

Order of Accuracy of Functional Fitting Runge-Kutta-Nyström Formula

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1 Introduction

In this paper we introduce functional fitting Runge-Kutta-Nyström method which integrates some set of functions exactly. The method proposed here is a generalization of exponentially or trigonometrically fitting Runge-Kutta-Nyström methods.

2 Functional fitting Runge-Kutta-Nyström method

Consider the variable coefficient Runge-Kutta-Nyström method

$$\begin{cases} y_{n+1} = y_n + h y'_n + h^2 \sum_{i=1}^s \bar{b}_i(t_n, h) f(t_n + c_i h, Y_i), \\ y'_{n+1} = y'_n + h \sum_{i=1}^s b_i(t_n, h) f(t_n + c_i h, Y_i), \\ Y_i = y_n + c_i h y'_n + h^2 \sum_{j=1}^s \bar{a}_{i,j}(t_n, h) f(t_n + c_j h, Y_j), \end{cases} \quad (1)$$

for solving the second order ODE of the form

$$y''(t) = f(t, y), \quad y(0) = y_0, \quad y'(0) = y'_0, \quad t \in [t_0, T]. \quad (2)$$

We will call the method *functional fitting Runge-Kutta-Nyström* (FRKN) method, when the method is designed to integrate some functions exactly. The coefficients $\bar{a}_{i,j}$, b_i and $\bar{b}_i$ of the FRKN to be considered here are determined by the simultaneous equation

$$\begin{cases} u_m(t+h) = u_m(t) + h u'_m(t) + h^2 \sum_{i=1}^s \bar{b}_i(t, h) u''_m(t + c_i h), \\ u'_m(t+h) = u'_m(t) + h \sum_{i=1}^s b_i(t, h) u''_m(t + c_i h), \\ u_m(t + c_i h) = u_m(t) + c_i h u'_m(t) + h^2 \sum_{j=1}^s \bar{a}_{i,j}(t, h) u''_m(t + c_j h), \quad i = 1, 2, \dots, s, \end{cases} \quad (3)$$

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where the functions $u_m''(t) = \varphi_m(t)$ are linearly independent on $[t_0, T]$, that is the Wronskian matrix W given by

$$W(\varphi_1, \varphi_2, \dots, \varphi_s) \equiv \begin{pmatrix} \varphi_1(t) & \varphi_1'(t) & \dots & \varphi_1^{(s-1)}(t) \\ \varphi_2(t) & \varphi_2'(t) & \dots & \varphi_2^{(s-1)}(t) \\ \dots & \dots & \dots & \dots \\ \varphi_s(t) & \varphi_s'(t) & \dots & \varphi_s^{(s-1)}(t) \end{pmatrix} \quad (4)$$

is nonsingular for all $t \in [t_0, T]$. For the uniqueness of the coefficients we have:

Theorem 1 *The coefficients $\bar{a}_{i,j}(t, h)$, $b_i(t, h)$ and $\bar{b}_i(t, h)$ determined by (3) are unique for small $h > 0$, if the functions $\varphi_m(t)$ are sufficiently smooth and linearly independent.*

Proof. It is clear from (3) that these coefficients are unique, if the matrix given by

$$\Phi(t, h) = \begin{pmatrix} \varphi_1(t + c_1 h) & \varphi_1(t + c_2 h) & \dots & \varphi_1(t + c_s h) \\ \varphi_2(t + c_1 h) & \varphi_2(t + c_2 h) & \dots & \varphi_2(t + c_s h) \\ \vdots & \vdots & \dots & \vdots \\ \varphi_s(t + c_1 h) & \varphi_s(t + c_2 h) & \dots & \varphi_s(t + c_s h) \end{pmatrix}$$

is nonsingular. The matrix Φ can be expressed as

$$\Phi(t, h) = W(\varphi_1, \varphi_2, \dots, \varphi_s) \begin{pmatrix} 1 & 1 & \dots & 1 \\ c_1 h & c_2 h & \dots & c_s h \\ \vdots & \vdots & \dots & \vdots \\ \frac{(c_1 h)^{s-1}}{(s-1)!} & \frac{(c_2 h)^{s-1}}{(s-1)!} & \dots & \frac{(c_s h)^{s-1}}{(s-1)!} \end{pmatrix} + O(h^s).$$

Since W is assumed to be nonsingular, and the second matrix in the right-hand side is also nonsingular from the assumption that c_i are different from each other, then we have the nonsingularity of $\Phi(t, h)$ for sufficiently small $h > 0$. ■

Hereafter, we simply denote the coefficients by $\bar{a}_{i,j}$, b_i and $\bar{b}_i$, and denote their power series expansions in h by

$$\begin{aligned} \bar{a}_{i,j} &= \bar{a}_{i,j}^{(0)} + \bar{a}_{i,j}^{(1)} h + \bar{a}_{i,j}^{(2)} h^2 + \dots, & b_i &= b_i^{(0)} + b_i^{(1)} h + b_i^{(2)} h^2 + \dots, \\ \bar{b}_i &= \bar{b}_i^{(0)} + \bar{b}_i^{(1)} h + \bar{b}_i^{(2)} h^2 + \dots \end{aligned} \quad (5)$$

If we take $u_m''(t) = t^{m-1}$ ($m = 1, 2, \dots, s$) in (3), then $\bar{a}_{i,j} = \bar{a}_{i,j}^{(0)}$, $b_i = b_i^{(0)}$, and $\bar{b}_i = \bar{b}_i^{(0)}$, and the method reduces to the direct collocation Runge-Kutta-Nyström method proposed by Van der Houwen et al [5].

Here we consider the order of accuracy of the FRKN method. The order of accuracy of the FRKN is defined to be $p = \min\{p_1, p_2\}$, where p_1 and p_2 are the integers satisfying

$$E := y_{n+1} - y(t_{n+1}) = O(h^{p_1+1}), \quad E' := y'_{n+1} - y'(t_{n+1}) = O(h^{p_2+1}), \quad h \rightarrow 0, \quad (6)$$

and the stage order is defined to be the minimum of the r_i ($i = 1, 2, \dots, s$) satisfying

$$e_i \equiv Y_i - y(t_n + c_i h) = O(h^{r_i+1}), \quad h \rightarrow 0, \quad i = 1, 2, \dots, s. \quad (7)$$

In these definitions, like in the case of constant coefficient methods, the localizing assumption $y_n = y(t_n)$, $y'_n = y'(t_n)$ is of course made, and unlike in that case, the errors are considered in the situation that the coefficients are being changed as the functions of h , when $h \rightarrow 0$.

In order to analyze the order of accuracy of the FRKN, let us define the quantities:

$$\begin{aligned} B(q) &= \sum_{i=1}^s b_i c_i^{q-1} - \frac{1}{q}, & \bar{B}(q) &= \sum_{i=1}^s \bar{b}_i c_i^{q-1} - \frac{1}{q(q+1)}, \\ \bar{C}_i(q) &= \sum_{j=1}^s \bar{a}_{i,j} c_j^{q-1} - \frac{c_i^{q+1}}{q(q+1)}, & i &= 1, 2, \dots, s. \end{aligned} \quad (8)$$

In (3) expanding $u_m''(t) = \varphi_m(t)$ into their power series, we find

$$\sum_{q=1}^{\infty} \frac{B(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad (9)$$

$$\sum_{q=1}^{\infty} \frac{\bar{C}_i(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad i = 1, 2, \dots, s+1, \quad (10)$$

where we set $c_{s+1} = 1$, $\bar{a}_{s+1,j} = \bar{b}_j$ and $\bar{C}_{s+1}(q) = \bar{B}(q)$. For the orders of $B(q)$ and $\bar{C}_i(q)$ we have the following lemma:

Lemma 1 Let the orders of $B(q)$, $\bar{B}(q)$ and $\bar{C}_i(q)$ ($i = 1, 2, \dots, s$) be

$$B(q) = O(h^{\mu_q}), \quad \bar{B}(q) = O(h^{\bar{\mu}_q}), \quad \bar{C}_i(q) = O(h^{\nu_{i,q}}),$$

then for $q = 1, 2, \dots, s$

$$\mu_q \geq s+1-q, \quad \bar{\mu}_q \geq s+1-q, \quad \nu_{i,q} \geq s+1-q, \quad i = 1, 2, \dots, s. \quad (11)$$

Proof. Let us define the power series expansion of $B(q)$ as

$$B(q) = B^{(0)}(q) + B^{(1)}(q) + B^{(2)}(q)h^2 + \dots,$$

then (9) means

$$\sum_{l=1}^{\infty} \left(\sum_{q=1}^l \varphi_m^{(q-1)}(t) \frac{B^{(l-q)}(q)}{(q-1)!} \right) h^l = 0, \quad m = 1, 2, \dots, s.$$

Since the coefficients of h^l are 0 for all l , we have for $l = 1, 2, \dots, s$

$$W(\varphi_1, \varphi_2, \dots, \varphi_s) \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \cdots & \beta_{1,s} \\ 0 & \beta_{2,2} & \cdots & \cdots & \beta_{2,s} \\ 0 & 0 & \ddots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & \beta_{s,s} \end{pmatrix} = 0,$$

where we set $\beta_{q,l} = B^{(l-q)}(q)/(q-1)!$. From the nonsingularity of W , we have

$$\beta_{q,q} = \beta_{q,q+1} = \cdots = \beta_{q,s} = 0, \quad q = 1, 2, \dots, s,$$

which proves the first inequality in (11). The second and third ones are proved in the same way. ■

Corollary 1 *The constant terms of the expansions of $a_{i,j}$, b_i and $\bar{b}_i$ satisfy the so-called simplifying assumption:*

$$\sum_{j=1}^s b_i^{(0)} c_i^{q-1} = \frac{1}{q}, \quad q = 1, 2, \dots, s, \quad (12)$$

$$\sum_{j=1}^s \bar{b}_i^{(0)} c_i^{q-1} = \frac{1}{q(q+1)}, \quad q = 1, 2, \dots, s, \quad (13)$$

$$\sum_{j=1}^s \bar{a}_{i,j}^{(0)} c_j^{q-1} = \frac{c_i^{q+1}}{q(q+1)}, \quad i = 1, 2, \dots, s, \quad q = 1, 2, \dots, s. \quad (14)$$

These equations determine $\bar{a}_{i,j}^{(0)}$, $b_i^{(0)}$ and $\bar{b}_i^{(0)}$ uniquely, since c_i are assumed to be different from each other.

Lemma 2 *If $\nu_{i,q}$, μ_q and $\bar{\mu}_q$ are equal to their lower bounds in (11), i.e.*

$$\nu_{i,q} = \mu_q = \bar{\mu}_q = s + 1 - q, \quad q = 1, 2, \dots, s, \quad (15)$$

then for any sufficiently smooth function $g(t)$, we have

$$\begin{aligned} g(c_i h) &= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^s \bar{a}_{i,j} g''(c_j h) + O(h^{s+2}) \\ &= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^s \bar{a}_{i,j}^{(0)} g''(c_j h) + O(h^{s+2}), \quad i = 1, 2, \dots, s+1, \end{aligned} \quad (16)$$

$$g'(h) = g'(0) + h \sum_{i=1}^s b_i g''(c_i h) + O(h^{s+1}) = g'(0) + h \sum_{i=1}^s b_i^{(0)} g''(c_i h) + O(h^{s+1}). \quad (17)$$

Proof. Let $g(t)$ be a sufficiently smooth function, then

$$\begin{aligned} g(c_i h) &= g(0) + c_i h g'(0) + h^2 \sum_{q=1}^{\infty} \frac{1}{(q-1)!} (g'')^{(q-1)}(0) h^{q-1} \left(\sum_{j=1}^s \bar{a}_{i,j} c_j^{q-1} - \bar{C}_i(q) \right) \\ &= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^s \bar{a}_{i,j} g''(c_j h) - \sum_{q=1}^s \frac{\bar{C}_i(q)}{(q-1)!} h^{q+1} g^{(q+1)}(0) + O(h^{s+2}). \end{aligned} \quad (18)$$

In this expression we have from the assumption of this lemma $\bar{C}_i(q) h^{q+1} = O(h^{s+2})$, which leads to the first relations in (16). The second relation is also proved by noting that (18) is valid even for the case that $\varphi_m(t) = t^{m-1}$ ($m = 1, 2, \dots, s$), in which case $\bar{a}_{i,j} = \bar{a}_{i,j}^{(0)}$. The proof of (17) is done by the straightforward manner. ■

Next we consider the stage order of the FRKN for the case that (15) holds. If the solution $y(t)$ of (2) is sufficiently smooth, then we have from the result of Lemma 2

$$e_i = (1 - h^2 f_y \bar{a}_{i,i})^{-1} h^2 \sum_{\substack{j=1 \\ j \neq i}}^s (\bar{a}_{i,j} e_j f_y + O(e_j^2)) + O(h^{s+2}), \quad i = 1, 2, \dots, s, \quad (19)$$

where f_y is the partial derivative of f with respect to y , and is assumed to be bounded. Since $e_i = O(h^{r_i+1})$ we have from (19) $r = \min_i \{r_i\} = \min\{r+2, s+1\}$, which means $r = s+1$, i.e. the stage order of the method is $s+1$.

Next we consider the order of accuracy of the FRKN. We have also from (15)

$$\begin{aligned} E &= y_1 - y(h) = h^2 f_y \sum_{i=1}^s \bar{b}_i e_i + O(h^{s+2}) = O(h^{s+2}), \\ E' &= y'_1 - y'(h) = h f_y \sum_{i=1}^s b_i e_i + O(h^{s+1}) = O(h^{s+1}), \end{aligned} \quad (20)$$

which means that the order of accuracy of the method is s . Thus we have proved:

Theorem 2 *The stage order of the functional fitting Runge-Kutta-Nyström method is $s+1$, and the order of accuracy of the method is s , when (15) holds.*

3 Higher order formula

Here we consider the order of accuracy of the FRKN for the cases that the relations

$$\mu_q \geq s+1-q, \quad \bar{\mu}_q \geq s+1-q$$

hold for $q = 1, 2, \dots, s$. We note again that the constant terms $b_i^{(0)}$, $\bar{b}_i^{(0)}$ and $\bar{a}_{ij}^{(0)}$, which are determined uniquely by (12), (13) and (14), respectively, are the coefficients of the direct collocation Runge-Kutta-Nyström method proposed by Van der Houwen et. al [5]. According to [1] and [5], if we take the abscissae c_i such that

$$\int_0^1 t^{q-1} \prod_{i=1}^s (t - c_i) dt = 0, \quad q = 1, 2, \dots, \nu, \quad 1 \leq \nu \leq s, \quad (21)$$

then for the $b_i^{(0)}$ determined by (12), the stronger relation is in fact valid:

$$\sum_{i=1}^s b_i^{(0)} c_i^{q-1} = \frac{1}{q}, \quad q = 1, 2, \dots, s + \nu. \quad (22)$$

Moreover for the $\bar{b}_i^{(0)}$ determined (13) are related to the $b_i^{(0)}$ by

$$\bar{b}_i^{(0)} = b_i^{(0)} (1 - c_i), \quad i = 1, 2, \dots, s. \quad (23)$$

As a result we have instead of (13)

$$\sum_{i=1}^s \bar{b}_i^{(0)} c_i^{q-1} = \frac{1}{q(q+1)}, \quad q = 1, 2, \dots, s + \nu - 1. \quad (24)$$

Thus from (22) and (24) we have

$$B^{(0)}(q) = 0, \quad q = 1, 2, \dots, s + \nu, \quad \bar{B}^{(0)}(q) = 0, \quad q = 1, 2, \dots, s + \nu - 1. \quad (25)$$

When $\nu > 1$ in (21) we have the following lemma:

Lemma 3 If $\nu > 1$ then for $1 \leq \xi \leq \nu - 1$,

$$\sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}^{(0)} = b_j^{(0)} \left(\frac{c_j^{\xi+1}}{\xi(\xi+1)} - \frac{c_j}{\xi} + \frac{1}{\xi+1} \right). \quad (26)$$

Proof. Let α_j be

$$\alpha_j = \sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}^{(0)} - b_j^{(0)} \left(\frac{c_j^{\xi+1}}{\xi(\xi+1)} - \frac{c_j}{\xi} + \frac{1}{\xi+1} \right), \quad j = 1, 2, \dots, s,$$

then from (22) we have $\sum_{j=1}^s \alpha_j c_j^{q-1} = 0$ ($q = 1, 2, \dots, s$), which means $\alpha_j = 0$ ($j = 1, 2, \dots, s$), since c_i are different from each other. ■

Next we define the quantity $D(q, \xi)$ by

$$D(q, \xi) = \sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} \bar{C}_i(q) = \sum_{j=1}^s d_j c_j^{q-1} - \frac{1}{q(q+1)} \sum_{i=1}^s b_i^{(0)} c_i^{q+\xi}, \quad (27)$$

where we set $d_j = \sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}$ and $d_j = d_j^{(0)} + d_j^{(1)}h + \dots$. For $D(q, \xi)$ we can find the two relations which are similar to (10) and (25). The first one is

$$\sum_{q=1}^{\infty} \frac{D(q, \xi)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad (28)$$

which is easily derived by multiplying both sides of (10) by $b_i^{(0)} c_i^{\xi-1}$ and summing over i . The second one is

$$D^{(0)}(q, \xi) = 0, \quad q = 1, 2, \dots, s + \nu - \xi - 1, \quad \xi = 1, 2, \dots, \nu - 1, \quad \nu > 1, \quad (29)$$

which can be shown by Lemma 3.

Lemma 4 Consider the function $F(q)$ defined by

$$F(q) = \sum_{i=1}^s f_i(h) c_i^{q-1} - \eta_q, \quad q = 1, 2, \dots, s,$$

where $f_i(h)$ is analytic at $h = 0$, and η_q depends only on q . If the function $F(q)$ satisfies

$$\sum_{q=1}^{\infty} \frac{F(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad m = 1, 2, \dots, s, \quad (30)$$

and for some $\kappa > 0$

$$F^{(0)}(q) = 0, \quad q = 1, 2, \dots, s + \kappa,$$

then we have

$$F(q) = O(h^{\tau_q}), \quad \tau_q = \max\{s + \kappa + 1 - q, \kappa + 1\}, \quad q = 1, 2, \dots, s + \kappa,$$

$$\sum_{i=1}^s (f_i^{(0)} - f_i(h)) g(c_i h) = O(h^{s+\kappa}), \quad (31)$$

where $f_i^{(0)}$ are the constant terms of the power series expansions of $f_i(h)$, and $g(t)$ is a sufficiently smooth function.

Proof. The proof of this theorem is done in the same manner as in the proof of Lemma 4 in [3].

Corollary 2 For the orders of $B(q)$, $\bar{B}(q)$ and $D(q, \xi)$, we have for $\nu > 1$

$$\begin{aligned} B(q) &= O(h^{\mu_q}), \quad \mu_q = \max\{s + \nu + 1 - q, \nu + 1\}, \quad q = 1, 2, \dots, s + \nu, \\ \bar{B}(q) &= O(h^{\bar{\mu}_q}), \quad \bar{\mu}_q = \max\{s + \nu - q, \nu\}, \quad q = 1, 2, \dots, s + \nu - 1, \\ D(q, \xi) &= O(h^{\lambda_{q,\xi}}), \quad \lambda_{q,\xi} = \max\{s + \nu - \xi - q, \nu - \xi\}, \quad q = 1, 2, \dots, s + \nu - \xi - 1, \end{aligned}$$

and from (31) we have

$$\begin{aligned} \sum_{i=1}^s (b_i^{(0)} - b_i) g(c_i h) &= O(h^{s+\nu}), \quad \sum_{i=1}^s (\bar{b}_i^{(0)} - \bar{b}_i) g(c_i h) = O(h^{s+\nu-1}), \\ \sum_{i=1}^s (d_i^{(0)} - d_i) g(c_i h) &= O(h^{s+\nu-\xi-1}), \quad \nu > 1. \end{aligned} \quad (32)$$

Lemma 5 If relation (21) holds, then for any sufficiently smooth function $g(t)$

$$\begin{aligned} g(h) &= g(0) + h g'(0) + h^2 \sum_{j=1}^s \bar{b}_j g''(c_j h) + O(h^{s+\nu+1}) \\ &= g(0) + h g'(0) + h^2 \sum_{j=1}^s \bar{b}_j^{(0)} g''(c_j h) + O(h^{s+\nu+1}), \\ g'(h) &= g'(0) + h \sum_{i=1}^s b_i g''(c_i h) + O(h^{s+\nu+1}) = g'(0) + h \sum_{i=1}^s b_i^{(0)} g''(c_i h) + O(h^{s+\nu+1}). \end{aligned}$$

Proof. This lemma is proved in the same way as Lemma 2.

Lemma 6 If $\nu > 1$, then for $\xi = 1, 2, \dots, \nu - 1$,

$$\sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} e_i = (h^2 f_y) \sum_{i=1}^s b_i^{(0)} \left(\frac{c_i^{\xi+1}}{\xi(\xi+1)} - \frac{c_i}{\xi} + \frac{1}{\xi+1} \right) e_i + O(h^{s+\nu-\xi+1}).$$

Proof. Multiplying both sides of

$$y(c_i h) = y_0 + c_i h y'_0 + h^2 \sum_{j=1}^s \bar{a}_{i,j} y''(c_j h) - \sum_{q=1}^s \frac{h^{q+1}}{(q-1)!} \bar{C}_i(q) y^{(q+1)}(0), \quad (33)$$

by $b_i^{(0)} c_i^{\xi-1}$, and summing for i , we have

$$\begin{aligned} \sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} y(c_i h) &= \frac{1}{\xi} y_0 + \frac{1}{\xi+1} h y'_0 + h^2 \sum_{i,j=1}^s b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j} y''(c_j h) \\ &\quad - \sum_{q=1}^{s+\nu-\xi-1} \frac{h^{q+1}}{(q-1)!} D(q, \xi) y^{(q+1)}(0) + O(h^{s+\nu-\xi+1}). \end{aligned} \quad (34)$$

Taking into account the relation

$$q+1 + \lambda_{q,\xi} \geq s + \nu - \xi + 1, \quad \text{for } q = 1, 2, \dots, s + \nu - \xi - 1,$$

and using (32), we have

$$\sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} y(c_i h) = \frac{1}{\xi} y_0 + \frac{1}{\xi+1} h y'_0 + h^2 \sum_{i,j=1}^s b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}^{(0)} y''(c_j h) + O(h^{s+\nu-\xi+1}). \quad (35)$$

Therefore we have from Lemma 3

$$\begin{aligned} \sum_{i=1}^s b_i^{(0)} c_i^{\xi-1} e_i &= h^2 \sum_{i,j=1}^s b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}^{(0)} (f_y e_j + O(e_j^2)) + O(h^{s+\nu-\xi+1}) \\ &= (h^2 f_y) \sum_{j=1}^s b_j^{(0)} \left(\frac{c_j^{\xi+1}}{\xi(\xi+1)} - \frac{c_j}{\xi} + \frac{1}{\xi+1} \right) e_j + O(h^{s+\nu-\xi+1}), \end{aligned} \quad (36)$$

where $e_i = O(h^{s+2})$ is used. ■

Next we consider the order of accuracy of the method, for the two cases, $\nu = 1$ and $\nu > 1$. If $\nu = 1$ then we have from (23) and the result of Lemma 5

$$\begin{aligned} E' &= y'_1 - y'(h) = h f_y \sum_{i=1}^s b_i^{(0)} e_i + O(h^{s+2}) = O(h^{s+2}), \\ E &= y_1 - y(h) = h^2 f_y \sum_{i=1}^s \bar{b}_i^{(0)} e_i + O(h^{s+2}) = O(h^{s+2}), \end{aligned}$$

so that the method is of order $s+1$. For $\nu > 1$, we have from Lemma 6

$$\begin{aligned} E' &= h f_y \sum_{i=1}^s b_i^{(0)} e_i + O(h^{s+\nu+1}) \\ &= h^3 (f_y)^2 \sum_{i=1}^s b_i^{(0)} \left(\frac{c_i^2}{2} - c_i + \frac{1}{2} \right) e_i + O(h^{s+\nu+1}) \\ &= \dots \\ &= \begin{cases} h^{\nu+1} (f_y)^{\frac{\nu}{2}+1} \sum_{i=1}^s b_i^{(0)} Q_\nu(c_i) e_i + O(h^{s+\nu+1}), & \nu = \text{even} \\ h^\nu (f_y)^{\frac{\nu-1}{2}+1} \sum_{i=1}^s b_i^{(0)} Q_{\nu-1}(c_i) e_i + O(h^{s+\nu+1}), & \nu = \text{odd} \end{cases} \\ &= O(h^{s+\nu+1}), \end{aligned} \quad (37)$$

where $Q_\nu(c_i)$ is a polynomial in c_i of degree ν . On the other hand, E is given by

$$E = hE' - h^2 f_y \sum_{i=1}^s b_i^{(0)} c_i e_i + O(h^{s+\nu+1}).$$

Evaluating the sum in this expression in the same way, we have $E = O(h^{s+\nu+1})$. Thus we have:

Theorem 3 *If the abscissae c_i are taken to satisfy (21), then the order of accuracy of the FRKN is $s + \nu$, for any ν ($1 \leq \nu \leq s$).*

Note that this theorem is a generalization of the theorem (Theorem 3.4 of [5]) which has proved that the order of accuracy of the direct collocation Runge-Kutta-Nyström method with the same abscissae is being $s + \nu$.

Corollary 3 *The attainable order of the FRKN method is $2s$.*

4 Numerical examples

Consider the 3-stage FRKN method with the abscissae $c_1 = 0$, $c_2 = 0.5$, $c_3 = 1$, and with $\varphi_1(t) = \cos \omega t$, $\varphi_2(t) = \sin \omega t$, $\varphi_3(t) = 1$, which are linearly independent functions when $\omega > 0$. This method is expected to be of order 4, since orthogonal condition (21) holds with $s = 3$ and $\nu = 1$. The equation to be solved is

$$y'' = -y + \varepsilon \cos t, \quad y(0) = 1, \quad y'(0) = 1, \quad (38)$$

which has the exact solution $y(t) = \cos t + \frac{1}{2} \varepsilon t \sin t$. We solve the equation by the method with $\omega = 1$ and obtain the global errors at $t = 20$ (see Table 1). We can easily see from Table 1 that the order of accuracy of the method is being 4 for $\varepsilon = 0.05$, and that for $\varepsilon = 0.0$ the method is exact; the values in the column headed with $\varepsilon = 0.0$ must be the accumulations of the round-off errors, since the rounding unit of our computer is $2^{-52} \simeq 2.22 \times 10^{-16}$.

Next we consider the well-known two-body problem [2]:

$$\begin{aligned} y_1'' &= -y_1/r^3, \quad y_2'' = -y_2/r^3, \quad r = \sqrt{y_1^2 + y_2^2} \\ y_1(0) &= 1 - e, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = \sqrt{\frac{1+e}{1-e}}, \end{aligned} \quad (39)$$

where e ($0 \leq e < 1$) is an eccentricity. The exact solution of this system is given by

$$y_1(t) = \cos u - e, \quad y_2(t) = \sqrt{1 - e^2} \sin u, \quad (40)$$

where u is the solution of Kepler's equation $u = t + e \sin u$. Here we calculate the global errors at $t = 20$ of the two methods, 3-stage FRKN method with $\omega = 1$ and 2-stage Gauss Runge-Kutta method, for various h (see Table 2). From the table we can see that the FRKN method is accurate compared with the 2-stage Gauss Runge-Kutta method.

Table 1. Global errors at $t = 20$ of problem (38).

i	$\varepsilon = 0.05$		$\varepsilon = 0.0$
	$\log_2 R_i$	$\log_2(R_i/R_{i-1})$	$\log_2 R_i$
1	-15.1		-50.4
2	-19.0	-3.99	-49.5
3	-23.0	-4.00	-50.1
4	-27.0	-4.00	-51.4
5	-31.0	-4.00	-48.5
6	-35.0	-4.00	-49.4
7	-39.0	-4.00	-51.2
8	-43.0	-3.99	-48.8
9	-46.6	-3.62	-48.5
10	-46.1	.500	-47.2

$h = 2^{-i}$, $R_i = |y_n - y(nh)|$, where $nh = 20$.

Table 2. Global errors at $t = 20$ of the two-body problem.

	FRKN method			Gauss RK method		
	$h=0.200$	$h=0.100$	$h=0.050$	$h=0.200$	$h=0.100$	$h=0.050$
$e=0.00$	1.119e-13	4.186e-14	2.242e-13	5.839e-04	3.658e-05	2.290e-06
$e=0.01$	1.886e-05	1.182e-06	7.402e-08	5.939e-04	3.623e-05	2.266e-06
$e=0.10$	2.280e-04	1.429e-05	8.938e-07	8.345e-04	5.238e-05	3.278e-06
$e=0.50$	5.665e-03	7.101e-04	4.897e-05	2.121e-02	1.493e-03	9.551e-05

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