

Interfaces of Solutions to an Inhomogeneous Filtration Equation with Absorption *

Alberto Tesei[†]

Abstract

We review some recent results, obtained jointly with R. Kersner and G. Reyes, concerning qualitative properties of solutions to the Cauchy problem for the equation $\rho(x)u_t = (u^m)_{xx} - c_0 u^p$, where $m > 1$ and $c_0, p > 0$. The initial data are nonnegative with compact support and the density $\rho(x) > 0$ satisfies suitable decay conditions as $|x| \rightarrow \infty$.

1 Introduction

We discuss some recent results (see [KRT]) concerning support properties of solutions to the Cauchy problem:

$$(1.1) \quad \begin{cases} \rho(x)u_t = (u^m)_{xx} - c_0 u^p & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}, \end{cases}$$

where

$$(i) \quad m > 1, \quad c_0 > 0, \quad p > 0;$$

$$(ii) \quad \rho \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \rho > 0 \text{ in } \mathbb{R};$$

$$(iii) \quad u_0 \in C_c(\mathbb{R}), \quad u_0 \geq 0 \text{ in } \mathbb{R}.$$

(by $C_c(\mathbb{R})$ we denote the set of continuous, compactly supported functions on $\mathbb{R}$). A typical choice for the density $\rho = \rho(x)$ is

$$(1.2) \quad \rho(x) = \frac{\bar{\rho}}{(1 + |x|)^k} \quad (\bar{\rho}, k > 0).$$

Physical motivations of the model can be found in [KR1], [KR2], [GuHP] and references therein; for instance, it arises in connection with a parabolic system suggested by plasma

*Work partially supported through TMR Programme NPE # FMRX-CT98-0201.

[†]Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", P.le A. Moro 5, I-00185 Roma, Italia (tesei@mat.uniroma1.it).

physics (see [BK]). Let us also mention that the related problem of *uniqueness* of solutions to the positive Cauchy problem for diffusion equations with variable density has raised much attention in recent years (e.g., see [E], [GuHP], [KKT], [T] and references therein).

Solutions to problem (1.1) are always meant in the following weak sense.

Definition 1.1 *By a solution to problem (1.1) we mean any bounded, nonnegative and continuous function u on $\mathbb{R} \times [0, \infty)$ such that*

$$\begin{aligned} \iint_{D^r} \{ \rho u \phi_t + u^m \phi_{xx} - c_0 u^p \phi \} dx dt = \\ = \int_{-\tau}^{\tau} \rho u(x, \tau) \phi(x, \tau) dx - \int_{-\tau}^{\tau} \rho u_0(x) \phi(x, 0) dx + \\ + \int_0^{\tau} [u^m(r, t) \phi_x(r, t) - u^m(-r, t) \phi_x(-r, t)] dt \end{aligned}$$

for any $r, T > 0$, $\tau \in [0, T]$ and $\phi \in C_{x,t}^{2,1}(\overline{D^T})$, $\phi \geq 0$ such that $\phi(-r, t) = \phi(r, t) = 0$ in $[0, T]$ (here $D^r := (-r, r) \times (0, \tau]$, $\tau \in [0, T]$).

Subsolutions of problem (1.1) are similarly defined, replacing " $=$ " by " $\geq$ " in the above equality. On the other hand, supersolutions are meant in the following more restricted sense (introduced in [B] to deal with the case $0 < p < 1$).

Definition 1.2 *By a supersolution to problem (1.1) we mean any solution $\bar{u}$ to the problem*

$$\begin{cases} \rho(x) u_t = (u^m)_{xx} - c_0 u^p + h & \text{in } \mathbb{R} \times (0, T] \\ u(x, 0) = \hat{u}_0(x), & x \in \mathbb{R}, \end{cases}$$

for some $h \in L^\infty(S^T)$, $h \geq 0$, $\hat{u}_0 \geq u_0$ and $T > 0$.

Finally, let $u \geq 0$ be any solution to problem (1.1); its *interfaces* are defined as follows:

$$\zeta^+(t) := \sup\{x : u(x, t) > 0\}, \quad \zeta^-(t) := \inf\{x : u(x, t) > 0\} \quad (t \geq 0).$$

We also set:

$$\zeta(t) := \sup\{|x| : u(x, t) > 0\} = \max\{|\zeta^-(t)|, |\zeta^+(t)|\}.$$

2 Previous Results

Let us recall some well-known results for problem (1.1) in two particular cases: $\rho = 1$, $c_0 > 0$, respectively $\rho = \rho(x)$, $c_0 = 0$.

2.1 Porous Medium Equation with Absorption

When $\rho = 1$, $c_0 > 0$ it is well known that (see [BKP], [CMM], [Ka]):

- the following estimates hold:

$$|\zeta^\pm(t)| \leq \text{constant} \quad \text{for } 1 \leq p < m,$$

$$|\zeta^\pm(t)| \sim \log t \quad \text{for } p = m,$$

$$|\zeta^\pm(t)| \sim t^\alpha, \quad \alpha > 0 \quad \text{for } p > m$$

(*localization* of the solution if $p < m$, respectively *positivity* if $p \geq m$);

- if $0 < p < 1$, there is *extinction* of the solution in finite time - namely, there exists $T^* \in (0, \infty)$ such that $u \equiv 0$ in $\mathbb{R} \times (T^*, \infty)$.

Let us mention that the case $\rho = 1$, $c_0 = c_0(x)$ was also investigated (see [PT1], [PT2]).

2.2 Inhomogeneous Porous Medium Equation

New interesting phenomena arise when ρ depends on the space variable. Consider the case $\rho = \rho(x)$, $c_0 = 0$. The main qualitative novelty is that, if $\rho(x) \rightarrow 0$ "fast enough" as $|x| \rightarrow \infty$, interfaces can *run off in finite time* - namely there possibly exists $\bar{T} \in (0, \infty)$ such that

$$\zeta(t) \rightarrow \infty \quad \text{as } t \rightarrow \bar{T}^-.$$

In fact, the following holds ([KK], [GuHP], [GKK]; see also [GK], [P]).

Theorem 2.1 *Let $c_0 = 0$, $|x| \rho(x) \in L^1(\mathbb{R})$. Then for any solution to problem (1.1) there exists $\bar{T} \in (0, \infty)$ such that*

$$\zeta(t) \rightarrow \infty \quad \text{as } t \rightarrow \bar{T}^-.$$

The above theorem is related to the following convergence result ([KR2]).

Theorem 2.2 *If $\rho \in L^1(\mathbb{R})$, there holds*

$$u(\cdot, t) \rightarrow \bar{u} := \frac{\|\rho u_0\|_1}{\|\rho\|_1} \quad \text{as } t \rightarrow \infty,$$

the convergence being uniform on compact subsets of $\mathbb{R}$.

Sketch of the Proof of Theorem 2.1: If $\text{supp } u(\cdot, t)$ is compact for any $t > 0$, the following equalities can be proven:

$$\int_{\mathbb{R}} \rho(x) u(x, t) dx = \int_{\mathbb{R}} \rho(x) u_0(x) dx,$$

$$\frac{1}{t} \int_0^\infty x \rho(x) [u(x, t) - u_0(x)] dx = \frac{1}{t} \int_0^t u^m(0, \tau) d\tau.$$

($t > 0$). Since $u(\cdot, t) \rightarrow \bar{u} > 0$ as $t \rightarrow \infty$ by Theorem 2.2, a contradiction arises if $\zeta(t) < \infty$ for any $t > 0$. This proves the result. $\square$

Remark 2.3 The above proof cannot be adapted to the case $c_0 > 0$.

The criterion of Theorem 2.1 is extended to higher space dimension as follows. Let $\rho \sigma \in L^1(\mathbb{R}^n)$, where

$$\sigma(x) := \begin{cases} |x| & \text{if } n = 1, \\ \log(|x|) & \text{if } n = 2, \\ |x|^{\frac{2-n}{m}} & \text{if } n \geq 3; \end{cases}$$

then $\text{supp } u$ becomes unbounded in finite time ([GuHP], [GiT]).

For the choice

$$\rho(x) = \frac{\bar{\rho}}{(1 + |x|)^k}$$

it can be proved by comparison methods that blow-up occurs if and only if $k > 2$ (see [KK]). In fact, for any $k \leq 2$ there exist $a_0, b_0 > 0$ such that:

$$|\zeta^\pm(t)| \sim a_0 t^{\frac{1}{2-k}} \text{ as } t \rightarrow \infty \text{ if } k < 2,$$

$$|\zeta^\pm(t)| \sim e^{b_0 t} \text{ as } t \rightarrow \infty \text{ if } k = 2.$$

Hence *critical value* for the case $c_0 = 0$ is $k = 2$.

3 Results

Now the question arises: How do absorption and variable density cooperate to influence the situation depicted in Section 2? In particular, which phenomena arising when $\rho = \rho(x)$ and $c_0 = 0$ are *structurally stable* with respect to the parameter $c_0 \geq 0$?

As we shall see below, the following answer can be given:

(i) As in the case of constant ρ , there is localization of the solution if $p < m$, positivity if $p > m$. In the latter event, the behaviour of interfaces is the same as in the case $c_0 = 0$, yet with a different critical value of k , namely

$$k^* := 2 \frac{p-1}{p-m}$$

(observe that $k^* \rightarrow 2$ as $p \rightarrow \infty$).

(ii) The convergence result in Theorem 2.2 is not structurally stable; in fact, solutions to problem (1.1) go to zero uniformly as $t \rightarrow \infty$ for any $c_0 > 0$.

The results summarized above are proved by comparing solutions to problem (1.1) with suitable sub- and supersolutions (possibly suggested by a proper splitting of domains). We only sketch below the proof of Theorem 3.9, referring the reader to [KRT] for complete proofs. As already remarked (see Remark 2.3) the techniques used for the case $c_0 = 0$ (which rely on mass conservation; see [KK]), cannot be adapted to the present situation.

Concerning the critical values $k = 2$ (if $c_0 = 0$) and $k = k^*$ (if $c_0 > 0$, $p > m$), observe that:

(i') the function $u(x, t) = f(xt^{-\frac{1}{2-k}})$ is a similarity solution to the equation

$$|x|^{-k}u_t = (u^m)_{xx}$$

if $k = 2$;

(ii') the function $u(x, t) = t^\alpha f(xt^{-\beta})$ with

$$\alpha := -\frac{2}{(p-m)(k^*-k)}, \quad \beta := \frac{1}{k^*-k}$$

is a similarity solution to the equation

$$|x|^{-k}u_t = (u^m)_{xx} - c_0u^p$$

in the case $c_0 > 0$, $p > m$ if $k = k^*$.

Analogous results hold at $k = 2$ and $k = k^*$.

3.1 Well-Posedness and Comparison

The basic theory for problem (1.1) - as well as for initial-boundary value problems related to the differential equation in (1.1) - was studied in [RT]; in particular, this includes comparison results for solutions to the first boundary value and to the Cauchy-Dirichlet problems, in regions whose lateral boundaries may be *curvilinear*, which are needed to prove several statements listed below. Let us mention the following results.

Theorem 3.1 *Let $m > 1$, $p > 0$, $\rho \in C^3(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\rho > 0$ and $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists a unique solution to the Cauchy problem (1.1).*

Theorem 3.2 *Let $\underline{u}$ be a subsolution, $\bar{u}$ a supersolution to the Cauchy problem (1.1). Then $\underline{u} \leq \bar{u}$.*

3.2 Asymptotic Behaviour

As already mentioned, the convergence result in Theorem 2.2 is not structurally stable; moreover, there is extinction of the solution in finite time if $0 < p < 1$. In fact, the following result can be proven.

Theorem 3.3 *Let u be any solution to problem (1.1). Then*

$$\|u(\cdot, t)\|_{\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Moreover, if $0 < p < 1$ there exists $T^ \in (0, \infty)$ such that $u(\cdot, t) = 0$ for $t > T^*$.*

3.3 Localization and Positivity

The results of this subsection show that there is localization in the range $p < m$, positivity in the range $p > m$; namely, in this respect the qualitative situation is the same as in the case $\rho = 1$ (see Subsection 2.1).

Theorem 3.4 *Let $p < m$. Then for any solution u to problem (1.1) there exists $L > 0$ (depending on m, p, c_0, u_0) such that*

$$|\zeta^{\pm}(t)| \leq L \quad \text{for any } t \geq 0.$$

Theorem 3.5 *Let $p > m$. Then for any solution u to problem (1.1) there exist $a, b > 0$ (depending on $m, p, \|\rho\|_{\infty}, c_0, u_0$) such that*

$$|\zeta^{\pm}(t)| \geq b [\log(at + 3)]^{1/2} \quad \text{for any } t \geq 0.$$

3.4 Global Existence of Interfaces

Now suppose $p > m$, so that positivity prevails (see Theorem 3.5 above). Does the support of the solution remain bounded for any positive time? On the strength of the case $c_0 = 0$ (see Subsection 2.2), it is expected that the dependence of the support on time is influenced by the decay rate of the density ρ as $|x| \rightarrow \infty$. In fact, if the exponent k in (1.2) is "small" - namely, if $k \leq k^*$ - both interfaces exist for any $t > 0$; this is the content of the following two theorems. On the other hand, the interfaces can blow up in finite time if the exponent k is "large" (namely, if $k > k^*$), as we shall see in the following subsection.

Theorem 3.6 *Let $p > m$. Moreover, let ρ satisfy the condition:*

$$(3.3) \quad \frac{\rho_1}{(1 + |x|)^k} \leq \rho(x) \leq \rho_0, \quad (\rho_0, \rho_1 > 0)$$

where $0 < k < k^ := 2(p - 1)/(p - m)$. Then there exists $C_1 > 0$ such that*

$$|\zeta^{\pm}(t)| \leq C_1 t^{\frac{1}{k^* - k}} \quad \text{for any } t \geq 0.$$

Theorem 3.7 Let $p > m$. Moreover, let ρ satisfy condition (3.3) with $k = k^*$. Then there exist $C_2, \beta > 0$ such that

$$|\zeta^\pm(t)| \leq C_2 e^{\beta t} \quad \text{for any } t \geq 0.$$

Theorem 3.8 Let $p = m$. Moreover, let ρ satisfy condition (3.3) with $k > 0$. Then for any $\beta > 0$ there exists $C_3 > 0$ such that

$$|\zeta^\pm(t)| \leq C_3 t^\beta \quad \text{for any } t \geq 0.$$

3.5 Blow-Up of Interfaces

In contrast with the previous situation, we prove below that the interfaces can blow up in finite time if $k > k^*$, at least for a suitable class of initial data.

Theorem 3.9 Let $p > m$. Moreover, let ρ satisfy the inequalities

$$\frac{\rho_1}{(1 + |x|)^k} \leq \rho(x) \leq \frac{\rho_2}{(1 + |x|)^k}, \quad (\rho_1, \rho_2 > 0),$$

with $k > k^*$. Then for any $h > 0$ there exists $b_0 = b_0(h) > 0$ such that, if

$$\frac{m}{m-1} u_0^{m-1}(x) \geq h \left[1 - \frac{|x|}{b} \right]_+$$

with $b > b_0$, then

$$u(x, t) > 0 \quad \text{for any } x \in \mathbb{R}, t > 1.$$

Sketch of the Proof: Consider the auxiliary function

$$w(x, t) := (1 + at)^{-\alpha} \left[b^2 - \frac{x^2}{(1 + at)^{2\beta}} \right]_+,$$

where a, b, α and β are positive parameters.

The following claim can be proved: There exist $a, b, \alpha, \beta > 0$ such that

$$v \leq w \quad \text{in } G := \left\{ |x| > \sqrt{\frac{\alpha}{\alpha + \beta}} b(1 + at)^\beta, t > 0 \right\},$$

where $v := \frac{m}{m-1} u^{m-1}$. In fact, we can achieve

$$-M \leq \mathcal{L}w \equiv -\rho(x)w_t + (m-1)ww_{xx} + w_x^2 - cw^q \leq 0.$$

This implies

$$\text{supp } v(\cdot, t) \subseteq (-b(1 + at)^\beta, b(1 + at)^\beta)$$

for any $t \geq 0$. It is possible to choose $\beta = \frac{1}{k^* - k}$; then the conclusion follows. $\square$

References

- [B] M. Bertsch, *A Class of Degenerate Diffusion Equations with a Singular Nonlinear Term*, Nonlinear Anal. **7** (1983), 117-127.
- [BK] M. Bertsch and S. Kamin, *A System of Degenerate Parabolic Equations*, SIAM J. Math. Anal. **21** (1998), 905-916..
- [BKP] M. Bertsch, R. Kersner and L. A. Peletier, *Positivity Versus Localization in a Degenerate Diffusion Equation*, Nonlinear Anal. **9** (1985), 987-1008.
- [CMM] X.-Y. Chen, H. Matano and M. Mimura, *Finite-Point Extinction and Continuity of Interfaces in a Nonlinear Diffusion Equation with Strong Absorption*, J. Reine Angew. Math. **459** (1995), 1-36.
- [E] D. Eidus, *The Cauchy Problem for the Nonlinear Filtration Equation in an Inhomogeneous Medium*, J. Differential Equations **84** (1990) , 309-318.
- [GK] V. Galaktionov and J. R. King, *On Behaviour of Blow-Up Interfaces for an Inhomogeneous Filtration Equation*, IMA J. Appl. Math. **57** (1996), 53-77.
- [GKK] V.A. Galaktionov, S. Kamin and R. Kersner, *On Compactly Supported Solutions to Nonlinear Filtration Equations in an Inhomogeneous Medium* (unpublished manuscript, 1993).
- [GiT] B.H. Gilding and A. Tesei, unpublished result.
- [GuHP] M. Guedda, D. Hilhorst and M. A. Peletier, *Disappearing Interfaces in Nonlinear Diffusion*, Adv. Math. Sci. Appl. **7** (1997), 695-710.
- [Ka] A.S. Kalashnikov, *Some Problems of the Qualitative Theory of Non-Linear Degenerate Second-Order Parabolic Equations*, Usp. Mat. Nauk **42** (1987), 135-176. English translation in: Russian Math. Surveys **42** (1987), 169-222.
- [K] S. Kamin, *Heat Propagation in an Inhomogeneous Medium*. In: *Progress in Partial Differential Equations: the Metz Surveys 4*, pp. 229-237 (Pitman Research Notes Math. Ser. **345**; Longman, Harlow, 1996).
- [KK] S. Kamin and R. Kersner, *Disappearance of Interfaces in Finite Time*, Meccanica **28** (1993), 117-120.
- [KKT] S. Kamin, R. Kersner and A. Tesei, *On the Cauchy Problem for a Class of Parabolic Equations with Variable Density*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **9** (1998), 279-298.
- [KR1] S. Kamin and P. Rosenau, *Propagation of Thermal Waves in an Inhomogeneous Medium*, Comm. Pure Appl. Math. **34** (1981), 831-852.
- [KR2] S. Kamin and P. Rosenau, *Nonlinear Diffusion in a Finite Mass Medium*, Comm. Pure Appl. Math. **35** (1982), 113-127.
- [KRT] R. Kersner, G. Reyes and A. Tesei, *On a Class of Parabolic Equations with Variable Density and Absorption*, Adv. Differential Equations (to appear).
- [P] M.A. Peletier, *A Supersolution for the Porous Media Equation with Nonuniform Density*, Appl. Math. Lett. **7** (1994), 29-32.

- [PT1] L.A. Peletier and A. Tesei, *Diffusion in Inhomogeneous Media: Localization and Positivity*, Ann. Mat. Pura Appl. **141** (1985), 307-330.
- [PT2] L.A. Peletier and A. Tesei, *Global Bifurcation and Attractivity of Stationary Solutions of a Degenerate Diffusion Equation*, Adv. Appl. Math. **7** (1986), 435-454.
- [RT] G. Reyes and A. Tesei, *Basic Theory for a Diffusion-Absorption Equation in an Inhomogeneous Medium*, Nota Scientifica 01/1, Dip. Mat. Università "La Sapienza", Roma (2001).
- [T] A. Tesei, *On Uniqueness of the Positive Cauchy Problem for a Class of Parabolic Equations*. In: *Problemi Attuali dell'Analisi e della Fisica Matematica*, pp. 145-160 (Aracne, Roma, 2000).