

Backward global solutions characterizing annihilation dynamics of travelling fronts

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Abstract We consider a reaction-diffusion equation $u_t = u_{xx} + f(u)$, where f has exactly three zeros 0 , α and 1 ($0 < \alpha < 1$), $f_u(0) < 0$, $f_u(1) < 0$ and $\int_0^1 f(u)du \geq 0$. Then, the equation has a travelling wave solution $u(x, t) = \phi(x - ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Known results suggest that for an initial state $u_0(x)$ with $\lim_{x \rightarrow \pm\infty} u_0(x) > \alpha$ having two interfaces at a large distance, $u(x, t)$ approaches a pair of travelling wave solutions $\phi(x - p_1(t)) + \phi(-x + p_2(t))$ for a long time, and then the travelling fronts eventually disappear by colliding with each other. While our results establish this process, they show that *there is a (backward) global solution $\psi(x, t)$ and that the annihilation process is approximated by a solution $\psi(x - x_0, t - t_0)$.*

Keywords: bistable reaction-diffusion equation, entire solution, travelling wave, collision, collapse, invariant manifold.

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1 Introduction

In this paper, we consider the scalar bistable reaction-diffusion equation

$$(1.1) \quad \begin{cases} u_t = u_{xx} + f(u), & t > 0, \quad x \in \mathbf{R}, \\ u(0) = u_0 \in BU(\mathbf{R}), \end{cases}$$

where $BU(\mathbf{R})$ is the space of bounded uniformly continuous functions from $\mathbf{R}$ to $\mathbf{R}$ with the supremum norm, and the reaction term f satisfies the following conditions:

- 1 $f \in C^2(\mathbf{R})$,
- 2 f has exactly three zeros 0, α and 1 ($0 < \alpha < 1$),
- 3 $f_u(0) < 0$, $f_u(1) < 0$,
- 4 $\int_0^1 f(u) du \geq 0$.

It is known (e.g. [4, Section 4.4]) that the reaction-diffusion equation (1.1) has a unique (except for translation) travelling wave solution $u(x, t) = \phi(x - ct)$, where (ϕ, c) satisfies

$$(1.2) \quad \phi''(z) + c\phi'(z) + f(\phi(z)) = 0$$

with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Then $c \leq 0$ holds from $\int_0^1 f(u) du \geq 0$. We normalize the definition of ϕ by requiring $\phi(0) = 1/2$.

This solution is linearly stable except for neutral translational perturbations. Specifically, the following is known (e.g. [10, Section 5.4]).

Theorem A (1) *The operator $-(\frac{\partial^2}{\partial z^2} + c\frac{\partial}{\partial z} + f_u(\phi(z))) : BU(\mathbf{R}) \rightarrow BU(\mathbf{R})$ is a sectorial one with a simple eigenvalue 0. The remainder of the spectrum has real part greater than some positive constant.*

(2) *There exist δ , C and $\gamma > 0$ such that for any $u_0 \in BU(\mathbf{R})$ with $\|u_0(x) - \phi(x)\|_{C^0} \leq \delta$, there exists $x_0 \in \mathbf{R}$ satisfying*

$$\|u(x, t) - \phi(x - x_0 - ct)\|_{C^0} \leq Ce^{-\gamma t} \|u_0(x) - \phi(x)\|_{C^0}$$

for all $t \geq 0$.

Moreover, Fife and McLeod [6] showed the following theorem, which gives a global stability result for the travelling wave solution $\phi(x - ct)$.

Theorem B *If $\overline{\lim}_{x \rightarrow -\infty} u_0(x) < \alpha$ and $\underline{\lim}_{x \rightarrow +\infty} u_0(x) > \alpha$ hold, then*

$$\inf_{x_0 \in \mathbf{R}} \|u(x, t) - \phi(x - x_0)\|_{C^0} \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

holds.

Also, Fife and McLeod [6] showed the following, which means that the pair of the travelling wave solutions going to $x = \pm\infty$ has strong attractivity.

Theorem C *Suppose that $c < 0$, $\overline{\lim}_{x \rightarrow \pm\infty} u_0(x) < \alpha$, $u_0(x) \geq \eta$ ($|x| < L$) for some $\eta > \alpha$ and $u_0(x) \geq \zeta$ ($|x| < \infty$) for some $\zeta > -\infty$ hold. If L is large enough depending on η and ζ , then $u(x, t)$ approaches (uniformly in x and exponentially in t) a pair of diverging travelling wave solutions*

$$\phi(x - x_1 - ct) + \phi(-x - x_2 - ct) - 1.$$

On the other hand, when $\underline{\lim}_{x \rightarrow \pm\infty} u_0(x) > \alpha$ holds, the following is known (e.g. [5]).

Proposition D *If $\underline{\lim}_{x \rightarrow \pm\infty} u_0(x) > \alpha$ holds, then $\lim_{t \rightarrow +\infty} \|u(x, t) - 1\|_{C^0} = 0$ holds.*

For an initial state $u_0(x)$ with $\underline{\lim}_{x \rightarrow \pm\infty} u_0(x) > \alpha$ having two interfaces at a large distance, Theorems A, B and C suggest that $u(x, t)$ approaches a pair of travelling wave solutions

$$\phi(x - p_1(t)) + \phi(-x + p_2(t))$$

for a long time. Then, Proposition D suggests that the travelling fronts eventually disappear by colliding with each other. While our main results (Theorem 1.1 and Corollary 1.4) establish this process, they show that *there is a (backward) global solution $\psi(x, t)$ and that the annihilation process is approximated by a solution $\psi(x - x_0, t - t_0)$.*¹

Theorem 1.1 *There exists a solution $\psi \in C(\mathbf{R}, BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ satisfying $\lim_{t \rightarrow +\infty} \|\psi(t) - 1\|_{C^0(\mathbf{R})} = 0$, $\psi(-x, t) = \psi(x, t)$ and the following.*

¹ For mathematical studies on motion and collapse of fronts in (1.1) from other aspects, we can refer to, e.g., [1], [2], [3], [7], [8], [9], [11] and [12].

(1) There exists $p \in C^1(\mathbf{R})$ such that

$$p(-\infty) = +\infty, \dot{p}(-\infty) = c$$

and

$$\lim_{t \rightarrow -\infty} \|\psi(x, t) - (\phi(x - p(t)) + \phi(-x - p(t)))\|_{C^0(\mathbf{R})} = 0$$

hold.

(2) There exist $\delta > 0$, $C > 0$ and $\gamma > 0$ such that for any $t_0 \in \mathbf{R}$ and $u_0 \in BU(\mathbf{R})$ satisfying $\|u_0 - \psi(t_0)\|_{C^0(\mathbf{R})} \leq \delta$, there exist $x_0, t'_0 \in \mathbf{R}$ and a solution $u \in C([0, +\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ such that

$$\|u(x, t) - \psi(x - x_0, t - t'_0)\|_{C^0(\mathbf{R})} \leq Ce^{-\gamma t} \|u_0(x) - \psi(x, t_0)\|_{C^0(\mathbf{R})}$$

holds for all $t \geq 0$.

Theorem 1.1 leads to the following. This is a uniqueness result for the global solution $\psi(x, t)$.

Corollary 1.2 For any $T \in [-\infty, +\infty)$ and solution $\bar{\psi} \in C((T, +\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$, if there exist $\{p_n\}_{n=1}^\infty, \{q_n\}_{n=1}^\infty \subset \mathbf{R}$ and $\{T_n\}_{n=1}^\infty \subset (T, +\infty)$ such that

$$\lim_{n \rightarrow \infty} (p_n - q_n) = +\infty$$

and

$$(1.3) \quad \lim_{n \rightarrow \infty} \|\bar{\psi}(x, T_n) - (\phi(x - p_n) + \phi(-x + q_n))\|_{C^0(\mathbf{R})} = 0$$

hold, then $T = -\infty$ holds and there exist x_0 and $t_0 \in \mathbf{R}$ satisfying

$$\psi(x, t) = \bar{\psi}(x + x_0, t + t_0).$$

Proof. By Theorem 1.1 (1), there exists $\{t'_n\}_{n=1}^\infty \subset \mathbf{R}$ with $\lim_{n \rightarrow \infty} t'_n = -\infty$ such that

$$\lim_{n \rightarrow \infty} \|\psi(x, t'_n) - (\phi(x - \frac{p_n - q_n}{2}) + \phi(-x - \frac{p_n - q_n}{2}))\|_{C^0(\mathbf{R})} = 0$$

holds. Hence, from (1.3),

$$\lim_{n \rightarrow \infty} \|\bar{\psi}(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t'_n)\|_{C^0(\mathbf{R})} = 0$$

holds. By Theorem 1.1 (2), if $n \in \{1, 2, \dots\}$ is sufficiently large, then there exist x_n and $t_n \in \mathbf{R}$ such that

$$\begin{aligned} & \|\bar{\psi}(x, t + T_n) - \psi(x - x_n, t + T_n - t_n)\|_{C^0(\mathbf{R})} \\ & \leq Ce^{-\gamma t} \|\bar{\psi}(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t'_n)\|_{C^0(\mathbf{R})} \end{aligned}$$

holds for all $t \geq 0$. Therefore, we obtain

$$(1.4) \quad \lim_{n \rightarrow \infty} \sup_{t \geq T_n - t_n} \|\bar{\psi}(x + x_n, t + t_n) - \psi(x, t)\|_{C^0(\mathbf{R})} = 0.$$

Hence, from (1.3),

$$\lim_{n \rightarrow \infty} \|\psi(x, T_n - t_n) - (\phi(x - (p_n - x_n)) + \phi(-x + (q_n - x_n)))\|_{C^0(\mathbf{R})} = 0$$

holds. Because $\lim_{n \rightarrow \infty} ((p_n - x_n) - (q_n - x_n)) = +\infty$ also holds, by Theorem 1.1 (1), we obtain $\lim_{n \rightarrow \infty} (T_n - t_n) = -\infty$.

Now, we show that there exists $\bar{t}_0 \in \mathbf{R}$ such that $\lim_{n \rightarrow \infty} t_n = \bar{t}_0$ holds. Assume that there exist $\{N_n\}_{n=1}^\infty$ and $\{M_n\}_{n=1}^\infty \subset \{1, 2, \dots\}$ such that $\lim_{n \rightarrow \infty} N_n = \lim_{n \rightarrow \infty} M_n = \infty$ and $\inf_{n=1,2,\dots} (t_{N_n} - t_{M_n}) > 0$ hold. Then, by (1.4),

$$\lim_{n \rightarrow \infty} \|\psi(x, t) - \psi(x + x_{N_n} - x_{M_n}, t + t_{N_n} - t_{M_n})\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. This is contradiction with $\inf_{n=1,2,\dots} (t_{N_n} - t_{M_n}) > 0$. Hence, $\lim_{n \rightarrow \infty} t_n = \bar{t}_0 \in \mathbf{R}$ holds.

Because $\lim_{n \rightarrow \infty} (T_n - t_n) = -\infty$ and $\lim_{n \rightarrow \infty} t_n = \bar{t}_0 \in \mathbf{R}$ hold, we obtain $T = \lim_{n \rightarrow \infty} T_n = -\infty$. Also, by (1.4),

$$\lim_{(n,m) \rightarrow (\infty, \infty)} \|\psi(x, t - \bar{t}_0) - \psi(x + x_n - x_m, t - \bar{t}_0)\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. Hence, we have $\lim_{(n,m) \rightarrow (\infty, \infty)} |x_n - x_m| = 0$. There exists $\bar{x}_0 \in \mathbf{R}$ such that $\lim_{n \rightarrow \infty} x_n = \bar{x}_0$ holds. Therefore, by (1.4), we obtain $\bar{\psi}(x + \bar{x}_0, t + \bar{t}_0) = \psi(x, t)$. q.e.d.

Definition 1 For $l > 0$, $\delta \in (0, \min\{\alpha, 1 - \alpha\})$ and $L > 0$, a closed subset $\Xi_{l,\delta,L}$ of $BU(\mathbf{R})$ is defined by

$$\begin{aligned} \Xi_{l,\delta,L} = \{ & u \in BU(\mathbf{R}) \mid 0 \leq u(x) \leq \alpha - \delta \ (|x| < l - L), \\ & 0 \leq u(x) \leq 1 \ (l - L \leq |x| \leq l + L), \ \alpha + \delta \leq u(x) \leq 1 \ (l + L < |x|)\}. \end{aligned}$$

For $\bar{l} > 0$, $\bar{\delta} \in (0, \min\{\alpha, 1 - \alpha\})$ and $\bar{L} > 0$, a closed subset $\Pi_{\bar{l}, \bar{\delta}, \bar{L}}$ of $BU(\mathbf{R})$ is defined by

$$\Pi_{\bar{l}, \bar{\delta}, \bar{L}} = \bigcup_{l \geq \bar{l}} \Xi_{l, \bar{\delta}, \bar{L}}.$$

The following proposition is proved in Section 6.

Proposition 1.3 *For any $\bar{\delta}_0 \in (0, \min\{\alpha, 1 - \alpha\})$, $\bar{L}_0 > 0$ and $\varepsilon > 0$, there exist $\bar{l}_0 > 0$, $L > 0$ and $T > 0$ such that for any $l \geq \bar{l}_0$ and $u_0 \in \Xi_{l, \bar{\delta}_0, \bar{L}_0}$, there exist $x_1, x_2 \in [l - L, l + L]$ and a solution $u \in C([0, +\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ such that*

$$\|u(x, T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))\|_{C^0(\mathbf{R})} < \varepsilon$$

holds.

Theorem 1.1 and Proposition 1.3 lead to the following.

Corollary 1.4 *For any $\bar{\delta}_0 \in (0, \min\{\alpha, 1 - \alpha\})$, $\bar{L}_0 > 0$, $T_0 \in \mathbf{R}$ and $\varepsilon > 0$, there exists $\bar{l}_0 > 0$ such that for any $u_0 \in \Pi_{\bar{l}_0, \bar{\delta}_0, \bar{L}_0}$, there exist $x_0 \in \mathbf{R}$, $t_0 \geq -T_0$ and a solution $u \in C([0, +\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ such that*

$$\sup_{t \geq T_0} \|u(x + x_0, t + t_0) - \psi(x, t)\|_{C^0(\mathbf{R})} < \varepsilon$$

holds.

Proof. We first show that there exist $M > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for any p, q and $t \in \mathbf{R}$, if

$$p + q \geq M$$

and

$$(1.5) \quad \|\psi(x, t) - (\phi(x - p) + \phi(-x - q))\|_{C^0(\mathbf{R})} < \left(1 + \frac{1}{2C}\right) \varepsilon'$$

hold, then $t \leq T_0$ holds. Assume that there exist $\{p_n\}_{n=1}^\infty, \{q_n\}_{n=1}^\infty \subset \mathbf{R}$ and $\{t_n\}_{n=1}^\infty \subset (T_0, +\infty)$ such that

$$\lim_{n \rightarrow \infty} (p_n + q_n) = +\infty$$

$$\lim_{n \rightarrow \infty} \|\psi(x, t_n) - (\phi(x - p_n) + \phi(-x - q_n))\|_{C^0(\mathbf{R})} = 0$$

hold. Then, from Corollary 1.2, $T_0 = -\infty$ holds. This is contradiction for $T_0 \in \mathbf{R}$.

By Proposition 1.3, there exist L , T and $\bar{l}'_0 > 0$ such that for any $l \geq \bar{l}'_0$ and $u_0 \in \Xi_{l, \bar{\delta}_0, \bar{L}_0}$, there exist x_1 and $x_2 \geq l - (L - cT)$ such that

$$(1.6) \quad \|u(x, T) - (\phi(x - x_1) + \phi(-x - x_2))\|_{C^0(\mathbf{R})} < \min \left\{ \frac{\varepsilon'}{2C}, \frac{\delta}{2} \right\}$$

holds. Then, let $\bar{l}_0 > 0$ be sufficiently large. Because $\frac{x_1 + x_2}{2} > 0$ is sufficiently large, by Theorem 1.1 (1), there exists $t'_0 \in \mathbf{R}$ such that

$$\begin{aligned} & \|\psi(x, t'_0) - (\phi(x - \frac{x_1 + x_2}{2}) + \phi(-x - \frac{x_1 + x_2}{2}))\|_{C^0(\mathbf{R})} \\ & < \min \left\{ \frac{\varepsilon'}{2C}, \frac{\delta}{2} \right\} \end{aligned}$$

holds. Therefore, we have

$$\|u(x + \frac{x_1 - x_2}{2}, T) - \psi(x, t'_0)\|_{C^0(\mathbf{R})} < \min\{\varepsilon'/C, \delta\}.$$

Hence, by Theorem 1.1 (2), there exist x_0 and $t_0 \in \mathbf{R}$ such that

$$(1.7) \quad \sup_{t \geq T} \|u(x, t) - \psi(x - x_0, t - t_0)\|_{C^0(\mathbf{R})} < \varepsilon'$$

holds. Hence, from (1.6), we have

$$\begin{aligned} & \|\psi(x, T - t_0) - (\phi(x - (x_1 - x_0)) + \phi(-x - (x_2 + x_0)))\|_{C^0(\mathbf{R})} \\ & < \left(1 + \frac{1}{2C}\right) \varepsilon'. \end{aligned}$$

Because $(x_1 - x_0) + (x_2 + x_0)$ is sufficiently large and (1.5) holds, $T - t_0 \leq T_0$ holds. Hence, from (1.7), $\sup_{t \geq T_0} \|u(x + x_0, t + t_0) - \psi(x, t)\|_{C^0(\mathbf{R})} < \varepsilon$ holds.

In order to prove Theorem 1.1, we need to construct a *global* invariant manifold with asymptotic stability. Here, the word of *global* means that the invariant manifold includes a solution having two interfaces at any sufficiently large distance. In Section 2, we construct a semilinear parabolic system. The system concludes a part of the reaction-diffusion equation. This is the part which consists of solutions near pairs of the travelling wave solutions at a large distance. Further, such pairs are contained in a two-dimensional *linear* subspace of the system. Hence, we can construct a global invariant manifold near the subspace by a standard technique. While we do it in Section 5, we state the result in the end of Section 2. In Section 3, we prove that there is a solution in the invariant manifold of the system and the solution satisfies Theorem 1.1 (1) in the reaction-diffusion equation, i.e., it becomes the pair of the travelling wave solutions as $t \rightarrow -\infty$. This solution is denoted by $\psi(x, t)$. In Section 4, we show that the set of solutions $\psi(x - x_0, t - t_0)$ by translation of $\psi(x, t)$ corresponds the invariant manifold of the system. This argument is rather troublesome. Then, we show Theorem 1.1 (2), i.e., the set has asymptotic stability in the reaction-diffusion equation. This is also a little troublesome, as the topologies of the equation and the system are different. Proposition 1.3 is proved in Section 6.

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