

ON THE THINNESS OF A CLOSED SET IN THE NEIGHBORHOOD OF THE POINT AT INFINITY

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1. Introduction

Let X be a locally compact and non-compact Hausdorff space with countable basis and $G = G(x, y)$ be a continuous function-kernel on X satisfying the complete maximum principle.

For any compact K and for any set A in X , the G -capacity, $cap_G(K)$, of K and the inner G -capacity, $cap_G^i(A)$, of A are defined as usual.

If $cap_G^i(A) < +\infty$, then there exists a compact $K \subset A$ such that

$$(1.1) \quad cap_G^i(A) - \varepsilon < cap_G(K) \leq cap_G^i(A)$$

But then, the following inequality

$$(1.2) \quad cap_G^i(A \setminus K) < \varepsilon$$

does not necessarily hold. Because the inner G -capacity is, indeed, subadditive but not additive in general.

In this paper, we first define the several notions of the thinness of A in the neighbourhood of the point at infinity and investigate the mutual relations holding among them, when A is an unbounded closed set.

Then we consider the conditions on the kernel G and on A under that the inequality (1.2) also holds.

2. Preliminaries

A non-negative function $G = G(x, y)$ on $X \times X$ is called a *continuous function-kernel* if $G(x, y)$ is continuous in the extended sense on $X \times X$ and satisfies

$$0 \leq G(x, y) < +\infty \quad \text{for } \forall (x, y) \in X \times X \quad \text{s.t. } x \neq y,$$

$$0 < G(x, x) \leq +\infty \quad \text{for } \forall x \in X.$$

We denote by M the set of all positive measures on X . The G -potential $G\mu(x)$ and the G -energy $\|\mu\|$ of μ is defined by

$$G\mu(x) = \int G(x, y) d\mu(y),$$

$$\|\mu\|^2 = \int G\mu(x) d\mu(x)$$

respectively.

Put

$$M_o = \{\mu \in M ; \text{support } S\mu \text{ of } \mu \text{ is compact} \},$$

$$E_o = E_o(G) = \{\mu \in M_o ; \|\mu\| < +\infty\},$$

$$F_o = F_o(G) = \{\mu \in E_o(G) ; G\mu(x) \text{ is finite and continuous on } X\}.$$

A Borel measurable set B is said to be G -negligible if $\mu(B) = 0$ for every $\mu \in E_o(G)$.

We say that a property holds G -nearly everywhere on a subset A of X (written simply G -n.e. on A), when it holds on A except for a G -negligible set.

A non-negative lower semi-continuous function u on X is said to be G -superharmonic, when $u(x) < +\infty$ G -n.e. on X and for any $\mu \in E_o(G)$, the inequality $G\mu(x) \leq u(x)$ G -n.e. on $S\mu$ implies the same inequality on the whole space X .

We denote by $S(G)$ the totality of G -superharmonic functions on X and by P_{M_o} (resp. $P_{E_o}(G)$) the totality of G -potentials of measures in M_o (resp. $E_o(G)$).

The potential theoretic principles are stated as follows.

- (i) We say that G satisfies the *domination principle* and write simply $G \prec G$ when $P_{M_o}(G) \subset S(G)$.
- (ii) We say that G satisfies the *maximun principle* and write simply $G \prec 1$ when $1 \in S(G)$.
- (iii) We say that G satisfies the *complete maximun principle* when, for any $c \geq 0$, $P_{M_o}(G) \cup \{c\} \subset S(G)$.
- (iv) We say that G satisfies the *balayage principle* if, for any compact K , there exists a measure $\mu'_K \in M_o$, called a *balayaged measure* of μ on K and supported by K satisfying

$$\begin{aligned} G\mu(x) &= G\mu'_K(x) && G\text{-n.e. on } K, \\ G\mu'_K(x) &\leq G\mu(x) && \text{on } X. \end{aligned}$$

- (v) We say that G satisfies the *equilibrium principle* if, for any compact K , there exists a measure $\gamma_K \in M_o$, called an *equilibrium measure* of K and supported by K satisfying

$$\begin{aligned} G\gamma_K(x) &= 1 && G\text{-n.e. on } K, \\ G\gamma_K(x) &\leq 1 && \text{on } X. \end{aligned}$$

- (vi) We say that G satisfies the *continuity principle* if, for $\mu \in M_o$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space X .

3. Thinness at infinity δ of a closed set with finite capacity

In this section, we define the several notions of thinness of a closed set at δ , the point at infinity, and shall obtain the mutual relations holding among them.

For any compact K and any set A in X , the G -capacity $cap_G(K)$ of K and the G -inner capacity $cap_G^i(A)$ of A are defined respectively by

$$cap_G(K) = \inf \left\{ \int d\mu ; \mu \in M_o, G\mu(x) \geq 1 \text{ } G\text{-n.e. on } K \text{ and } S\mu \subset K \right\},$$

$$cap_G^i(A) = \sup \{ cap_G(K) ; K \text{ is compact set contained in } A \}.$$

For a Borel function u and a closed set F , the G -reduced function of u on F and the G -reduced function of u on F at infinity δ , are defined respectively by

$$R_G^F(u)(x) = \inf \{ v(x) ; v \in S(G), v(x) \geq u(x) \text{ } G\text{-n.e. on } X \},$$

$$R_G^{F,\delta}(u)(x) = \inf_{\omega \in \Omega_o} R_G^{F \cap C\omega} u(x).$$

where Ω_o denotes the totality of all relatively compact open sets in X .

Definition 1. We say that a subset A of X is G -thin at infinity δ in the sense of capacity (written simply G -cap. thin at δ) when we have

$$\inf_{\omega \in \Omega_o} cap_G^i(A \cap C\omega) = 0.$$

For any set $A \subset X$, the subset $S_o(F; G)$ of $S(G)$ is defined by

$$S_o(F; G) = \{ u \in S(G) ; R_G^{F,\delta} u(x) = 0 \text{ } G\text{-n.e. on } X \}.$$

In the following, the class $S_o(F; G)$ plays an important role.

Definition 2. We say that a subset A of X is G -1-thin at infinity δ when $1 \in S_o(A; G)$.

Definition 3. We say that a subset A of X is G -thin at infinity δ , when $P_{M_o}(G) \subset S_o(A; G)$.

Definition 4. We say that, on a subset A , a function u on X converges to 0 in capacity at infinity δ , if the equality

$$\inf_{\omega \in \Omega_o} \text{cap}_G^i(A \cap E \cap C\omega) = 0$$

holds for $\forall c > 0$, where $E = E(u \geq c) = \{x \in X ; u(x) \geq c\}$.

Throughout the rest of this paper, G denotes a continuous function-kernel on X for which every non-empty open set in X is not negligible. For simplicity we assume further that G is symmetric.

First we compare the notions of thinness of a closed set with finite G -inner capacity at infinity δ .

Theorem 1. Suppose that G satisfies the complete maximum principle. Then, for any closed set F in X , the following four statements are equivalent :

- (1) F is G -cap. thin at infinity δ .
- (2) (i) $\text{cap}_G^i(F) < +\infty$, and
 (ii) on F , $G\mu(x)$ converges to 0 in capacity at infinity δ on F for $\forall \mu \in M_o$.
- (3) (i) $\text{cap}_G^i(F) < +\infty$, and
 (iii) F is G -1-thin at infinity δ .
- (4) (i) $\text{cap}_G^i(F) < +\infty$, and
 (iv) F is G -thin at infinity δ .

Corollary. *Suppose that G satisfies the complete maximum principle. Then for any closed set with finite G -inner capacity, the following five statements are equivalents :*

- (1) F is G -cap. thin at infinity δ .
- (2) Given $\varepsilon > 0$, there exists a compact K satisfying

$$\text{cap}_G^i(F) - \varepsilon < \text{cap}_G(K) \leq \text{cap}_G^i(F),$$

$$\text{cap}_G^i(F \setminus K) < \varepsilon.$$

- (3) On F , $G\mu(x)$ converges to 0 in capacity at infinity δ for any $\mu \in M_K$.
- (4) F is G -1-thin at infinity δ .
- (5) F is G -thin at infinity δ .

To prove our theorem, first we recall the following lemma obtained in [2].

Lemma 1. *Suppose that G satisfies the domination principle. Then, for a closed set F , every function $u \in S_o(G; F)$ can be balayaged on F , namely, there exists a measure $\mu'_F \in M$ supported by F satisfying*

$$G\mu'_F(x) = u(x) \quad G\text{-n.e. on } F,$$

$$G\mu'_F(x) \leq u(x) \quad \text{in } X.$$

Proof of Theorem 1. The equivalences (1) $\longleftrightarrow$ (3) $\longleftrightarrow$ (4) have been obtained in [3] by using Lemma 1.

The implication (1) $\longrightarrow$ (2) is trivial and therefore it suffices to obtain the implication (2) $\longrightarrow$ (3).

Suppose (2). For any measure $\nu \in M_o$ and $c > 0$, we put

$$E = E(G\nu(x) \geq c) = \{x \in X ; G\nu(x) \geq c\}.$$

Given a compact K and an open ω , we denote by $\gamma_{F \cap C\omega \cap K}$ (resp. $\gamma_{F \cap C\omega \cap E \cap K}$).

By (ii), we can find, for any $\varepsilon > 0$, an open set $\omega_o \in \Omega_o$ verifying

$$(3.1) \quad \int d\gamma_{F \cap C\omega \cap E \cap K} < \varepsilon \text{ for any open } \omega \supset \omega_o.$$

Then we have, for $\forall \nu \in F_o(G)$,

$$(3.2) \quad \int R_G^{F \cap C\omega \cap K}(1) d\nu = \int G\nu d\gamma_{F \cap C\omega \cap K} = \int_E + \int_{CE}.$$

We shall estimate the last two integrals. By (3.1), there exists $M > 0$ such that

$$(3.3) \quad \int_E \leq \int G\nu d\gamma_{F \cap C\omega \cap E \cap K} < M \cdot \varepsilon \text{ for any open } \omega \supset \omega_o.$$

On the other hand, the second integral is estimated as follows.

$$(3.4) \quad \int_{CE} = \int_{CE} G\nu d\gamma_{F \cap C\omega \cap K} < c \cdot \text{cap}_G^i(F).$$

Let K and ω tend to X and we have

$$(3.5) \quad \int R_G^{F,\delta}(1) d\nu \leq M \cdot \varepsilon + c \cdot \text{cap}_G^i(F).$$

Further letting c and ε tend to 0, we obtain

$$(3.6) \quad \int R_G^{F,\delta}(1) d\nu = 0,$$

and hence (iii). This completes the proof. ■

4. Thinness at infinity δ of a closed set with infinite capacity

For a closed set, the following characterizations of G -thinness at infinity δ have been already obtained (cf. [1], [2], [3] and [4]).

Theorem 2. *Suppose that G satisfies the complete maximum principle and that G is non-degenerate, namely, the potentials $G\varepsilon_{x_1}(x)$ and $G\varepsilon_{x_2}(x)$ are not proportional if $x_1 \neq x_2$. Then for any closed set F , the following statements are equivalent :*

- (1) F is G -thin at infinity δ .
- (2) On F , for $\forall \mu \in M_K$, $G\mu(x)$ converges to 0 at infinity δ .
- (3) G has the so called dominated convergence property :

$\{\mu_n\}_{n=1}^{\infty} \subset M$, $S\mu_n \subset F$ and $\mu_n \rightarrow \mu_o$ vaguely as $n \rightarrow +\infty$, and

$\exists \nu \in M_o$ such that $G\mu_n(x) \leq G\nu(x)$ on X for all n .

$\Rightarrow$

$G\mu_o(x) = \liminf_{n \rightarrow \infty} G\mu_n(x)$ G -n.e. on X .

(4) G is strongly balayable, namely, for $\forall u \in S(G)$ dominated by a potential in $P_{M_o}(G)$ and for every closed set $F' \subset F$, there exists a positive measure μ' supported by F' and verifying

$$G\mu'(x) = u(x) \text{ } G\text{-n.e. on } F',$$

$$G\mu'(x) \leq u(x) \text{ on } X.$$

By the same methods used in the proof of Theorem 2, we can also characterize the G -1-thinness at infinity δ of a closed set with infinite G -inner capacity.

Theorem 3. Suppose that G satisfies the complete maximum principle and that G is non-degenerate. Then, for any closed set F in X , the following three statements are equivalent :

(1) F is G -1-thin at infinity δ .

(2) G has the following dominated convergence property :

$$\{\mu\}_{n=1}^{\infty} \subset M, S\mu_n \subset F, \mu_n \longrightarrow \mu_0 \text{ vaguely as } n \longrightarrow +\infty,$$

$$\{G\mu_n(x)\}_{n=1}^{\infty} \text{ is uniformly bounded on } X$$

$\implies$

$$G\mu_0(x) = \liminf_{n \rightarrow +\infty} G\mu_n(x) \text{ } G\text{-n.e. on } X.$$

(3) Every bounded G -superharmonic function can be balayaged on every closed set contained in F .

For the proof of Theorem 3, it suffices to prepare the following two lemmata.

Lemma 2. Suppose that G satisfies the domination principle and that G is non-degenerate. Then for any closed set F , the following two statements are equivalent:

(1) $P_{F_0}(G) \subset S_0(F; G)$.

(2) Every G -superharmonic function dominated by a potential in $P_{F_0}(G)$ can be balayaged on every closed set contained in F .

Lemma 3. Suppose that G satisfies the complete maximum principle. Then for any closed set, the following two statements are equivalent :

(1) F is G -1-thin at infinity δ .

(2) (i) There exists an equilibrium measure of F , and

(ii) $P_{M_0}(G) \subset S_0(F; G)$.

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