

A new definition of resolvents for convex functions on complete geodesic spaces 完備測地距離空間上の凸関数に対するリゾルベント の新しい定義

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Abstract

In this paper, we propose a new resolvent in complete geodesic spaces and we show that it is well-defined as a single valued mapping. Moreover, we propose spherical nonspreadingness of sum-type and we show that the new resolvent satisfies this condition.

1 Introduction

Let X be a complete CAT(0) space and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. A resolvent for f is defined by

$$J_f x = \operatorname{argmin}_{y \in X} \{f(y) + d(y, x)^2\}$$

for all $x \in X$. In 1998, Mayer [7] proved its well-definedness; see also Jost [2]. It is known that J_f is nonspreading, that is,

$$2d(J_f x, J_f y)^2 \leq d(J_f x, y)^2 + d(x, J_f y)^2$$

for all $x, y \in X$. See [6] for more details.

Let X be a complete CAT(1) space with $d(v, v') < \pi/2$ for all $v, v' \in X$ and f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. In this case, a

resolvent of f is defined by

$$Q_fx = \operatorname{argmin}_{y \in X} \{f(y) + \tan d(y, x) \sin d(y, x)\}$$

for all $x \in X$. In 2016, Kimura and Kohsaka [5] proved its well-definedness. They also showed that the resolvent is spherically nonspreading of product-type, that is,

$$\cos^2 d(Q_fx, Q_fy) \geq \cos d(Q_fx, y) \cos d(x, Q_fy)$$

for all $x, y \in X$.

In this paper, we propose a new resolvent in a complete CAT(1) space and we show that it is well-defined as a single-valued mapping. Moreover, we propose spherical nonspreadingness of sum-type and we show that the new resolvent satisfies this condition.

2 Preliminaries

Let X be a metric space with metric d . We denote by $\mathcal{F}(T)$ the set of all fixed points of a mapping from X into itself. A continuous mapping $c : [0, l] \rightarrow X$ is called geodesic if c satisfies $c(0) = x$, $c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $x, y \in X$ and $s, t \in [0, l]$. Its image, which is denoted by $[x, y]$, is called geodesic segment with endpoints x and y . X is said to be a geodesic space if there exists $[x, y]$ for all $x, y \in X$. In this paper, when X is a geodesic space, its geodesic is always assumed to be unique.

Let X be a geodesic space. There exists a unique point $z \in [x, y]$ such that $d(x, z) = (1 - \alpha)d(x, y)$ and $d(z, y) = \alpha d(x, y)$ for all $x, y \in X$ and $\alpha \in [0, 1]$. This point is called convex combination of x and y , which is denoted by $\alpha x \oplus (1 - \alpha)y$. A subset $C \subset X$ is said to be convex if $[x, y] \subset C$ for all $x, y \in C$. A geodesic triangle of vertices $x, y, z \in X$ is defined by $[x, y] \cup [y, z] \cup [z, x]$, which is denoted by $\triangle(x, y, z)$.

Let M_κ^2 be a two dimensional model space for all $\kappa \in \mathbb{R}$. For example, $M_0^2 = \mathbb{R}^2$, $M_1^2 = \mathbb{S}^2$ and $M_{-1}^2 = \mathbb{H}^2$. A comparison triangle to $\triangle(x, y, z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ is defined by $[\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$ with $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, x) = d(\bar{y}, \bar{x})$ and $d(z, x) = d(\bar{z}, \bar{x})$, which is denoted by $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$. $\bar{z} \in [\bar{x}, \bar{y}]$ is called comparison point of $z \in [x, y]$ if $d(x, z) = d(\bar{x}, \bar{z})$ holds. For all $\kappa \in \mathbb{R}$, X is called a CAT(κ) space if $d(p, q) \leq d(\bar{p}, \bar{q})$ holds whenever $\bar{p}, \bar{q} \in \bar{\triangle}$ are comparison points for $p, q \in \triangle$. In general, if $\kappa < \kappa'$, then the CAT(κ) spaces are CAT(κ') spaces [1]. The following lemma is important to show the main theorem.

Lemma 2.1 ([3]). *Let X be a complete CAT(1) space, $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$, and $\alpha \in [0, 1]$. Then*

$$\begin{aligned} & \cos d(\alpha x \oplus (1 - \alpha)y, z) \sin d(x, y) \\ & \geq \cos d(x, z) \sin \alpha d(x, y) + \cos d(y, z) \sin(1 - \alpha)d(x, y). \end{aligned}$$

Lemma 2.2 ([4]). *Let X, x, y, z , and α be the same as in Lemma 2.1. Then*

$$\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \cos \frac{1}{2}d(x, y) \geq \frac{1}{2} \cos d(x, z) + \frac{1}{2} \cos d(y, z).$$

Lemma 2.3 ([5]). *Let X, x, y, z , and α be the same as in Lemma 2.1. If $d(x, z) < \pi/2$ and $d(y, z) < \pi/2$, then*

$$\cos d(\alpha x \oplus (1 - \alpha)y, z) \geq \alpha \cos d(x, z) + (1 - \alpha) \cos d(y, z).$$

Let X be a geodesic space and f a function from X into $]-\infty, \infty]$. The function f is said to be lower semicontinuous if the set $\{x \in X \mid f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$. If f is continuous, then it is lower semicontinuous. The domain of f is defined by $\{x \in X \mid f(x) \in \mathbb{R}\}$, which is denoted by $\text{dom} f$. The function f is said to be proper if $\text{dom} f$ is nonempty. The function is said to be convex if

$$f(\alpha x \oplus (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in X$ and $\alpha \in]0, 1[$.

Lemma 2.4 ([5]). *Let X be a complete CAT(1) space with $d(v, v') < \pi/2$ for all $v, v' \in X$, f a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ and p an element of X . Suppose that $f(x_n) \rightarrow \infty$ whenever $\{x_n\}$ is a sequence of X with $d(p, x_n) \rightarrow \pi/2$. Then $\text{argmin}_X f$ is nonempty. Further, if*

$$x, y \in \text{dom} f, x \neq y \Rightarrow f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

then $\text{argmin}_X f$ consists of one point.

Lemma 2.5 ([5]). *Let X be a complete CAT(1) space with $d(v, v') < \pi/2$ for all $v, v' \in X$. Then every proper lower semicontinuous convex function from X into $]-\infty, \infty]$ is bounded below.*

3 Resolvents for convex functions in complete CAT(1) spaces

Let X be a metric space and T a mapping from X into itself. Then T is said to be spherically nonspreading of sum-type if

$$2 \cos d(Tx, Ty) \geq \cos d(Tx, y) + \cos d(x, Ty)$$

for all $x, y \in X$. It is obvious that if T is spherically nonspreading of sum-type, then T is spherically nonspreading of product-type.

In this section, we show that a new resolvent

$$R_f x := \underset{y \in X}{\text{argmin}} \{f(y) - \log(\cos d(y, x))\}$$

is well-defined, where f is a proper lower semicontinuous convex function. Moreover we show the fundamental properties of the new resolvent.

Throughout this section, we suppose that X is a complete CAT(1) space with $d(v, v') < \pi/2$ for all $v, v' \in X$.

Lemma 3.1. *Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. If*

$$g(\cdot) = f(\cdot) - \log(\cos d(\cdot, p))$$

for each $p \in X$, then g is a proper lower semicontinuous convex function from X into $]-\infty, \infty]$.

Proof. Let $x, y \in X$ and $\alpha \in]0, 1[$. From Lemma 2.3, we know that

$$\cos d(\alpha x \oplus (1 - \alpha)y, p) \geq \alpha \cos d(x, p) + (1 - \alpha) \cos d(y, p)$$

holds for all $p \in X$. Since $-\log t$ is decreasing and convex for all $t \geq 0$, we get

$$\begin{aligned} -\log(\cos d(\alpha x \oplus (1 - \alpha)y, p)) &\leq -\log(\alpha \cos d(x, p) + (1 - \alpha) \cos d(y, p)) \\ &\leq -\alpha \log(\cos d(x, p)) - (1 - \alpha) \log(\cos d(y, p)). \end{aligned}$$

Thus g is convex. On the other hand, it is obvious that g is proper and lower semicontinuous. $\square$

Theorem 3.2. *Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. If*

$$g(\cdot) = f(\cdot) - \log(\cos d(\cdot, p))$$

for each $p \in X$, then $\operatorname{argmin}_X g$ consists of one point.

Proof. Let $\{x_n\}$ be a sequence of X with $\lim_{n \rightarrow \infty} d(x_n, p) = \pi/2$ for each $p \in X$. Then, it is obvious that $\lim_{n \rightarrow \infty} (-\log(\cos d(x_n, p))) \rightarrow \infty$. On the other hand, from Lemma 2.5, we know that there exists $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in X$. So, we get

$$g(x_n) \geq K + \log(\cos d(x_n, p)) \rightarrow \infty$$

and hence $g(x_n) \rightarrow \infty$. From Lemma 2.4 and 3.1, $\operatorname{argmin}_X g$ is nonempty.

We next show that $\operatorname{argmin}_X g$ consists of one point. Suppose that $x, y \in \operatorname{dom} f$ with $x \neq y$. Then, Lemma 2.2 implies that

$$\begin{aligned} \cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right) &> \cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right) \cos \frac{1}{2}d(x, y) \\ &\geq \frac{1}{2} \cos d(x, p) + \frac{1}{2} \cos d(y, p) \end{aligned}$$

for all $p \in X$. Further, since $-\log t$ is decreasing and convex for all $t > 0$, we get

$$-\log\left(\cos d\left(\frac{1}{2}x \oplus \frac{1}{2}y, p\right)\right) < -\log\left(\frac{1}{2} \cos d(x, p) + \frac{1}{2} \cos d(y, p)\right)$$

$$\leq -\frac{1}{2} \log(\cos d(x, p)) - \frac{1}{2} \log(\cos d(y, p)).$$

Since f is convex, the inequality

$$g\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) < \frac{1}{2}g(x) + \frac{1}{2}g(y)$$

holds. Thus $\operatorname{argmin}_X g$ consists of one point. $\square$

Definition 3.3. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$. Then we define a new resolvent $R_f : X \rightarrow X$ by

$$R_f x = \operatorname{argmin}_{y \in X} \{f(y) - \log(\cos d(y, x))\}$$

for all $x \in X$. From Theorem 3.2, we know that R_f is well-defined.

Remark 3.4. Let C be a nonempty closed convex subset of X . If $f = i_C$, then $R_f = P_C$, where P_C is metric projection from X onto C . In fact,

$$\begin{aligned} R_f x &= \operatorname{argmin}_{y \in C} \{f(y) - \log(\cos d(y, x))\} \\ &= \operatorname{argmin}_{y \in C} \{-\log(\cos d(y, x))\} = \operatorname{argmin}_{y \in C} d(y, x) = P_C x \end{aligned}$$

for all $x \in X$.

Theorem 3.5. Let f be a proper lower semicontinuous convex function from X into $]-\infty, \infty]$ and R_f a resolvent of f . Then the following properties hold:

- (i) R_f is spherically nonspreading of sum-type;
- (ii) $\mathcal{F}(R_f) = \operatorname{argmin}_X f$.

Proof. Put $T = R_f$. We first show (i). Let $x, y \in X$ with $Tx \neq Ty$ and put $z_t = tTx \oplus (1-t)Ty$ for all $t \in]0, 1[$. Then, by the definition of T and convexity of f , we have

$$\begin{aligned} f(Ty) - \log(\cos d(Ty, y)) &\leq f(z_t) - \log(\cos d(z_t, y)) \\ &\leq tf(Tx) + (1-t)f(Ty) - \log(\cos d(z_t, y)) \end{aligned}$$

and hence

$$\begin{aligned} t(f(Tx) - f(Ty)) &\geq \log(\cos d(z_t, y)) - \log(\cos d(Ty, y)) \\ &= \log\left(\frac{\cos d(z_t, y)}{\cos d(Ty, y)}\right). \end{aligned}$$

So, using Lemma 2.1 and putting $D = d(Tx, Ty)$, we get

$$e^{t(f(Tx) - f(Ty))} \sin D$$

$$\begin{aligned}
&\geq \frac{\cos d(z_t, y) \sin D}{\cos d(Ty, y)} \\
&\geq \frac{\cos d(Tx, y) \sin tD + \cos d(Ty, y) \sin(1-t)D}{\cos d(Ty, y)} \\
&= \frac{\cos d(Tx, y) \sin tD + \cos d(Ty, y) \sin D \cos tD - \cos d(Ty, y) \cos D \sin tD}{\cos d(Ty, y)} \\
&= \sin tD \frac{\cos d(Tx, y) - \cos d(Ty, y) \cos D}{\cos d(Ty, y)} + \sin D \cos tD \\
&= \sin tD \frac{\cos d(Tx, y) - \cos d(Ty, y) \cos D}{\cos d(Ty, y)} + \sin D - 2 \sin D \sin^2 \frac{t}{2} D
\end{aligned}$$

and hence

$$\begin{aligned}
&\sin D \left(\frac{e^{t(f(Tx) - f(Ty))} - 1}{t} \right) \\
&\geq \frac{\sin tD}{t} \frac{\cos d(Tx, y) - \cos d(Ty, y) \cos D}{\cos d(Ty, y)} - \frac{2}{t} \sin D \sin^2 \frac{t}{2} D.
\end{aligned}$$

Letting $t \downarrow 0$, we obtain

$$\sin D(f(Tx) - f(Ty)) \geq \frac{D}{\cos d(Ty, y)} (\cos d(Tx, y) - \cos d(Ty, y) \cos D).$$

From this inequality, we also know that

$$\sin D(f(Ty) - f(Tx)) \geq \frac{D}{\cos d(Tx, x)} (\cos d(x, Ty) - \cos d(Tx, x) \cos D)$$

holds. Adding these inequalities, we get

$$\begin{aligned}
0 &\geq \frac{1}{\cos d(Ty, y)} (\cos d(Tx, y) - \cos d(Ty, y) \cos D) \\
&\quad + \frac{1}{\cos d(Tx, x)} (\cos d(x, Ty) - \cos d(Tx, x) \cos D).
\end{aligned}$$

So we have

$$\begin{aligned}
2 \cos D &\geq \frac{1}{\cos d(Ty, y)} \cos d(Tx, y) + \frac{1}{\cos d(Tx, x)} \cos d(x, Ty) \\
&\geq \cos d(Tx, y) + \cos d(x, Ty).
\end{aligned}$$

Thus we get the conclusion.

We next show (ii). Let $u \in \operatorname{argmin}_X f$. Then we have

$$f(u) - \log(\cos d(u, u)) = f(u) \leq f(y) \leq f(y) - \log(\cos d(y, u))$$

for all $y \in X$. Hence we get

$$f(u) - \log(\cos d(u, u)) = \inf_{y \in X} \{f(y) - \log(\cos d(y, u))\}.$$

This implies that $u \in \mathcal{F}(T)$. Inversely, let $u \in \mathcal{F}(T)$ and $t \in]0, 1[$. Then, by the definition of T and the convexity of f , we have

$$\begin{aligned} f(u) &= f(u) - \log(\cos d(u, u)) \\ &\leq f(ty \oplus (1-t)u) - \log(\cos d(ty \oplus (1-t)u, u)) \\ &= f(ty \oplus (1-t)u) - \log(\cos td(y, u)) \\ &\leq tf(y) + (1-t)f(u) - \log(\cos td(y, u)) \end{aligned}$$

and hence

$$f(u) \leq f(y) - \frac{\log(\cos td(y, u))}{t}$$

for all $y \in X$. Letting $t \downarrow 0$, we obtain $f(u) \leq f(y)$. This implies that $u \in \operatorname{argmin}_X f$. $\square$

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