

Squaring operations in the Hermitian symmetric spaces

By

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§ 0. Introduction

In this paper we calculate the squaring operations in the mod 2 cohomology of the irreducible Hermitian symmetric spaces of compact type. Each of them is obtained as a quotient of an appropriate compact simple Lie group by the centralizer of an appropriate 1-dimensional torus, and they are divided into six classes:

AIII	$W(m, n)$	$= U(m+n)/(U(m) \times U(n))$	$(m, n \geq 1)$
BDI	Q_n	$= SO(n+2)/(SO(2) \times SO(n))$	$(n \geq 3)$
CI		$Sp(n)/U(n)$	$(n \geq 3)$
DIII		$SO(2n)/U(n)$	$(n \geq 4)$
EIII		$= E_6/(Spin(10) \cdot T^1)$	$(Spin(10) \cap T^1 \cong \mathbf{Z}_4)$
EVII		$= E_7/(E_6 \cdot T^1)$	$(E_6 \cap T^1 \cong \mathbf{Z}_3)$

Their cohomology rings have been obtained by several authors:

$$H^*(W(m, n); \mathbf{Z}) = \mathbf{Z}[a_1, \dots, a_m, b_1, \dots, b_n] / \left(\sum_{i+j=k} a_i b_j; k \geq 1 \right);$$

$$H^*(Q_n; \mathbf{Z}) = \begin{cases} \mathbf{Z}[t, e] / (t^m - 2e, e^2) & (n = 2m - 1), \\ \mathbf{Z}[t, s] / (t^{m+1} - 2st, s^2 - \delta_m st^m) & (n = 2m); \end{cases}$$

$$H^*(Sp(n)/U(n); \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n] / \left(\sum_{i+j=2k} (-1)^i c_i c_j; k \geq 1 \right);$$

$$H^*(SO(2n)/U(n); \mathbf{Z}) = \mathbf{Z}[e_2, e_4, \dots, e_{2n-2}] / \left(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i} e_{4k-2i} \right)$$

(it should be understood that $e_{2j} = 0$ if $j \geq n$);

$$H^*(\text{EIII}; \mathbf{Z}) = \mathbf{Z}[t, w] / (t^9 - 3w^2t, w^3 + 15w^2t^4 - 9wt^8);$$

$$H^*(\text{EVII}; \mathbf{Z}) = \mathbf{Z}[u, v, w] / (v^2 - 2wu, u^{14} - 2A, w^2 - 2B),$$

where $\delta_m = \frac{1 + (-1)^m}{2}$, A and B are appropriate integral cohomology classes, and

$$|a_i| = |b_i| = |c_i| = |e_{2i}| = 2i, \quad |t| = 2, \quad |s| = |e| = 2m$$

$$|u| = 2, \quad |v| = 10 \quad \text{and} \quad |w| = 8 \quad \text{for EIII}; = 18 \quad \text{for EVII}.$$

For details see [1], [6], [7], [9] and § 1.4 in this paper.

For AIII and CI the cohomology rings are generated by Chern classes and for DIII by the suspension images of Stiefel-Whitney classes, whence the squaring operations are obtained by the Wu formula.

For BDI ($n = 2m$) we calculate in $H^*\left(\frac{SO(2m+2)}{U(1) \times U(m)}\right)$ through the homomorphism induced by the projection $\frac{SO(2m+2)}{U(1) \times U(m)} \rightarrow Q_{2m}$, and obtain

$$\textbf{Theorem 1.4.} \quad \text{Sq}^{2i}s = \binom{m+1}{i} st^i \quad (i \geq 0).$$

$$\textbf{Corollary 1.5.} \quad \text{Sq}^{2i}e = \binom{m+1}{i} et^i \quad (i \geq 0).$$

For the exceptional types EIII and EVII we calculate in G/T (T a maximal torus of G) using the fibration $G/T \xrightarrow{\tilde{i}} B\tilde{T} \rightarrow B\tilde{G}$, where $B\tilde{G}$ is the 4-connective cover of BG and $B\tilde{T}$ is defined in a similar way (for details see [3]). The results are

Theorem 2.5. (i) In EIII we have

$$\text{Sq}^2w' = w't, \quad \text{Sq}^4w' = t^6, \quad \text{Sq}^8w' = w'^2 \quad (\text{where } w' = w + t^4).$$

(ii) In EVII

$$\begin{aligned} \text{Sq}^2v &= 0, & \text{Sq}^4v &= vu^2 + u^7, & \text{Sq}^8v &= w + vu^4 + u^9; \\ \text{Sq}^2w &= u^{10}, & \text{Sq}^4w &= vu^6 + u^{11}, & \text{Sq}^8w &= vu^8 + u^{13}, & \text{Sq}^{16}w &= vu^{12}. \end{aligned}$$

As an application we give in the final section the Stiefel-Whitney classes of EIII and EVII by use of the Wu classes.

Throughout the paper $H^*(\)$ denotes exclusively the mod 2 cohomology (integral cohomology is always denoted by $H^*(\ ; \mathbf{Z})$). $\mathbf{F}_2$ denotes the prime field of characteristic 2. For an integral element x its mod 2 reduction is denoted by $\rho(x)$, or simply by x unless there is danger of confusion. $\sigma_i(x_1, \dots, x_n)$ denotes the i -th elementary symmetric polynomial in $x_1, \dots, x_n$ ($i \geq 0$). $\Delta(a_1, \dots, a_n)$ denotes an algebra with simple system of generators $a_1, \dots, a_n$.

§ 1. Classical types

First recall the Wu formula. Let $x_1, \dots, x_n$ be elements of degree d with $\text{Sq}^i x_j = 0$ ($0 < i < d$), and put $c_j = \sigma_j(x_1, \dots, x_n)$ ($j \geq 0$). Then we have

$$(1.1) \quad \text{Sq}^{di} c_j = \sum_{0 \leq r \leq i} \binom{j-i+r-1}{r} c_{j+r} c_{i-r}.$$

1.1. The Grassmannian $W(m, n)$. We have the fibration

$$W(m, n) \xrightarrow{L} BU(m) \times BU(n) \longrightarrow BU(m+n).$$

Put $a_i = \iota^*(c_i \times 1)$ and $b_i = \iota^*(1 \times c_i)$, where c_i is the i -th universal Chern class in $BU(m)$ or $BU(n)$ ($i \geq 0$). Then

$$H^*(W(m, n); \mathbf{Z}) = \mathbf{Z}[a_1, \dots, a_m, b_1, \dots, b_n] / \left(\sum_{i+j=k} a_i b_j; \quad k \geq 1 \right),$$

whence by the naturality the operation of Sq^i is obtained from the Wu formula (1.1).

1.2. The space $Sp(n)/U(n)$. The operation of Sq^i is again obtained from the Wu formula since

$$H^*(Sp(n)/U(n); \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n] / \left(\sum_{i+j=2k} (-1)^i c_i c_j; \quad k \geq 1 \right),$$

where c_i is the i -th Chern class ($i \geq 0$).

1.3. The space $SO(2n)/U(n)$. We extract from [6], Chap. 3, §6. Using the fibration

$$SO(2n)/U(n) \xrightarrow{\iota} BU(n) \longrightarrow BSO(2n),$$

we have unique elements $e_{2i} = \frac{1}{2} \iota^* c_i \in H^{2i}(SO(2n)/U(n); \mathbf{Z})$ ($1 \leq i \leq n-1$). Let $p: SO(2n) \rightarrow SO(2n)/U(n)$ be the projection and $\sigma: H^*(BSO(2n)) \rightarrow H^*(SO(2n))$ the suspension. Then

$$(1.2) \quad p^*(e_{2i}) = \sigma(w_{2i+1}) \quad (w_j \text{ the Stiefel-Whitney classes});$$

$$(1.3) \quad H^*(SO(2n)/U(n)) = \Delta(e_2, e_4, \dots, e_{2n-2}), \quad e_{2i}^2 = e_{4i} \\ \text{(it should be understood that } e_{2j} = 0 \text{ if } j \geq n).$$

It follows that p^* is injective. So we calculate in $SO(2n)$:

$$p^*(Sq^{2i} e_{2k}) = \sigma(Sq^{2i} w_{2k+1}) = \sigma \left(\sum_{0 \leq r \leq 2i} \binom{2k-2i+r}{r} w_{2k+1+r} w_{2i-r} \right)$$

by the Wu formula. Since σ annihilates decomposables, we have

$$\textbf{Proposition 1.1.} \quad Sq^{2i} e_{2k} = \binom{k}{i} e_{2k+2i} \quad (i, k \geq 0).$$

1.4. The complex quadric $Q_n = SO(n+2)/(SO(2) \times SO(n))$. We have the fibration

$$Q_n \xrightarrow{\iota} BSO(2) \times BSO(n) \longrightarrow BSO(n+2).$$

Let $t \in H^2(BSO(2); \mathbf{Z})$ be the canonical generator, and put $t = \iota^*(t \times 1)$. Here we distinguish the two cases (a) n is even, and (b) n is odd.

(a) $n = 2m$. Let $\chi \in H^{2m}(BSO(2n); \mathbf{Z})$ be the Euler class and $p_i \in H^{2i}(BSO(n); \mathbf{Z})$ the i -th Pontrjagin class. Using the fibration above we see that

$$\iota^*(t \times \chi) = 0. \quad \iota^*(1 \times p_m) = (-1)^m t^{2m} \quad \text{and} \\ \iota^*(1 \times \chi + t^m \times 1) \equiv 0 \pmod{(2)}.$$

Since $H^*(Q_{2m}; \mathbf{Z})$ has no torsion we have a unique element $s \in H^{2m}(Q_{2m}; \mathbf{Z})$ with $2s = \iota^*(1 \times \chi + t^m \times 1)$. Then the relations above yield

$$2st = t^{m+1} \quad \text{and} \quad 4s^2 = 2(1 + (-1)^m) st^m.$$

Considering the Serre spectral sequence for the fibration $SO(2m+2)/SO(2m) \rightarrow Q_{2m} \rightarrow BSO(2)$, we obtain

$$\textbf{Theorem 1.2.} \quad H^*(Q_{2m}; \mathbf{Z}) = \mathbf{Z}[t, s] / (t^{m+1} - 2st, s^2 - \delta_m st^m).$$

Now consider the diagram :

$$\begin{array}{ccccc}
 & & SO(2m)/U(m) & & \\
 & & \downarrow & & \\
 P_m(\mathbf{C}) = \frac{U(m+1)}{U(1) \times U(m)} & \xrightarrow{j} & \frac{SO(2m+2)}{U(1) \times U(m)} & \xrightarrow{q} & SO(2m+2)/U(m+1) \\
 (1,4) & & \downarrow p & \searrow \iota' & \downarrow \\
 & & Q_{2m} = \frac{SO(2m+2)}{SO(2) \times SO(2m)} & \xrightarrow{\iota} & BSO(2) \times BSO(2m) \\
 & & & & \downarrow \\
 & & & & BU(1) \times BU(m) \xrightarrow{q'} BU(m+1)
 \end{array}$$

Define $t, e_{2i} (i \geq 1)$ and $e \in H^*\left(\frac{SO(2m+2)}{U(1) \times U(m)}; \mathbf{Z}\right)$ by

$$t = p^*(t) = \iota'^*(c_1 \times 1), \quad e_{2i} = q^*(e_{2i}) \quad \text{and} \quad e = \sum_{i=1}^m (-1)^i e_{2i} t^{m-i}.$$

Then

Lemma 1.3. (i) q^* induces an isomorphism of algebras

$$H^*\left(\frac{SO(2m+2)}{U(1) \times U(m)}; \mathbf{Z}\right) \cong H^*\left(\frac{SO(2m+2)}{U(m+1)}; \mathbf{Z}\right)[t]/(t^{m+1} + 2te).$$

$$(ii) \quad p^*(s) = \delta_m t^m + (-1)^m e.$$

Proof. By the definition of t we see that j^*t generates the ring $H^*(P_m(\mathbf{C}); \mathbf{Z})$. Since the spectral sequence for the row in (1.4) collapses, (i) holds as an isomorphism of modules. Let c, c' and c'' be the total Chern classes of $BU(m), BU(1)$ and $BU(m+1)$, respectively. Then $q'^*(c'') = c' \times c$. Applying ι'^* we have

$$1 + 2e_2 + 2e_4 + \cdots + 2e_{2m} = (\iota'^*(1 \times c)) \cdot (1 + t),$$

whence

$$\iota'^*(1 \times c_m) = (-1)^m (t^m + 2e).$$

Then $(t^m + 2e)t = (-1)^m (\iota'^*(1 \times c_m)) \cdot t = 0$, which completes the proof of (i).

Next from the diagram (1.4) we have

$$2p^*(s) = \iota'^*(c_m \times 1) + t^m = 2\delta_m t^m + (-1)^m 2e,$$

which proves (ii) since $H^*\left(\frac{SO(2m+2)}{U(1) \times U(m)}; \mathbf{Z}\right)$ is torsion free. q.e.d.

Theorem 1.4. $\text{Sq}^{2i}s = \binom{m+1}{i} st^i \quad (i \geq 0).$

Proof. As is well known $\text{Sq} = 1 + \text{Sq}^1 + \text{Sq}^2 + \cdots$ is an algebra homomorphism, and by 1.2 we can put $\text{Sq}(s) = s(1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots)$ ($\varepsilon_i \in \mathbf{F}_2$). Applying p^* we have

$$(1.5) \quad \text{Sq}(\delta_m t^m + e) = (\delta_m t^m + e) (1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots).$$

Now put $A = H^*(P_m(\mathbf{C}) \times P_m(\mathbf{C})) = \mathbf{F}_2[a, b]/(a^{m+1}, b^{m+1})$. Then the correspondence $t \mapsto a, e_{2i} \mapsto b^i (i \geq 1)$ extends to an algebra homomorphism $\varphi : H^*\left(\frac{SO(2m+2)}{U(1) \times U(m)}\right) \rightarrow A$, which commutes with the squaring operations. Apply φ to (1.5):

$$(1.6) \quad \text{Sq}((\delta_m + 1)a^m + c) = ((\delta_m + 1)a^m + c) (1 + \varepsilon_1 a + \varepsilon_2 a^2 + \cdots),$$

where $c = a^m + a^{m-1}b + \cdots + b^m$. First calculating in the quotient field of $\mathbf{F}_2[[a, b]]$, we obtain

$$\text{Sq}(c) = \sum_{i=0}^m \binom{m+1}{i} (a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i}) \cdot \sum_{j \geq 0} (a+b)^j,$$

and then in A using the equalities $a^{m+i} + a^{m+i-1}b + \cdots + b^{m+i} = ca^i$ and $c(a+b) = 0$,

$$\text{Sq}(c) = c(1+a)^{m+1}.$$

Comparing the coefficients of $a^m b^i$ in both sides of (1.6), we obtain $\varepsilon_i = \binom{m+1}{i}$, which proves the theorem. q.e.d.

(b) $n = 2m - 1$. According to [6]

$$H^*(Q_{2m-1}; \mathbf{Z}) = \mathbf{Z}[t, e]/(t^m - 2e, e^2),$$

where t is the same as ours. The inclusion $SO(2m+1) \subset SO(2m+2)$ ($X \mapsto X \oplus 1$) yields a commutative diagram

$$\begin{array}{ccc} Q_{2m-1} & \xrightarrow{f} & Q_{2m} \\ \iota \downarrow & & \downarrow \iota \\ BSO(2) \times BSO(2m-1) & \xrightarrow{f'} & BSO(2) \times BSO(2m). \end{array}$$

From $f'^*(1 \times \chi) = 0$ and $f'^*(t \times 1) = t \times 1$ it follows that $f^*(t) = t$ and $f^*(s) = e$, and we have

$$\text{Corollary 1.5.} \quad \text{Sq}^{2i}e = \binom{m+1}{i} et^i \quad (i \geq 0).$$

§ 2. Exceptional types

In this section $\ell = 6$ or 7 . Let G be the simply connected exceptional Lie group of type E_ℓ and T a maximal torus of G . Take the root system $\{\alpha_1, \dots, \alpha_\ell\}$ as in [2], and define K to be the centralizer of the 1-dimensional torus defined by the equations $\alpha_i = 0$ ($i \neq \ell$). Then the quotient space G/K is the irreducible Hermitian symmetric space EIII ($\ell = 6$) or EVII ($\ell = 7$).

Consider the fibration $K/T \rightarrow G/T \xrightarrow{p} G/K$. By the classical theorem of Bott the odd dimensional parts of the cohomology of both the fibre and the base vanish. Hence

the spectral sequence for the fibration collapses, and $p^* : H^*(G/K; A) \rightarrow H^*(G/T; A)$ is injective for any coefficient ring A . Therefore the action of Sq^i in G/K is derived from that in G/T .

First we fix a system of generators of $H^*(BT; \mathbf{Z})$ after [7] and [9]. Let $\{w_1, \dots, w_\ell\}$ be the fundamental weights of G . Being regarded as elements of $H^2(BT; \mathbf{Z})$, they form a basis of it. Let R_j be the reflection in the plane $\alpha_j = 0$, and put

$$t_\ell = w_\ell, \quad t_i = R_{i+1}(t_{i+1}) \quad (\ell > i > 1), \quad t_1 = R_1(t_2) \quad \text{and} \quad c_i = \sigma_i(t_1, \dots, t_\ell) \quad (i \geq 0).$$

Then

$$H^*(BT; \mathbf{Z}) = \mathbf{Z}[t_1, \dots, t_\ell, x] / (c_1 - 3x).$$

As the canonical mapping $i : G/T \rightarrow BT$ does not induce a surjection in $H^*(\quad)$, we introduce $B\tilde{G}$ the 4-connective fibre space over BG to have commutative diagram with two fibrations

$$\begin{array}{ccccc} G/T & \xrightarrow{\tilde{i}} & B\tilde{T} & \longrightarrow & B\tilde{G} \\ \parallel & & \downarrow g & & \downarrow \\ G/T & \xrightarrow{i} & BT & \longrightarrow & BG \end{array}$$

In $H^*(B\tilde{T}; \mathbf{Z}_{(2)})$ we have new generators g_i ($i = 3, 5, 9$) with

$$2g_3 = c_3, \quad 2g_5 = c'_5 = c_5 + c_4c_1 \quad \text{and} \quad 2g_9 = c'_9 = c_7c_1^2 + c_6c_1^3$$

(Note that the symbol g^* is omitted here). We put

$$\gamma_3 = \rho(g_3), \quad \gamma_5 = \text{Sq}^4\gamma_3 \quad \text{and} \quad \gamma_9 = \text{Sq}^8\gamma_5 \quad \in H^*(B\tilde{T}).$$

Then

$$(2.1) \quad \begin{aligned} \gamma_5 &= \rho(g_5 + g_3c_1^2 + c_4c_1) \quad \text{and} \\ \gamma_9 &= \rho(g_9 + g_5(c_4 + c_1^4) + g_3(c_6 + c_4c_1^2 + c_1^6) + c_7c_1^2 + c_4^2c_1 + c_4c_1^5). \end{aligned}$$

For details see [3]. Note that our γ_i ($i = 5, 9$) are slightly different from those in [9].

Recall that the generator of maximum degree is w in each case (see §0). So it is sufficient for us to consider in the range of degree $\leq d_\ell$, where $d_6 = 14$ and $d_7 = 34$. Define polynomials

$$\begin{aligned} I_6 &= \gamma_3^2 + c_4c_1^2 + c_1^6, & I_8 &= c_6c_1^2 + c_4^2 + c_4c_1^4 + c_1^8, \\ I_{10} &= \gamma_5^2 + c_6c_1^4 + c_1^{10}, & I_{12} &= c_6^2 + c_6c_4c_1^2 + c_4^2c_1^4 + c_4c_1^8, \\ I_{14} &= c_7^2 + c_6c_4c_1^4 + c_6c_1^8 \end{aligned}$$

and sets

$$R_6 = \{c_2, c_3, c'_5, I_6\}, \quad R_7 = \{c_2, c_3, c'_5, I_6, I_8, c'_9, I_{10}, I_{12}, I_{14}\}$$

Then from §3 in [4] we have

Lemma 2.1. (1) *Up to degree d_ℓ*

$$H^*(G/T) = \begin{cases} \mathbf{F}_2[t_1, \dots, t_6, \gamma_3] / (R_6) & (\ell = 6), \\ \mathbf{F}_2[t_1, \dots, t_7, \gamma_3, \gamma_5, \gamma_9] / (R_7) & (\ell = 7). \end{cases}$$

$$\begin{aligned}
(2) \quad & \text{Sq}^2\gamma_3 = c_4, & \text{Sq}^4\gamma_3 = \gamma_5 \ (= c_4c_1 + c_1^5 \text{ if } \ell = 6) ; \\
& \text{Sq}^2\gamma_5 = c_4c_1^2 + c_1^6, & \text{Sq}^4\gamma_5 = c'_7, \quad \text{Sq}^8\gamma_5 = \gamma_9 ; \\
& \text{Sq}^2\gamma_9 = c_4c_1^6, & \text{Sq}^4\gamma_9 = 0, \quad \text{Sq}^8\gamma_9 = c'_7c_6, \quad \text{Sq}^{16}\gamma_9 = c'_7c_6c_4
\end{aligned}$$

where $c'_7 = c_7 + c_6c_1$.

Now we interpret the results in [7] and [9] to our situation.

$$(2.2) \quad H^*(\text{EIII}) = \mathbf{F}_2[t, w'] / (w'^2t, w'^3 + t^{12}),$$

where $t, w' = w + t^4 \in H^*(E_6/T; \mathbf{Z})$ satisfy

$$t \equiv c_1 + t_1, \quad w' \equiv c_4 + (\gamma_3 + c_1^2t + c_1t^2)t \pmod{(2)}.$$

$$(2.3) \quad H^*(\text{EVII}) = \mathbf{F}_2[u, v, w] / (v^2, u^{14}, w^2),$$

where $u = t_7, v, w \in H^*(E_7/T; \mathbf{Z})$ satisfy

$$2v \equiv \bar{c}_5 - \bar{c}_4\chi + \bar{c}_3\chi^2 - \bar{c}_2\chi^3 + 2\chi^5 + 2u^5 \pmod{(4)},$$

$$\begin{aligned}
2w \equiv & \bar{c}_6\bar{c}_3 + \bar{c}_5\bar{c}_4 + 2\bar{c}_5\bar{c}_2^2 + (2\bar{c}_6\bar{c}_2 - \bar{c}_4^2 + 2\bar{c}_4\bar{c}_2^2)\chi \\
& - (\bar{c}_5\bar{c}_2 - \bar{c}_4\bar{c}_3 + 2\bar{c}_3\bar{c}_2^2)\chi^2 + (\bar{c}_6 + 2\bar{c}_2^3)\chi^3 - (\bar{c}_5 + \bar{c}_3\bar{c}_2)\chi^4 \\
& - (\bar{c}_4 - \bar{c}_2^2)\chi^5 - \bar{c}_3\chi^6 - \bar{c}_2\chi^7 + 2\chi^9 + 2vu^4 \pmod{(4)}
\end{aligned}$$

with $\chi = \frac{1}{3}c_1 - u$ and $\bar{c}_i = \sigma_i\left(t_1 - \frac{1}{3}u, \dots, t_6 - \frac{1}{3}u\right)$ ($i \geq 0$).

We must describe v and w modulo 2 in terms of the t_i and the γ_j . For EVII the results are a little complicated. So we calculate modulo (c_1) . Note that $c_2 \equiv 0 \pmod{(4)}$ (see [8]) and recall the relation (2.1). Then after some calculations modulo $(4, 2c_1)$ we obtain the following:

Lemma 2.2. *Modulo (c_1)*

$$t \equiv t_1, \quad w' \equiv \gamma_3t_1 + c_4 \pmod{(\ell = 6)};$$

$$u \equiv t_7, \quad v \equiv \gamma_5 + \gamma_3t_7^2 + c_4t_7, \quad w \equiv \gamma_9 + \gamma_3t_7^6 + c_6t_7^3 \pmod{(\ell = 7)}.$$

Fortunately we have

Lemma 2.3. (i) *In $H^*(G/T)$ the ideal (c_1) is closed under the operation of the Sq^i .*

(ii) *Up to degree d_ℓ the composition of $p^*: H^*(G/K) \rightarrow H^*(G/T)$ and the projection $\pi: H^*(G/T) \rightarrow H^*(G/T)/(c_1)$ is injective.*

Proof. (i) This follows from the Cartan formula.

(ii) For $\ell = 6$ put $b_i = \sigma_i(t_2, \dots, t_6)$ ($i \geq 0$), and let

$$A = \mathbf{F}_2[t_1, \dots, t_6, \gamma_3] \quad \text{and} \quad B = \mathbf{F}_2[t_1, b_1, \dots, b_5, \gamma_3].$$

Then A is free as a B -module, and hence for any subset $R \subset B$ the canonical map $B/BR \rightarrow A/AR$ is injective, where BR denotes the ideal generated by R in B , and similarly for AR .

Note that B coincides with $\mathbf{F}_2[t_1, c_1, \dots, c_5, \gamma_3]$. So it contains the set $R = \{c_1\} \cup R_6 = \{c_1, c_2, c_3, c'_5, I_6\}$, and B/BR can be regarded as a subset of $A/AR = H^*(G/T)/(c_1)$. Then by 2.2 the image of $\pi \circ p^*$ is contained in $B/BR = \mathbf{F}_2[t_1, c_4, \gamma_3]/(\gamma_3^2)$, and it is easily seen that up to degree d_ℓ the map $H^*(G/T) \rightarrow B/BR$ is injective.

Similarly for $\ell = 7$.

q.e.d.

Now the operation of the Sq^i are obtained from 2.2 and 2.1,(2) by calculating modulo (c_1) . The results are as follows:

Theorem 2.4. (i) In $H^*(\text{EIII})$ we have

$$\text{Sq}^2 w' = w' t, \quad \text{Sq}^4 w' = t^6, \quad \text{Sq}^8 w' = w'^2$$

(ii) In $H^*(\text{EVII})$

$$\begin{aligned} \text{Sq}^2 v &= 0, & \text{Sq}^4 v &= v u^2 + u^7, & \text{Sq}^8 v &= w + v u^4 + u^9; \\ \text{Sq}^2 w &= u^{10}, & \text{Sq}^4 w &= v u^6 + u^{11}, & \text{Sq}^8 w &= v u^8 + u^{13}, & \text{Sq}^{16} w &= v u^{12}. \end{aligned}$$

§ 3. Stiefel-Whitney classes

For the Hermitian symmetric spaces of classical type the Chern classes have been obtained in [1], whence so have the Stiefel-Whitney classes by mod 2 reduction. In this section as an application of the previous section we give the Stiefel-Whitney classes of EIII and EVII by using the Wu classes.

For a compact n -manifold M the i -th Wu class $u_i \in H^i(M)$ is characterised by

$$u_i \cdot x = \text{Sq}^i x \quad \text{for any } x \in H^{n-i}(M).$$

Using the Wu classes the Stiefel-Whitney classes w_i are given by $w_i = \sum_{j \geq 0} \text{Sq}^{i-j} u_j$, or equivalently

$$(3.1) \quad W = \text{Sq} U,$$

where $W = \sum w_i$, and $U = \sum u_i$ (see [5], for example).

From the previous section it follows that

(3.2) $H^*(\text{EIII})$ has $\{w^m t^i, w'^2 \mid n = 0, 1; 0 \leq i \leq 12\}$ as a basis, and

$$\text{Sq}^{2r} w' t^{12-r} = w' t^{12} \quad (r = 0, 2, 6), \quad \text{Sq}^{16} w^2 = w t^{12},$$

$$\text{Sq}^{2r} b = 0 \quad \text{for the other } b \text{ of degree } 32 - 2r \text{ in the basis};$$

(3.3) $H^*(\text{EVII})$ has $\{w^n v^m u^i \mid n, m = 0, 1; 0 \leq i \leq 13\}$ as a basis, and

$$\text{Sq}^{2r} w v u^{13-r} = w v u^{13} \quad (r = 0, 4, 12),$$

$$\text{Sq}^{2r} b = 0 \quad \text{for the other } b \text{ of degree } 54 - 2r \text{ in the basis}.$$

Therefore

Theorem 3.1. The non-zero Wu classes are given as follows:

$$\text{for EIII}, \quad u_0 = 1, \quad u_4 = t^2, \quad u_{12} = t^6, \quad u_{16} = w'^2;$$

$$\text{for EVII}, \quad u_0 = 1, \quad u_8 = u^4, \quad u_{24} = u^{12}.$$

Then by use of the formula (3.1) we obtain

Corollary 3.2. *The total Stiefel-Whitney class W is given as follows:*

$$\text{for EIII,} \quad W = 1 + t^2 + t^4 + t^6 + (w'^2 + t^8) + t^{10} + w't^{12};$$

$$\text{for EVII,} \quad W = 1 + u^4 + u^8 + u^{12}.$$

Remark 3.3. For EIII the result coincides with the mod 2 reduction of that in [8].

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