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1. Introduction

Let L^2 be the Hilbert space of all square Lebesgue integrable functions defined on the unit circle, and L^∞ the Banach algebra of all essentially bounded functions defined on the unit circle. We denote the Hardy spaces corresponding to L^2 and L^∞ by H^2 and H^∞ respectively. If h is a function in some Hardy space, then by the Poisson integral formula h is extended to an analytic function in the open unit disc.

Given ϕ in L^∞ , let M_ϕ be the operator on L^2 defined by

$$M_\phi h = \phi h \quad \text{for } h \text{ in } L^2.$$

Let P' be the projection from L^2 onto H^2 . Then an operator on H^2 defined by

$$T_\phi = P' M_\phi |_{H^2}$$

is called a Toeplitz operator with symbol ϕ . In particular, if ϕ is the identity function, then T_ϕ is called the shift operator. From Beurling's theorem, each invariant subspace for the shift operator is of the form ψH^2 with some inner function ψ (i.e., ψ is in H^∞ and $|\psi(e^{it})| = 1$ a.e.)

For an inner function well known Hilbert space $H(\psi)$ is determined by

$$H(\psi) = H^2 \ominus \psi H^2.$$

From now we fix an inner function ψ and hence Hilbert space $H(\psi)$.

We denote the projection from H^2 onto $H(\psi)$ by P .

Definition. For ϕ in L^∞ , we define the general Toeplitz operator $\phi(S(\psi))$ in the sense of [1] by $\phi(S(\psi)) = PT_\phi|_{H(\psi)}$.

If ϕ belongs to H^∞ , then $\phi(S(\psi))$ was defined in [3] and [4], and its properties are well known. Sarason showed that if ϕ is in H^∞ , then $\phi(S(\psi))$ is compact if and only if $\bar{\psi}\phi$ belongs to $H^\infty + C$, where C is the Banach algebra of all continuous functions defined on the unit circle. In this paper we extend this result to ϕ in $H^\infty + C$.

2. Trivial Results

We denote the inner product in $H(\psi)$, H^2 and L^2 by $(\cdot, \cdot)$, $(\cdot, \cdot)'$ and $(\cdot, \cdot)''$ respectively. Let $B(H(\psi))$ be the Banach algebra of all bounded operators on $H(\psi)$, and x a mapping from L^∞ to $B(H(\psi))$ defined by $x(\phi) = \phi(S(\psi))$.

Proposition 1. x is a contractive star linear mapping.

Proof. For f and g in $H(\psi)$, we have

$$(\bar{\phi}(S(\psi))f, g) = (PT_\phi f, g) = (T_\phi f, g)' = (f, T_\phi g)' = (f, PT_\phi g)' = (f, \phi(S(\psi))g).$$

Thus $x(\bar{\phi}) = \phi(S(\psi))^*$. The rest is trivial.

Proposition 2. If ϕ is an invertible function in L^∞ whose essential range is contained in the open right half plane, then $\phi(S(\psi))$ is invertible.

Proof. There exists an $\epsilon > 0$ such that $\|\epsilon \phi^{-1}\| < 1$ (c.f. [2]). From proposition 1, it follows that

$$\|\epsilon \phi(S(\psi)) - I\| < 1,$$

which implies that $\phi(S(\psi))$ is invertible.

From this proposition, we can obtain the next proposition by the same techniques as the proof of 7.19 of [2]. Therefore we omit the proof.

Proposition 3. $\sigma(\phi(S(\psi)))$ is included in the closed convex hull of essential range of ϕ .

Proposition 4. If ϕ is a real valued function in L^∞ , then

$$\sigma(\phi(S(\psi))) \subset \sigma(T_\phi).$$

Proof. Hartmann-Wintner showed that

$$\sigma(T_\phi) = [\text{ess inf } \phi, \text{ess sup } \phi],$$

which is a closed convex hull of the essential range of ϕ . Thus the assertion follows from proposition 3.

3. Main Results

We denote the identity operators on $H(\psi)$, H^2 and L^2 by I , I' and I'' .

Lemma 1. For ϕ in $H^\infty + C$, $(I'' - P')M_\phi P'$ is a compact operator on L^2 .

Proof. Let $\phi = \phi^1 + \phi^2$ be a decomposition of ϕ such that ϕ^1 is in H^∞ and ϕ^2 is in C . Then it follows that

$$(I'' - P')M_\phi P' = (I'' - P')M_{\phi^2} P'.$$

Take trigonometric polynomials q_n ($n=1, 2, \dots$) whose sequence uniformly converges to ϕ^2 . Then, since

$$\| (I'' - P')M_{q_n} P' - (I'' - P')M_{\phi^2} P' \| \leq \| M_{q_n} - M_{\phi^2} \| \leq \| q_n - \phi^2 \| \rightarrow 0 \quad (n \rightarrow \infty),$$

finiteness of the rank of $(I'' - P')M_{q_n} P'$ implies that $(I'' - P')M_{\phi^2} P'$ is compact.

Lemma 2. For ϕ in $H^\infty + C$, $PT_\phi(I'-P)$ is a compact operator.

Proof. This lemma follows from Lemma 1 and next relations;

$$\begin{aligned} PT_\phi(I'-P) &= PP'M_\phi(I'-P) = PP'M_\phi M_\psi M_{\overline{\psi}}(I'-P) = PP'M_\psi M_\phi M_{\overline{\psi}}(I'-P) = \\ &= PP'M_\psi(I''-P')M_\phi P'M_{\overline{\psi}}(I'-P). \end{aligned}$$

Proposition 5. If ϕ is in C and η is in L^∞ , then

$$\phi(S(\psi))\eta(S(\psi)) - (\phi\eta)(S(\psi))$$

$$\text{and } \eta(S(\psi))\phi(S(\psi)) - (\phi\eta)(S(\psi))$$

are compact.

Proof. Since $T_\phi T_\eta - T_{\phi\eta}$ is compact, we have

$$\begin{aligned} PT_\phi PT_\eta P - PT_{\phi\eta} P &= PT_\phi PT_\eta P - PT_\phi T_\eta P + \text{compact} = \\ &= PT_\phi(I'-P)T_\eta P + \text{compact}. \end{aligned}$$

Thus, by Lemma 2, $\phi(S(\psi))\eta(S(\psi)) - (\phi\eta)(S(\psi))$ is compact. Since

$$[\eta(S(\psi))\phi(S(\psi)) - (\eta\phi)(S(\psi))]^* = \overline{\phi(S(\psi))}\overline{\eta(S(\psi))} - \overline{(\phi\eta)(S(\psi))},$$

we can conclude the proof.

Set $H_0^p = \{f \in H^p; f(0)=0\}$. It is well known that if f belongs to H_0^1 then there exist f_1 in H^2 and f_2 in H_0^2 such that $f=f_1 f_2$ and $|f| = |f_1|^2 = |f_2|^2$ a.e..

Lemma 3. If ϕ is in $H^\infty + C$, then there exists a compact operator K from H^2 to $\overline{H}_0^2$ (conjugate space of H_0^2) such that

$$\frac{1}{2\pi} \int_0^{2\pi} \phi \overline{\psi} f dt = (Kf_1, f_2)'' + (\phi(S(\psi))Pf_1, P'\psi \overline{f}_2)$$

for every f in H_0^1 , f_1 in H^2 and f_2 in H_0^2 such that $f = f_1 f_2$.

Remark. It is not assumed that $|f| = |f_1|^2 = |f_2|^2$.

Proof. Since $\psi \overline{f}_2$ is orthogonal to ψH^2 , $P'\psi \overline{f}_2$ belongs to $H(\psi)$.

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f \, dt &= (\phi f_1, \bar{\psi} \bar{f}_2)'' = (P' \phi P f_1, \bar{\psi} \bar{f}_2)'' + \\
&+ (P' \phi (I' - P) f_1, \bar{\psi} \bar{f}_2)'' + ((I'' - P') \phi f_1, \bar{\psi} \bar{f}_2)'' \\
&= (P' \phi P f_1, P' \bar{\psi} \bar{f}_2)'' + (\bar{\psi} P P' \phi (I' - P) f_1, \bar{f}_2)'' + (\bar{\psi} (I'' - P') \phi f_1, \bar{f}_2)'' \\
&= (\phi (S(\psi)) P f_1, P' \bar{\psi} \bar{f}_2)'' + (\bar{\psi} P T_\phi (I' - P) f_1, \bar{f}_2)'' + (\bar{\psi} (I'' - P') M_\phi f_1, \bar{f}_2)''
\end{aligned}$$

Thus $K = M_{\bar{\psi}} P T_\phi (I' - P) + M_{\bar{\psi}} (I'' - P') M_\phi |H^2$ satisfies the conditions of this lemma.

The proof of next theorem is deeply depend to [3].

Theorem 1. Let ϕ be a function in $H^\infty + C$. Then $\phi(S(\psi))$ is compact if and only if $\bar{\psi}\phi$ belongs to $H^\infty + C$.

Proof. Suppose first that $\bar{\psi}\phi$ is in $H^\infty + C$. Then there exist η in H^∞ and ζ in C such that $\phi = \psi(\eta + \zeta)$. Since $(\psi\eta)(S(\psi)) = 0$, it follows that $\phi(S(\psi)) = (\psi\zeta)(S(\psi))$, which is compact [3].

Suppose next that $\phi(S(\psi))$ is compact. We wish to show that the kernel of functional of $\bar{\psi}\phi + H^\infty$ on H_0^1 is sequentially weak star closed. Let f_n be a sequence in the kernel of it which converges weak star to f . Let $f_n = f_{1n} f_{2n}$ be the factorization of f_n such that f_{1n} and f_{2n} belong to H^2 and H_0^2 , respectively, and that $|f_n| = |f_{1n}|^2 = |f_{2n}|^2$. Then, since each of sequences of $\{f_{1n}\}$ and $\{f_{2n}\}$ is bounded in L^2 , we may assume that each of sequences above weakly converges to f_1 and f_2 respectively in L^2 , and $f = f_1 f_2$ [3]. It is clear that f_1 is in H^2 and f_2 is in H_0^2 . From lemma 3, there is a compact operator K such that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f_n \, dt &= (K f_{1n}, \bar{f}_{2n})'' + (\phi(S(\psi)) P f_{1n}, P' \bar{\psi} \bar{f}_{2n}) \\
\text{and } \frac{1}{2\pi} \int_0^{2\pi} \phi \bar{\psi} f \, dt &= (K f_1, \bar{f}_2)'' + (\phi(S(\psi)) P f_1, P' \bar{\psi} \bar{f}_2).
\end{aligned}$$

Since both K and $\phi(S(\psi))$ are compact, it follows that

$$(K f_{1n}, \bar{f}_{2n})'' \longrightarrow (K f_1, \bar{f}_2)'' \text{ as } n \longrightarrow \infty$$

and $(\phi(S(\psi))Pf_{1n}, P'\psi\bar{f}_{2n}) \rightarrow (\phi(S(\psi))Pf_1, P'\psi\bar{f}_2)$ as $n \rightarrow \infty$.

Thus we have

$$\frac{1}{2\pi} \int_0^{2\pi} \phi\bar{\psi} f dt = 0.$$

Thus we can conclude the proof.

4. Miscellaneous Results

Let $\underline{K}$ be the ideal of all compact operators of $B(H(\psi))$.

Corollary 1. $\{\phi(S(\psi)) + \underline{K} ; \phi \in H^\infty + C\}$ is an algebra and the natural mapping from it onto $\{\phi + \psi(H^\infty + C); \phi \in H^\infty + C\}$ is an isomorphism.

Proof. From Proposition 5, $\{\phi(S(\psi)) + \underline{K}\}$ is an algebra. From Theorem 1, the natural mapping is well defined and one to one.

Corollary 2. If ϕ is in $H^\infty + C$ and $\phi(H^\infty + C) + \psi(H^\infty + C) = H^\infty + C$, then $\phi(S(\psi))$ is a Fredholm operator.

Proof. There exists η in $H^\infty + C$ such that $\phi\eta + \psi(H^\infty + C) = 1 + \psi(H^\infty + C)$. Therefore we have

$$\phi(S(\psi))\eta(S(\psi)) + \underline{K} = (\phi\eta)(S(\psi)) + \underline{K} = I + \underline{K}.$$

Consequently $\phi(S(\psi))$ is a Fredholm operator.

Corollary 3. If ϕ is in $H^\infty + C$ and T_ϕ is a Fredholm operator, then $\phi(S(\psi))$ is a Fredholm operator.

Proof. Since ϕ is invertible in $H^\infty + C$ [2], this corollary follows from Corollary 2.

If ϕ is in H^∞ , then we showd that $\phi(S(\psi))$ is a Fredholm operator

if and only if there is a factorization $\phi = \phi^1 \phi^2$, where $\phi^1(S(\psi))$ is invertible and ϕ^2 is a finite Blaschke function [5].

Theorem 2. If ϕ is in H^∞ , then next conditions are equivalent ;

- (a) $\phi(S(\psi))$ is a Fredholm operator,
- (b) there are $\epsilon > 0$ and $1 > \delta \geq 0$ such that $|\phi(z)| + |\psi(z)| \geq \epsilon$ for $1 > |z| \geq \delta$,
- (c) $\phi(H^\infty + C) + \psi(H^\infty + C) = H^\infty + C$.

Proof. First assume (a). Let $\phi = \phi^1 \phi^2$ be the factorization given above. Then there is an $\epsilon > 0$ such that

$$|\phi^1(z)| + |\psi(z)| \geq \epsilon \quad \text{for } |z| < 1,$$

because $\phi^1(S(\psi))$ is invertible. And there are $\epsilon' > 0$ and $1 > \delta \geq 0$ such that

$$|\phi^2(z)| \geq \epsilon' \quad \text{for } 1 > |z| \geq \delta,$$

because ϕ^2 is a finite Blaschke function. Therefore we have

$$|\phi(z)| + |\psi(z)| \geq |\phi^2(z)| \{ |\phi^1(z)| + |\psi(z)| \} \geq \epsilon \epsilon' \quad \text{for } 1 > |z| \geq \delta.$$

Thus we have (b).

Next assume (b). Let η be the greatest common inner divisor of ϕ and ψ . Then it is obvious that there is an $\epsilon' \geq 0$ such that

$$|\eta(z)| \geq \epsilon' \quad \text{for } 1 > |z| \geq \delta.$$

Consequently $1/\eta$ belongs to $H^\infty + C$ [2]. Set $\phi' = \phi/\eta$ and $\psi' = \psi/\eta$. Then it is clear that there is an $\epsilon'' > 0$ such that

$$|\phi'(z)| + |\psi'(z)| \geq \epsilon'' \quad \text{for } |z| < 1.$$

Hence, by corona theorem, we have $\phi' H^\infty + \psi' H^\infty = H^\infty$, which yields

$$\phi(H^\infty + C) + \psi(H^\infty + C) = H^\infty + C.$$

Last, by Corollary 2, (c) implies (a).

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