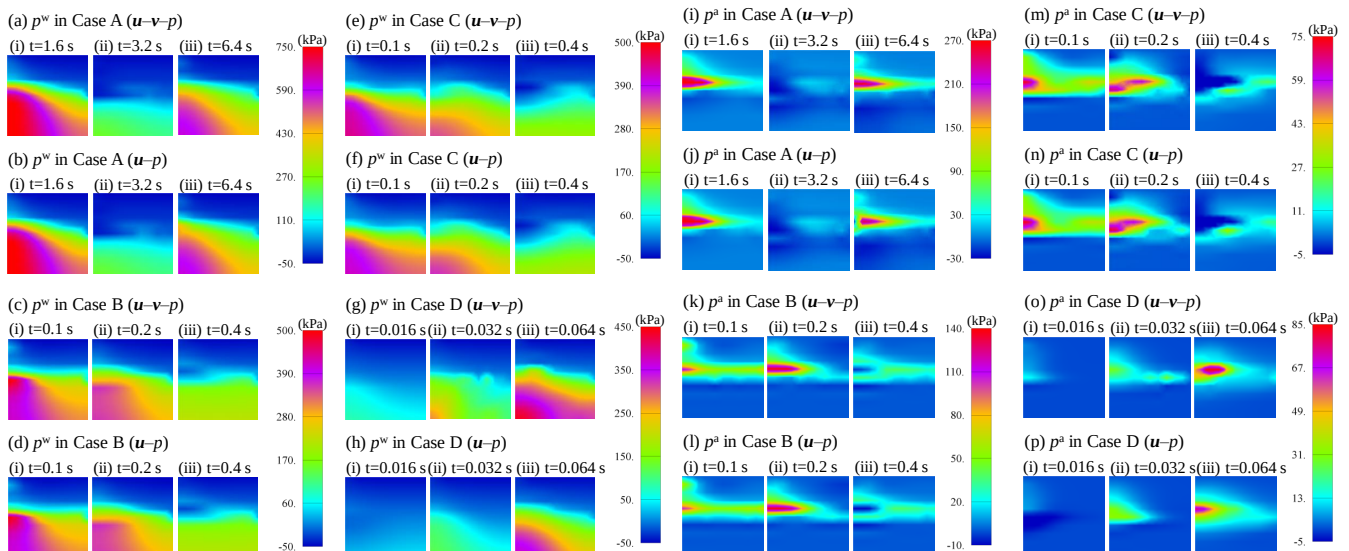


Figure 12 Distributions of pore water pressures and pore air pressures of soil layer during dynamic loading in Cases A, B, C, and D at different time steps

When it comes to the pore air pressures under various conditions in Figure 12i–p, the development takes place mostly in the soil near and above the ground water table where soil is unsaturated during the vertical sinusoidal loading. Different from the pore water pressure, the pore air pressure exhibits the greatest difference between two analyses in the transition area between unsaturation and saturation.

The solid displacement during dynamic loading in Cases A, B, C, and D is recorded in Figure 13. It is found that the solid displacement of soil above the ground water table does not show much difference in the prediction based on the $u-v-p$ and $u-p$ analyses in Figure 13a–h. However, the results of Case D in Figure 13p exhibit differences, although they are quite subtle.

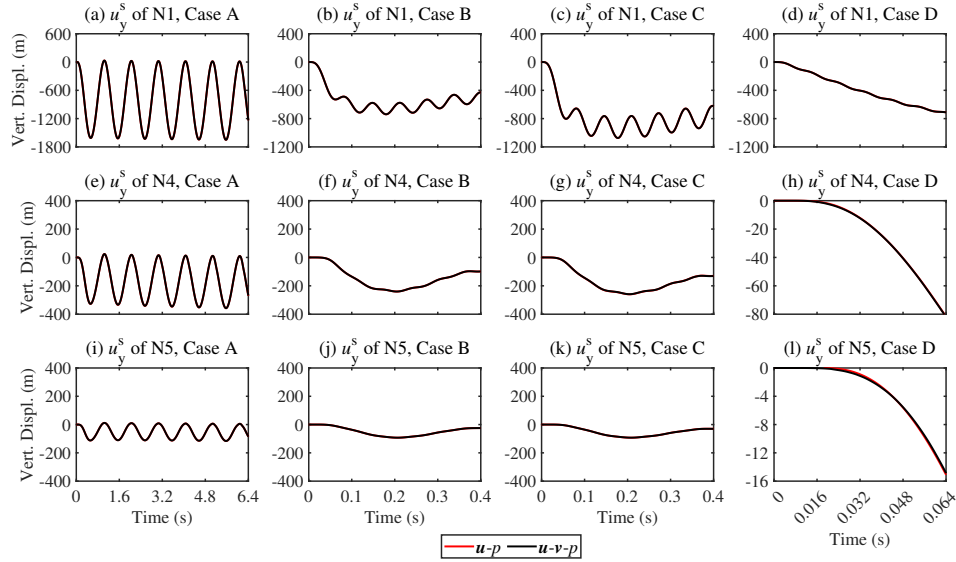


Figure 13 Vertical solid displacement of nodes N1, N4, and N5 during vertical dynamic loading in Cases A, B, C, and D

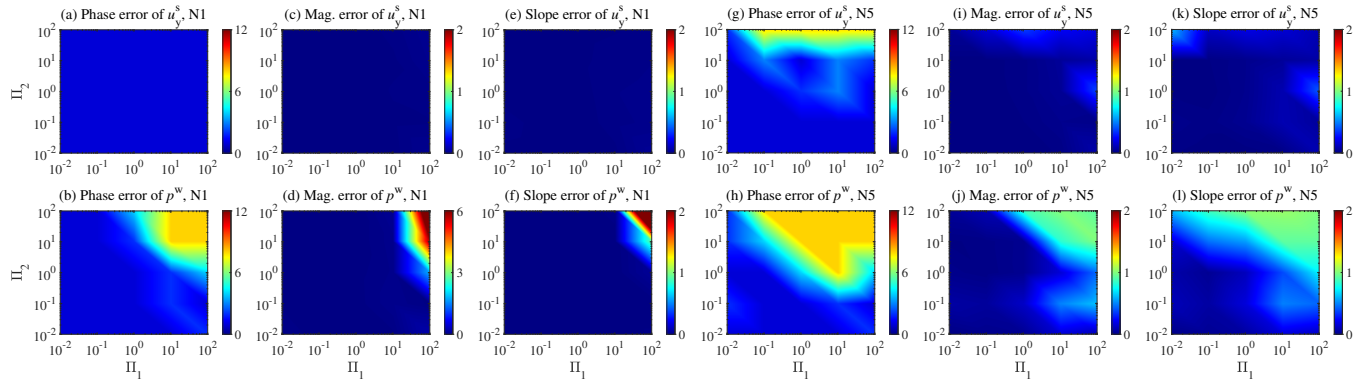


Figure 14 Errors of phase, magnitude, and slope of vertical solid displacement and pore water pressures of node N1 on soil surface and node N5 near soil bottom

For the purpose of further examining the performance of analyses based on the three-field $u-v-p$ formulation and two-field $u-p$ formulation and the differences in their prediction, more cases featured by different combinations of Π_1 and Π_2 in the range 0.01 and 100 are simulated. The quantitative differences in the vertical solid displacement and pore water pressure of nodes near the surface and bottom of soil layer in Figures 10 and 13 between two kinds of analyses are presented in Figure 14. Generally, the

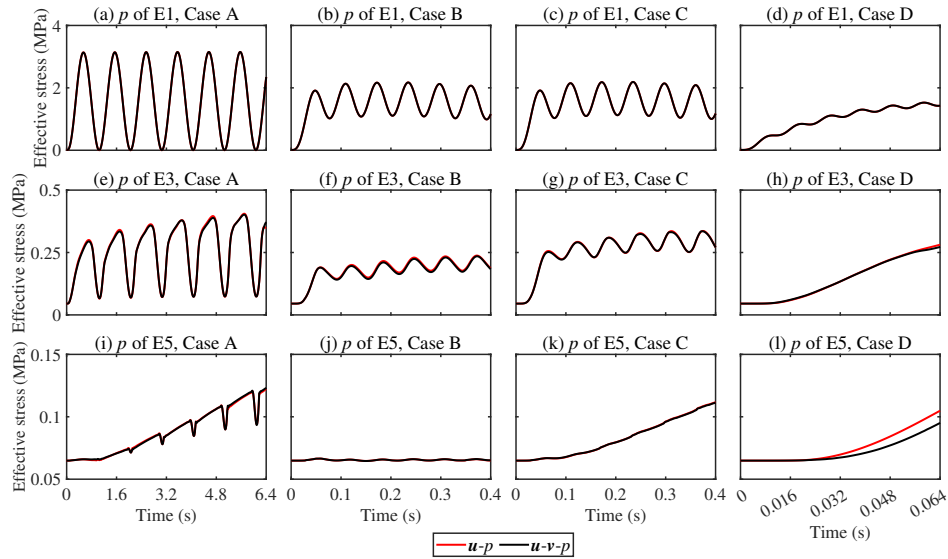


Figure 15 Mean effective stresses of soil elements during dynamic loading

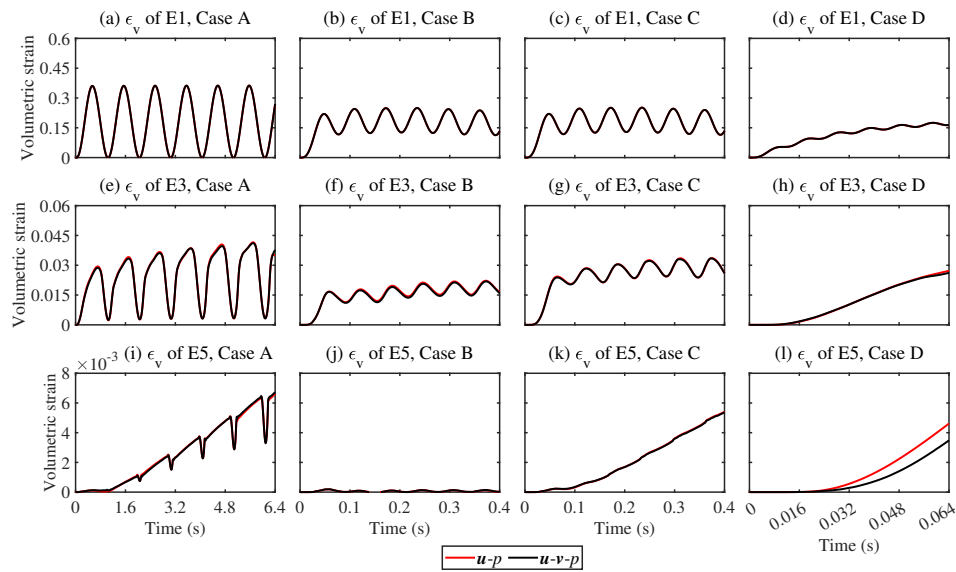


Figure 16 Volumetric strains of soil elements during dynamic loading

1 graphs are categorized into two zones and the differences in phase, magnitude, and slope of pore water pressure time histories
 2 of different nodes during dynamic loading are predominantly distributed within the upper right-hand corner above the diagonal
 3 line from point (0.01, 100) to point (100, 0.01), i.e., Zone III, which indicates the more drained condition characteristic of large
 4 soil permeability and loading frequency and stands for the zone where analyses based on the full and simplified formulations
 5 would give rise to noticeable discrepancy. This is also similar to the trends of pore water pressure histories shown in Figure 8 and
 6 consistent with the categorization in the original Π_1 – Π_2 chart proposed by Zienkiewicz et al.⁶, except that the highlighted area in
 7 Figure 8 accounts for much more space. However, the solid displacement of different nodes predicted by two analyses in Figure
 8 13 does not exhibit distinct differences between each other. This is may be due to the use of elastic stress-strain relationship for soil
 9 in the simulation, where the deformation is not comprehensively reproduced, leading to unnoticeable differences. Additionally,
 10 for the node N1 near soil surface in the unsaturated area, the large difference is more concentrated in area with larger Π_1 , in
 11 comparison with the node N5 near soil bottom in the saturated area. This results from the small permeability of unsaturated soil
 12 in comparison with that of saturated soil, in order for the relative accelerations of pore fluids of soil to be not negligible—thus

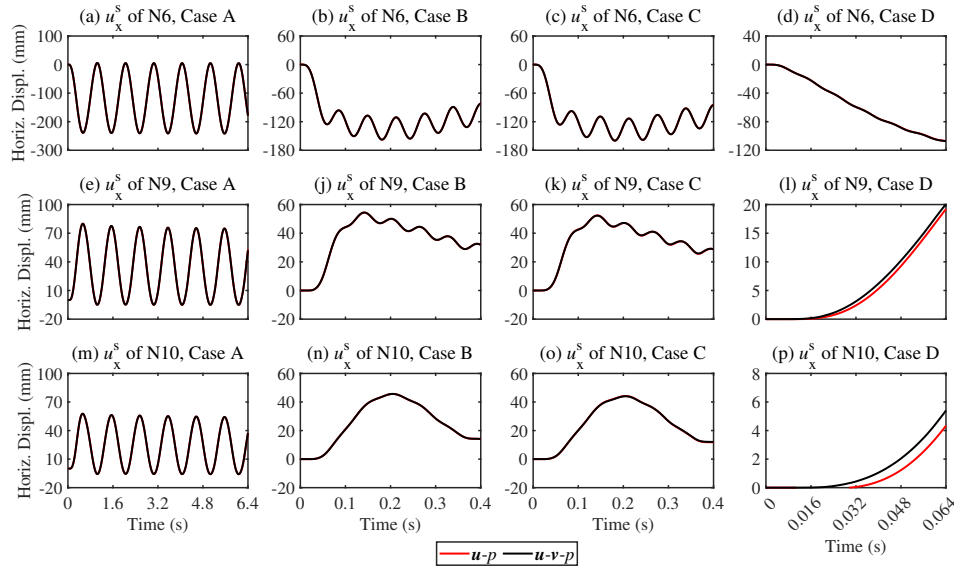


Figure 17 Horizontal solid displacement of nodes with different depths

showing major differences in its response predicted by two different analyses—the larger soil permeability or Π_1 is required. As a result, the highlighted differences of pore water pressure of N1 in Figure 14b, d, and f are more shifted to the upper-right side compared to those of N5 in Figure 14 h, j, and l. Therefore, the unsaturated soil is less likely to exhibit fast flow phenomena than the saturated soil.

It is also found that, regarding the partially saturated condition, since soil permeability is not as large as that in the fully saturated condition, fast flow motion is not necessarily generated during dynamic loading. It is confirmed in the solid displacement and fluid pressure results where the simulated dynamic responses using two kinds of analyses are quite similar. Therefore, for the partially saturated condition, the analysis based on the $u-p$ formulation is sufficient to predict the dynamics of poroelastic media. Regarding the fully saturated condition, although soil below the ground water table demonstrates different responses during high-frequency loading, the discrepancy is somehow minor. It is only when both Π_1 and Π_2 are sufficiently large can the difference between two formulations become significant. Consistent with the finding in the dynamic analysis of one-dimensional saturated soil column in Section 5, the $u-p$ formulation is sufficiently accurately to predict the response of poroelastic media under dynamic loading with commonly encountered frequency ranges in geotechnical engineering.

The mean effective stresses of different elements of soil layer subjected to vertical sinusoidal loading in Figure 15 obtained using analyses based on three-field and two-field formulations generally do not deviates substantially in four cases except in Case D (Zone III) where the difference is observed in the soil in saturated area (element E5). Also, the wave propagation is demonstrated in the upper layer in Cases A, B, and C where the loading frequency is not large, in comparison with the large frequency situation in Case D. This is consistent with the findings about pore fluid pressure, in which the large difference between two analyses are caused, thus leading to the result in Figure 15l.

As the wave propagation mainly takes place in the vertical direction due to the longitudinal dynamic loading, the compression deformation of the elastic soil layer in terms of volumetric strain is induced, which is captured in Figure 16. Since the compression is vertical, the two different analyses will make a great difference in the vertical soil response when the relative fluid flow becomes significant. However, given that the soil layer is two-dimensional, the volumetric compression of soil element can result in the horizontal deformation, so the difference of soil response in the horizontal direction caused by the use of different analyses is also found.

Figure 17 provides the development of horizontal solid displacement of nodes from top surface (N1) to middle (N6) and bottom (N10) of soil layer. Basically, the degree of saturation of these nodes increases with depth. During dynamic loading applied to soil layer, node N6, which is on the top surface and in close contact with the right edge of loading, develops obvious downward settlement, causing the horizontal displacement to progress toward the left-hand side. As a result, the horizontal displacement of N6 in Figure 17a–d shows negative values. Loading causes the compression of soil below, causing the horizontal displacement of nodes N9 and N10 to develop toward the right-hand side, as displayed in Figure 17e–p. As for the differences in horizontal

solid displacement of nodes N6, N9, and N10 obtained using full and simplified analyses, only Case D manifests inconsistent horizontal solid displacement predicted by two different kinds of analyses, and as the water saturation of soil grows, i.e., from N6 to N9 and N10 in Figure 17d, l, and p, the disparity becomes increasingly wider. Therefore, although dynamic loading is applied vertically, the induced soil response in the horizontal direction is not negligible for soil with the higher degree of saturation. Moreover, Figure 17 clearly demonstrates that it is necessary to include the effect of relative pore fluid accelerations on the soil response in the direction perpendicular to the loading direction when simulating the two-dimensional soil dynamic problem. In this case, the full formulation results in more accurate description than the simplified formulation to predict soil dynamic behavior especially for soil with a larger degree of saturation.

7 | CONCLUSIONS

In this study, the dynamics of poroelastic media is investigated using the coupled finite element analysis. The $u-v-p$ full analysis approach that considers the relative fluid acceleration with respect to solid is proposed, which incorporates governing equations written in terms of three fields—solid displacement u , relative velocity of pore fluid with respect to solid v , and pore fluid pressure p . The consolidation of soil column under constant and dynamic loading is examined by means of the numerical simulation approach proposed in this study, which is also compared with the analytical solution. After that, the analyses of saturated one-dimensional soil column and partially saturated two-dimensional soil layer subjected to dynamic loading are carried out to study the behavior of poroelastic media under various conditions of loading frequency and soil permeability, where the performance of the proposed $u-v-p$ full analysis approach is compared with that of $u-p$ simplified analysis approach. Main conclusions are drawn as follows:

1. The proposed method is verified in the consolidation analysis where the full analysis approach based on the $u-v-p$ formulation has predicted soil response in consistence with the analytical solution.
2. When subjected to vertical dynamic loading, the saturated one-dimensional soil column demonstrates different responses according to the combination of loading frequency and soil permeability, which can be basically divided into two different categories representing insignificant and significant fluid flow.
3. The dynamic response of partially saturated two-dimensional soil layer can also be classified into two different categories. Generally, in comparison with saturated soil, unsaturated soil shows responses that are less sensitive to the increase of loading frequency and soil permeability. For two-dimensional soil with elasticity, despite vertical loading, the volumetric deformation induced by loading can also cause the soil response in the horizontal direction predicted by full and simplified analyses to exhibit differences that are dependent on loading frequency and soil permeability.
4. The $u-v-p$ analysis approach shows good performance in predicting soil dynamics when both soil permeability and loading frequency are significantly large. The $u-p$ analysis approach, though being simplified, can still predicts soil dynamics well under conditions with high loading frequency and low soil permeability.

8 | ACKNOWLEDGMENT

This work was supported by JSPS KAKENHI Grant Number 21H04575, Natural Science Basic Research Program of Shaanxi (Program No. 2024JC-YBQN-0333), the Postdoctoral Fellowship Program of CPSF under Grant Number GZC20232230, the Postdoctoral Research Program of Shaanxi (2023BSHEDZZ217), and the Fundamental Research Funds for the Central Universities, CHD (300102214111).



APPENDIX

A DERIVATION OF GOVERNING FORMULATION

A.1 u - v - p formulation

The three-phase porous media consist of solid, water, and air. With the assumptions of incompressible soil particle, constant temperature, and no mass exchange between the solid particle and fluid, the key equations governing the mass and momentum balance⁴¹ are presented. The superscripts s, w, and a represent soil phase, water phase, and air phase in the following sections. Note that $D^i \square / Dt$ stands for the material time derivative with respect to the phase i (which can be s, w, and a).

The mass balance equation of each phase takes the form of

$$\frac{D^s \rho^s}{Dt} + \rho^s \operatorname{div} \mathbf{v}^s = 0 \quad (\text{A1})$$

$$\frac{D^w \rho^w}{Dt} + \rho^w \operatorname{div} \mathbf{v}^w = 0 \quad (\text{A2})$$

$$\frac{D^a \rho^a}{Dt} + \rho^a \operatorname{div} \mathbf{v}^a = 0 \quad (\text{A3})$$

where $\mathbf{v}^s$, $\mathbf{v}^w$, and $\mathbf{v}^a$ are the velocities of solid, pore water, and pore air; ρ^s , ρ^w , and ρ^a are the their corresponding densities, with expressions of

$$\rho^s = (1 - n) \rho^{sR} \quad (\text{A4})$$

$$\rho^w = n s^w \rho^{wR} \quad (\text{A5})$$

$$\rho^a = n s^a \rho^{aR} = n(1 - s^s) \rho^{aR} \quad (\text{A6})$$

$$\rho = \rho^s + \rho^w + \rho^a \quad (\text{A7})$$

where ρ^{sR} , ρ^{wR} , and ρ^{aR} are the intrinsic densities of solid, pore water, and pore air; ρ is the overall density of the mixture; n is the porosity; s^w and s^a are the degrees of water and air saturation with a relation of $s^w + s^a = 1$.

The momentum balance equation of each phase takes the form of

$$\rho^s \mathbf{a}^s = \operatorname{div} \boldsymbol{\sigma}^s + \rho^s \mathbf{b} + \hat{\mathbf{p}}^s \quad (\text{A8})$$

$$\rho^w \mathbf{a}^w = \operatorname{div} \boldsymbol{\sigma}^w + \rho^w \mathbf{b} + \hat{\mathbf{p}}^w \quad (\text{A9})$$

$$\rho^a \mathbf{a}^a = \operatorname{div} \boldsymbol{\sigma}^a + \rho^a \mathbf{b} + \hat{\mathbf{p}}^a \quad (\text{A10})$$

where $\mathbf{a}^s$, $\mathbf{a}^w$, and $\mathbf{a}^a$ are the accelerations of solid, pore water, and pore air; $\hat{\mathbf{p}}^s$, $\hat{\mathbf{p}}^w$, and $\hat{\mathbf{p}}^a$ are the interaction force for solid phase, water phase, and air phase, having a relation of $\hat{\mathbf{p}}^s + \hat{\mathbf{p}}^w + \hat{\mathbf{p}}^a = 0$; $\mathbf{b}$ is the body force; $\boldsymbol{\sigma}^s$, $\boldsymbol{\sigma}^w$, and $\boldsymbol{\sigma}^a$ are the partial Cauchy stress tensor of three phases, which are described as

$$\boldsymbol{\sigma}^s = \boldsymbol{\sigma}' - (1 - n) p^f \mathbf{I} \quad (\text{A11})$$

$$\boldsymbol{\sigma}^w = -n s^w p^w \mathbf{I} \quad (\text{A12})$$

$$\boldsymbol{\sigma}^a = -n s^a p^a \mathbf{I} = -n(1 - s^s) p^a \mathbf{I} \quad (\text{A13})$$

where $\boldsymbol{\sigma}'$ is the effective stress tensor; p^f , p^w , and p^a are the average pore pressure, pore water pressure, and pore air pressure with a relationship of $p^f = s^w p^f + s^s p^f$; $\mathbf{I}$ is the second-order identity tensor.

The total stress $\boldsymbol{\sigma}$ is defined as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^s + \boldsymbol{\sigma}^w + \boldsymbol{\sigma}^a \quad (\text{A14})$$

The acceleration terms $\mathbf{a}^s$, $\mathbf{a}^w$, and $\mathbf{a}^a$ are further defined as

$$\mathbf{a}^s = \frac{D^s \mathbf{v}^s}{Dt} = \frac{\partial \mathbf{v}^s}{\partial t} + (\operatorname{grad} \mathbf{v}^s) \mathbf{v}^s \quad (\text{A15})$$

where $\mathbf{v}^s$ is the velocity of solid skeleton.

The acceleration of pore water $\mathbf{a}^w$ can be expressed in terms of the material time derivative $D^s \square / Dt$ as

$$\mathbf{a}^w = \frac{D^w \mathbf{v}^w}{Dt} = \mathbf{a}^s + \frac{D^s \mathbf{v}^{ws}}{Dt} + \operatorname{grad}(\mathbf{v}^s + \mathbf{v}^{ws}) \mathbf{v}^{ws} \quad (\text{A16})$$

where $\mathbf{v}^{ws}$ is the relative velocity of pore water to the solid skeleton.

The acceleration of pore air $\mathbf{a}^a$ also can be expressed in terms of the material time derivative $D^s \square / Dt$ as

$$\mathbf{a}^a = \frac{D^a \mathbf{v}^a}{Dt} = \mathbf{a}^s + \frac{D^s \mathbf{v}^{as}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{as})\} \mathbf{v}^{as} \quad (\text{A17})$$

The interaction forces of a phase with other phases are defined by

$$\hat{\mathbf{p}}^s = -\hat{\mathbf{p}}^w - \hat{\mathbf{p}}^a \quad (\text{A18})$$

$$\hat{\mathbf{p}}^w = p^w \text{grad}(ns^w) - \boldsymbol{\mu}^w ns^w \mathbf{v}^{ws} \quad (\text{A19})$$

$$\hat{\mathbf{p}}^a = p^a \text{grad}(ns^a) - \boldsymbol{\mu}^a ns^a \mathbf{v}^{as} \quad (\text{A20})$$

where $\boldsymbol{\mu}^w$ and $\boldsymbol{\mu}^a$ are the material parameter tensors defined by the following equations:

$$\boldsymbol{\mu}^w = \frac{ns^w \rho^{wR} g}{k^w} \mathbf{I} \quad (\text{A21})$$

$$\boldsymbol{\mu}^a = \frac{ns^a \rho^{aR} g}{k^a} \mathbf{I} \quad (\text{A22})$$

where g is the gravity; k^w and k^a are the water and air permeability coefficients.

Under the constant temperature, the state equations of pore water and pore air are written as

$$\frac{D^s \rho^{w0}}{\rho^{w0} Dt} = \frac{D^s p^w}{K^w Dt} \quad (\text{A23})$$

$$\frac{D^s \rho^{a0}}{\rho^{a0} Dt} = \frac{D^s p^a}{K^a Dt} \quad (\text{A24})$$

where K^w and K^a are the bulk moduli of pore water and pore air.

Adding Equations A4, A5, A6 together and introducing Equations A7 and A14 results in the expression of the momentum balance equation of the mixture:

$$\rho \mathbf{a}^s + \rho^w \left[\frac{D^s \mathbf{v}^{ws}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{ws})\} \mathbf{v}^{ws} \right] + \rho^a \left[\frac{D^s \mathbf{v}^{as}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{as})\} \mathbf{v}^{as} \right] = \text{div} \boldsymbol{\sigma} + \rho \mathbf{b} \quad (\text{A25})$$

Substituting Equations A5, A16, A12, reinteraction-water, and A21 into Equation A9 leads to the detailed expression of the momentum balance equation of pore water:

$$\rho^{wR} \left[\mathbf{a}^s + \frac{D^s \mathbf{v}^{ws}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{ws})\} \mathbf{v}^{ws} \right] = -\text{grad} p^w + \rho^{wR} \mathbf{b} - ns^w \rho^{wR} g (k^w)^{-1} \mathbf{v}^{ws} \quad (\text{A26})$$

Substituting Equations A6, A17, A13, A20, and A21 into Equation A10 leads to the detailed expression of the momentum balance equation of pore air:

$$\rho^{aR} \left[\mathbf{a}^s + \frac{D^s \mathbf{v}^{as}}{Dt} + \{\text{grad}(\mathbf{v}^s + \mathbf{v}^{as})\} \mathbf{v}^{as} \right] = -\text{grad} p^a + \rho^{aR} \mathbf{b} - ns^a \rho^{aR} g (k^a)^{-1} \mathbf{v}^{as} \quad (\text{A27})$$

Substituting Equation A4 into Equation A1 and Equation A5 into Equation A2 results in the detailed expressions of the mass balance equations of solid and pore water, and combining them gives the continuity equation of pore water with respect to solid:

$$\left(\frac{ns^w \rho^{wR}}{K^w} - cn \rho^{wR} \right) \frac{D^s p^w}{Dt} + cn \rho^{wR} \frac{D^s p^a}{Dt} + s^w \rho^{wR} \text{div} \mathbf{v}^s + \text{div} (ns^w \rho^{wR} \mathbf{v}^{ws}) = 0 \quad (\text{A28})$$

Substituting Equation A4 into Equation A1 and Equation A6 into Equation A3 results in the detailed expressions of the mass balance equations of solid and pore water, and combining them gives the continuity equation of pore air with respect to solid:

$$\left(\frac{ns^a \rho^{aR}}{K^a} - cn \rho^{aR} \right) \frac{D^s p^a}{Dt} + cn \rho^{aR} \frac{D^s p^w}{Dt} + s^a \rho^{aR} \text{div} \mathbf{v}^s + \text{div} (ns^a \rho^{aR} \mathbf{v}^{as}) = 0 \quad (\text{A29})$$

A.2 u - p formulation

In the u - p formulation, the relative accelerations of pore fluids (pore water and pore air) with respect to solid and the convection terms in Equations A25, A26, and A27 are neglected, so the momentum balance equation of the mixture becomes

$$\rho \mathbf{a}^s = \text{div} \boldsymbol{\sigma} + \rho \mathbf{b} \quad (\text{A30})$$

The momentum balance equations of pore water and pore air become

$$ns^w \mathbf{v}^{ws} = \frac{k^w}{\rho^{wR} g} \{ -\text{grad } p^w + \rho^{wR} (\mathbf{b} - \mathbf{a}^s) \} \quad (\text{A31})$$

$$ns^a \mathbf{v}^{as} = \frac{k^a}{\rho^{aR} g} \{ -\text{grad } p^a + \rho^{aR} (\mathbf{b} - \mathbf{a}^s) \} \quad (\text{A32})$$

Substituting Equation A31 into Equation A28 leads to the mass and momentum balance equation of pore water with respect to solid:

$$\left(\frac{ns^w \rho^{wR}}{K^w} - cn \rho^{wR} \right) \frac{D^s p^w}{Dt} + cn \rho^{wR} \frac{D^s p^a}{Dt} + s^w \rho^{wR} \text{div } \mathbf{v}^s + \text{div} \left(\frac{k^w}{g} (-\text{grad } p^w + p^{wR} \mathbf{b} - \rho^{wR} \mathbf{a}^s) \right) = 0 \quad (\text{A33})$$

Substituting Equation A32 into Equation A29 leads to the mass and momentum balance equation of pore air with respect to solid:

$$\left(\frac{ns^a \rho^{aR}}{K^a} - cn \rho^{aR} \right) \frac{D^s p^a}{Dt} + cn \rho^{aR} \frac{D^s p^w}{Dt} + s^a \rho^{aR} \text{div } \mathbf{v}^s + \text{div} \left(\frac{k^a}{g} (-\text{grad } p^a + p^{aR} \mathbf{b} - \rho^{aR} \mathbf{a}^a) \right) = 0 \quad (\text{A34})$$

References

1. de Boer R. Contemporary progress in porous media theory. *Appl Mech Rev.* 2000;53(12):323-369.
2. Biot MA. General theory of three-dimensional consolidation. *J Appl Phys.* 1941;12:155-164.
3. Biot MA. Theory of propagation of elastic waves in a fluid saturated porous solid. *J Acoust Soc Am.* 1956;28(2):168-178.
4. Biot MA. Mechanics of deformation and acoustic propagation in porous media. *J Appl Phys.* 1962;33(2):1482-1498.
5. Biot MA. Generalized theory of acoustic propagation in porous dissipation media. *J Acoust Soc Am.* 1962;34(9):1254-1264.
6. Zienkiewicz OC, Chang CT, Bettess P. Drained, undrained consolidating and dynamic behaviour assumptions in soils. *Geotechnique.* 1980;30(4):385-395.
7. Gajo A, Denzer R. Finite element modelling of saturated porous media at finite strains under dynamic conditions with compressible constituents. *Int J Numer Anal Methods Geomech.* 2011;85(13):1705-1736.
8. Sabetamal H, Nazem M, Sloan SW, Carter JP. Frictionless contact formulation for dynamic analysis of nonlinear saturated porous media based on the mortar method. *Int J Numer Anal Methods Geomech.* 2016;40(1):25-61.
9. Lotfian Z, Sivaselvan MV. Mixed finite element formulation for dynamics of porous media. *Int J Numer Methods Eng.* 2018;115(2):141-171.
10. Noda T, Toyoda T. Development and verification of a soil–water coupled finite deformation analysis based on $\mathbf{u-w-p}$ formulation with fluid convective nonlinearity. *Soils Found.* 2019;59(4):888-904.
11. Toyoda T, Noda T. Numerical simulation based heuristic investigation of inertia-induced phenomena and theoretical solution based verification by the damped wave equation for the dynamic deformation of saturated soil based on the $\mathbf{u-w-p}$ governing equation. *Soils Found.* 2021;61(2):352-370.
12. Wu W, Wan T, Yang Y, Zheng H. Three-dimensional numerical manifold formulation with continuous nodal gradients for dynamics of elasto-plastic porous media. *Comput Methods Appl Mech Engrg.* 2022;388(1):114203.
13. Zienkiewicz OC, Shiomi T. Dynamic behaviour of saturated porous media: The generalized Biot formulation and its numerical solution. *Int J Numer Anal Methods Geomech.* 1984;8(1):71-96.
14. Gajo A, Saetta A, Vitaliani R. Evaluation of three- and two-field finite element methods for the dynamic response of saturated soil. *Int J Numer Anal Methods Geomech.* 1994;37(7):1231-1247.
15. Jeremić B, Cheng Z, Taiebat M, Dafalias Y. Numerical simulation of fully saturated porous materials. *Int J Numer Anal Methods Geomech.* 2008;32(13):1635-1660.

16. Simon SR, Wu SS, Zienkiewicz OC, Paul DK. Evaluation of $u - w$ and $u - \pi$ finite element methods for the dynamic response of saturated porous media using one-dimensional models. *Int J Numer Anal Methods Geomech.* 1986;10(5):461-482.
17. López-Querol S, Blázquez R. Liquefaction and cyclic mobility model in saturated granular media. *Int J Numer Anal Methods Geomech.* 2006;30:413-439.
18. Navas P, López-Querol S, Yu R, Li B. B-bar based algorithm applied to meshfree numerical schemes to solve unconfined seepage problems through porous media. *Int J Numer Anal Methods Geomech.* 2016;40:962-984.
19. Ulker MBC, Rahman MS. Response of saturated and nearly saturated porous media: Different formulations and their applicability. *Int J Numer Anal Methods Geomech.* 2009;33(5):633-664.
20. Monforte L, Navas P, Carbonell JM, Arroyo M, Gens A. Low-order stabilized finite element for the full Biot formulation in soil mechanics at finite strain. *Int J Numer Anal Methods Geomech.* 2019;43:1488-1515.
21. Jassim I, Stolle D, Vermeer P. Two-phase dynamic analysis by material point method. *Int J Numer Anal Methods Geomech.* 2013;37(15):2502-2522.
22. Yerro A, Alonso EE, Pinyol NM. The material point method for unsaturated soils. *Geotechnique.* 2015;65(3):201-217.
23. Gajo A, Mongioli L. An analytical solution for the transient response of saturated linear elastic porous media. *Int J Numer Anal Methods Geomech.* 1995;19(6):399-413.
24. Zhao C, Li W, Wang J. An explicit finite element method for Biot dynamic formulation in fluid-saturated porous media and its application to a rigid foundation. *J Sound Vib.* 2005;282(3-5):1169-1181.
25. Ravichandran N. Fully coupled finite element model for dynamics of partially saturated soils. *Soil Dyn Earthq Eng.* 2009;29(9):1294-1304.
26. Ravichandran N, Muraleetharan KK. Dynamics of unsaturated soils using various finite element formulations. *Int J Numer Anal Methods Geomech.* 2009;33(5):611-631.
27. Liang J, Zhang G, Ba Z, Liang J. Development of a 3D fluid-saturated element for dynamic analysis of two-phase media in ABAQUS based on $u - U$ formed equations. *Comput Geomech.* 2021;139:104377.
28. Ghaboussi J, Wilson EL. Variational formulation of dynamics of fluid saturated porous elastic solids. *J Eng Mech Div.* 1972;98(4):947-963.
29. Prevost J. Nonlinear transient phenomena in saturated porous media. *Comput Methods Appl Mech Engrg.* 1982;30(1):3-18.
30. Zienkiewicz OC, Chan AHC, Pastor M, Schrefler BA, Shiomi A. *Computational geomechanics with special reference to earthquake engineering.* New York: John Wiley & Sons; 1999.
31. Argani LP, Gajo A. A novel insight into vertical ground motion modelling in earthquake engineering. *Int J Numer Anal Methods Geomech.* 2022;46(1):164-186.
32. Schrefler BA, Zhan X. A fully coupled model for water flow and airflow in deformable porous media. *Water Resour Res.* 1993;29(1):155-167.
33. Muraleetharan KK, Mish KD, Arulanandan K. A fully coupled non-linear dynamic analysis procedure and its verification using centrifuge test results. *Int J Numer Anal Methods Geomech.* 1994;18(5):305-325.
34. Oka F, Yashima A, Shibata T, Kato M, Uzuoka R. FEM-FDM coupled liquefaction analysis of a porous soil using an elasto-plastic model. *Appl Sci Res.* 1994;52(3):209-245.
35. Li X, Thomas HR, Fan Y. Finite element method and constitutive modelling and computation for unsaturated soils. *Comput Methods Appl Mech Engrg.* 1999;169(1-2):135-159.
36. Noda T, Asaoka A, Nakano M. Soil-water coupled finite deformation analysis based on a rate-type equation of motion incorporating the SYS Cam-Clay model. *Soils Found.* 2008;48(6):771-790.

- 1 37. Ehlers W, Ammann TGM, Deformation and localization analysis of partially saturated soil. *Comput Methods Appl Mech*
2 *Engrg.* 2004;193(27-29):2885-2910.
- 3 38. Li C, Borja RI, Regueiro RA. Dynamics of porous media at finite strain. *Comput Methods Appl Mech Engrg.* 2004;193(36-
4 38):3837-3870.
- 5 39. Khoei AR, Mohammadnejad T. Numerical modeling of multiphase fluid flow in deforming porous media: A comparison
6 between two- and three-phase models for seismic analysis of earth and rockfill dams. *Comput Geomech.* 2011;38(2):142-
7 166.
- 8 40. Lenhof B, Larsson F, Runesson K. A comparison of variational formats for porous media subjected to dynamic loading. *Int*
9 *J Numer Anal Methods Geomech.* 2011;35:807-823.
- 10 41. Uzuoka R, Borja RI. Dynamics of unsaturated poroelastic solids at finite strain. *Int J Numer Anal Methods Geomech.*
11 2012;36(13):1535-1573.
- 12 42. Pedroso DM. A consistent u-p formulation for porous media with hysteresis. *Int J Numer Meth Engrg.* 2015;101(8):606-634.
- 13 43. Xu C, Song J, Du L, Zhong Z. A completely explicit finite element method for solving dynamic u-p equations of fluid-
14 saturated porous media. *Soil Dyn Earthq Eng.* 2017;97:364-376.
- 15 44. Xu J, Ueda K, Uzuoka R. Numerical modeling of seepage and deformation of unsaturated slope subjected to post-earthquake
16 rainfall. *Comput Geotech.* 2022;148:104791.
- 17 45. White JA, Borja RI. Stabilized low-order finite element for coupled solid-deformation/fluid-diffusion and their application
18 to fault zone transients. *Comput Methods Appl Mech Engrg.* 2008;197:4353-4366.
- 19 46. Dowding CH. *Blast vibration monitoring and control.* Prentice-Hall: Englewood Cliffs, NJ, 1985.
- 20 47. Adapa G, Ueda K, Uzuoka R. Seismic stability of embankments with different densities and upstream conditions related to
21 the water level. *Soils Found.* 2021;61(1):185-197.
- 22 48. Xu J, Ueda K, Uzuoka R. Evaluation of failure of slopes with shaking-induced cracks in response to rainfall. *Landslides.*
23 2022;19(1):119-136.
- 24 49. van Genuchten MT. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. *Soil Sci Soc Am*
25 *J.* 1980; 44:892-898.
- 26 50. Bonet J, Wood RD. *Nonlinear continuum mechanics for finite element analysis.* Cambridge University Press. 1997.
- 27 51. Simo JC, Hughes TJR. *Computational inelasticity.* Springer-Verlag New York, Inc., 1998.
- 28 52. Budhu M. *Soil mechanics and foundations, 3rd Edition.* Wiley: John Wiley & Sons, Inc., Hoboken, 2010.
- 29 53. Morris PH. Analytical solutions of linear finite-strain one-dimensional consolidation. *J Geotech Geoenviron Eng.*
30 2002;4:319-326.
- 31 54. Simon BR, Zienkiewicz OC, Paul DK. An analytical solution for the transient response of saturated porous elastic solids.
32 *Int J Numer Anal Methods Geomech.* 1984;8:381-398.
- 33 55. Sarin H, Kokkolaras M, Hulbert G, Papalambros P, Barbat S, Yang RJ. Comparing time histories for validation of simulation
34 models: Error measures and metrics. *J Dyn Sys Meas Control.* 2010;132(6):061401.
- 35 56. Ni T, Pesavento F, Zaccariotto M, Galvanetto U, Schrefler BA. Numerical simulation of forerunning fracture in saturated
36 porous solids with hybrid fem/peridynamic model. *Comput Geomech.* 2021;133:104024.
- 37 57. Khoei AR, Mortazavi SMS, Simoni L, Schrefler BA. Irregular and stepwise behaviour of hydraulic fracturing: insights from
38 linear cohesive crack modelling with maximum stress criterion. *Comput Geomech.* 2023;161:105570.
- 39 58. Pampillón P, Santillán D, Mosquera JC, Cueto-Felgueroso L. The role of pore fluids in supershear earthquake ruptures. *Sci*
40 *Rep.* 2023;13(1):398.