

$\bar{\partial}$ -PROBLEMS AND SOME APPLICATIONS

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0. Preliminaries.

Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. We denote by $C^{1,\infty}(\partial D \times D)$ the space of all functions in $C^1(\partial D \times D)$ which are C^∞ in the second variable. A $(1, 0)$ -form $W = \sum_{j=1}^n w_j(\zeta, z) d\zeta_j$ is called a generating form with coefficients in $C^{1,\infty}(\partial D \times D)$ if W satisfies the following conditions (1) and (2):

- (1) $w_j(\zeta, z) \in C^{1,\infty}(\partial D \times D)$.
- (2) $\sum_{j=1}^n w_j(\zeta, z)(\zeta_j - z_j) = 1$.

We define

$$\beta = |\zeta - z|^2, \quad B = \frac{\partial_\zeta \beta}{\beta}, \quad I = [0, 1].$$

The homotopy form on $(\partial D \times I) \times D$ associated to W is defined by

$$\hat{W}(\zeta, \lambda, z) = \lambda W(\zeta, z) + (1 - \lambda)B(\zeta, z).$$

Cauchy-Fantappiè kernel $\Omega_q(\hat{W})$ of order q generated by $\hat{W}$ is defined by

$$\Omega_q(\hat{W}) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} \hat{W} \wedge (\bar{\partial}_{\zeta, \lambda} \hat{W})^{n-q-1} \wedge (\bar{\partial}_z \hat{W})^q, \quad 0 \leq q \leq n-1.$$

$\Omega_q(W)$ is defined in the same way, with W instead of $\hat{W}$. We define $K_q = \Omega_q(B)$. Then we have the Cauchy-Fantappiè integral formula(cf. Range[22]):

Theorem 1. For $1 \leq q \leq n$, define the linear operator

$$T_q^W : C_{0,q}(\bar{D}) \rightarrow C_{0,q-1}(D)$$

by

$$T_q^W f = \int_{\partial D \times I} f \wedge \Omega_{q-1}(\hat{W}) - \int_D f \wedge K_{q-1},$$

and set $T_0^W = T_{n+1}^W = 0$. Then the following holds:

- (a) For $k = 0, 1, \dots, \infty$, if $f \in C_{0,q}^k(D) \cap C_{0,q}(\bar{D})$, then $T_q^W f \in C_{0,q-1}^k(D)$.
- (b) For $0 \leq q \leq n$, if $f \in C_{0,q}^1(\bar{D})$, then

$$f = \int_{\partial D} f \wedge \Omega_q(W) + \bar{\partial} T_q^W f + T_{q+1}^W \bar{\partial} f \quad \text{on } D.$$

Remark. If $W = \sum_{j=1}^n w_j(\zeta, z) d\zeta_j$ is holomorphic in z , then $\Omega_q(W) = 0$ for $q \geq 1$. In this case, if f is a $\bar{\partial}$ -closed $(0, q)$ -form, then it holds that $f = \bar{\partial}(T_q^W f)$.

In 1965, Hörmander obtained L^2 estimates for solutions of the $\bar{\partial}$ -problem in bounded pseudoconvex domains in $\mathbb{C}^n$. On the other hand, L^p and Hölder estimates for solutions of the $\bar{\partial}$ -problem using the above integral formula have been studied since 1970. We begin with the $\bar{\partial}$ -problem in strictly pseudoconvex domains in $\mathbb{C}^n$.

1. $\bar{\partial}$ -problems in bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary.

Theorem 2. (Henkin[10], Ramirez[19]) Suppose $D \Subset \mathbb{C}^n$ is strictly pseudoconvex with C^∞ boundary. There are a neighborhood U of ∂D , positive constants δ, c and γ , and a function $g \in C^\infty(U \times D_\delta)$ with the following properties:

- (i) $g(\zeta, z)$ is holomorphic in z on D_δ .
- (ii) $g(\zeta, \zeta) = 0$ for $\zeta \in U$.
- (iii) $\text{Reg}(\zeta, z) > 0$ for $(\zeta, z) \in U \times D_\delta$ with $r(\zeta) - r(z) + c|\zeta - z|^2 > 0$.
- (iv) On $\{(\zeta, z) \in U \times D_\delta : |\zeta - z| \leq \gamma\}$ there is a function $A \in C^\infty(U \times D_\delta)$ with $|A(\zeta, z)| \geq \frac{2}{3}$, so that $g = FA$, where F is the Levi polynomial.

Using Hefer's theorem, there are functions $g_j \in C^\infty(U \times D_\delta)$, with $g_j(\zeta, \cdot) \in \mathcal{O}(D_\delta)$ such that

$$g(\zeta, z) = \sum_{j=1}^n g_j(\zeta, z)(\zeta_j - z_j) \quad \text{on } U \times D_\delta$$

We define

$$W^{HR} = \sum_{j=1}^n \frac{g_j(\zeta, z)}{g(\zeta, z)} d\zeta_j.$$

Then W^{HR} is called the Henkin-Ramirez generating form. Using the above Henkin-Ramirez generating form, the following theorem was obtained (cf. Henkin[10], Kerzman[13], Lieb[16], Henkin-Romanov[11], Grauert-Lieb[9], Range-Siu[25]).

Theorem 3. Let $D \Subset \mathbb{C}^n$ be strictly pseudoconvex with smooth boundary. For $1 \leq q \leq n$, there are linear operators

$$\hat{S}_q : L_{0,q}^1(D) \rightarrow L_{0,q-1}^1(D)$$

and a constant C with the following properties:

- (i) $\|\hat{S}_q f\|_{L^p(D)} \leq C \|f\|_{L^p(D)}$ for $1 \leq p \leq \infty$.
- (ii) $\|\hat{S}_q\|_{\Lambda_{1/2}(D)} \leq C \|f\|_{L^\infty(D)}$.
- (iii) For $k = 0, 1, 2, \dots$, if $f \in L_{0,q}^1(D) \cap C^k(D)$, then $\hat{S}_q f \in C_{0,q-1}^k(D)$.
- (iv) If $f \in C_{0,q}^1(D) \cap L_{0,q}^1(D)$ and $\bar{\partial} f = 0$, then $\bar{\partial}(\hat{S}_q f) = f$ on D .

Krantz[14] obtained the optimal Lipschitz and L^p estimates for $\bar{\partial}$ in strictly pseudoconvex domains:

Theorem 4. *Let D be a bounded strictly pseudoconvex domain with C^5 boundary. Let $A_{(0,1)}^\infty(D)$ be the space of all $\bar{\partial}$ -closed $(0,1)$ -forms f whose coefficients are C^∞ in $\bar{D}$. Then there is a linear operator*

$$H : A_{(0,1)}^\infty(D) \rightarrow C^\infty(D)$$

satisfying $\bar{\partial}Hf = f$. Moreover Hf satisfies

- (i) $\|Hf\|_{L^{(2n+2)/(2n+1)-\epsilon}}} \leq A_\epsilon \|f\|_{L^1}$ for small enough $\epsilon > 0$
- (ii) if $1 < p < 2n+2$, then $\|Hf\|_{L^q} \leq A_p \|f\|_{L^p}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2n+2}$
- (iii) if $2n+2 < p \leq \infty$, then $\|Hf\|_{\Lambda_\alpha} \leq A_p \|f\|_{L^p}$, $\alpha = \frac{1}{2} - \frac{n+1}{p}$.

For $i \in \{1, \dots, N\}$, we denote by D_i a strictly pseudoconvex open sets in $\mathbb{C}^n$ with C^2 boundary. Let ρ_i be a defining function for D_i . For sufficiently small $\delta_i > 0$ we denote $V_i^\delta = \{-\delta < \rho_i(z) < \delta\}$. we assume that for $1 \leq i_1 < i_2 < \dots < i_l \leq N$, $d\rho_{i_1}, d\rho_{i_2}, \dots, d\rho_{i_l}$ are $\mathbb{R}$ -linearly independent at all points of $V_{i_1}^\delta \cap V_{i_2}^\delta \cap \dots \cap V_{i_l}^\delta$. We set $D = \cap_{i=1}^N D_i$. Then Menini[17] proved the following:

Theorem 5. *Let $f \in L_{0,q}^p(D)$ ($1 \leq q \leq n, 1 \leq p \leq \infty$) be $\bar{\partial}$ -closed. Then there exists a kernel K such that if one defines on D , $T_q f(z) = c_{q,n} \int_D f(\zeta) \wedge K(\zeta, z)$ then $\bar{\partial}(T_q f) = f$. Moreover*

- (i) for $1 \leq p < \infty$,

$$T_q : L_{(0,q)}^p(D) \rightarrow L_{0,q-1}^r(D)$$

is a bounded linear operator where $\frac{1}{r} = \frac{1}{p} + \frac{1}{1+\eta}$, $0 \leq \eta < \frac{1}{2n-1+2 \inf(N_0, n-1)}$, where N_0 is the maximal number of the common intersections,

- (ii) for $p = \infty$

$$T_q : L_{(0,q)}^p(D) \rightarrow \Lambda_{(0,q-1)}^{1/2-\epsilon}(D)$$

is a bounded linear operator for any $\epsilon > 0$.

2. $\bar{\partial}$ -problems in q -convex domains in a complex manifold.

Theorem 6. (Fischer-Lieb[7]) *Let X be a complex manifold and let $D \Subset X$ be a strongly q -convex domain (in the sense of Andreotti-Grauert) with C^3 boundary. Then there exists a constant K with the following properties:*

For each $\bar{\partial}$ -closed $(0, r)$ -form β on D with $r \geq q$ there exists a $(0, r-1)$ -form α on D with $\bar{\partial}\alpha = \beta$ and $|\alpha| \leq K|\beta|$.

Let X be an n -dimensional complex manifold. $D \Subset X$ is called a strictly q -convex C^2 intersection if there exists a finite number of real C^2 functions $\rho_1, \dots, \rho_N$ in a neighborhood U of $\bar{D}$ such that

$$D = \{z \in U : \rho_j(z) < 0 \text{ for } 1 \leq j \leq N\}$$

and the following condition is fulfilled: if $z \in \partial D$ and $1 \leq k_1 < \dots < k_l \leq N$ with $\rho_{k_1}(z) = \dots = \rho_{k_l}(z) = 0$, then

$$d\rho_{k_1}(z) \wedge \dots \wedge d\rho_{k_l}(z) \neq 0$$

and, for all $\lambda_1, \dots, \lambda_l \geq 0$ with $\lambda_1 + \dots + \lambda_l = 1$, the Levi form at z of the function

$$\lambda_1 \rho_{k_1} + \dots + \lambda_l \rho_{k_l}$$

has at least $q+1$ positive eigenvalues. D is called completely q -convex if there exists a real C^2 function φ on D whose Levi form has at least $q+1$ positive eigenvalues at each point in D and such that

$$\{z \in D : \varphi(z) < C\} \Subset D \text{ for all } C > 0.$$

Let E be a holomorphic vector bundle over X . Denote by $B_{n,r}^\beta(D, E)$, $\beta \geq 0$, $r = 0, 1, \dots, n$, the Banach space of E -valued continuous (n, r) -forms f on D such that

$$\sup_{z \in D} \|f(z)\| [\text{dist}(z, \partial D)]^\beta < \infty,$$

and denote by $C_{n,r}^\alpha(\bar{D}, E)$, $0 \leq \alpha \leq 1$, $r = 0, 1, \dots, n$, the Banach space of E -valued (n, r) -forms which are Hölder continuous with exponent α on $\bar{D}$. In this setting, Laurent-Thiébaud-Leiterer[15] proved the following:

Theorem 7. *Let $D \Subset X$ be a strictly q -convex C^2 intersection and completely q -convex. Then:*

(i) *If $0 \leq \beta < \frac{1}{2}$, then there exist linear operators*

$$T_r : B_{n,r}^\beta(D, E) \cap \ker d \rightarrow \cap_{0 < \varepsilon \leq 1/2 - \beta} C_{n,r-1}^{1/2 - \beta - \varepsilon}(\bar{D}, E),$$

$n - q \leq r \leq n$, which are compact as operators from $B_{n,r}^\beta(D, E) \cap \ker d$ to each $C_{n,r-1}^{1/2 - \beta - \varepsilon}(\bar{D}, E)$, $0 < \varepsilon \leq 1/2 - \beta$, and such that

$$dT_r f = f$$

- for all $n - q \leq r \leq n$ and $f \in B_{n,r}^\beta(D, E) \cap \ker d$.
(ii) If $1/2 \leq \beta < 1$, then there exist linear operators

$$T_r : B_{n,r}^\beta(D, E) \cap \ker d \rightarrow \cap_{0 < \varepsilon} B_{n,r-1}^{\beta+\varepsilon-1/2}(D, E),$$

$n - q \leq r \leq n$, which are compact as operators from $B_{n,r}^\beta(D, E) \cap \ker d$ to each $B_{n,r-1}^{\beta+\varepsilon-1/2}(D, E)$, $\varepsilon > 0$, and such that

$$dT_r f = f$$

for all $n - q \leq r \leq n$ and $f \in B_{n,r}^\beta(D, E) \cap \ker d$.

3. $\bar{\partial}$ -problems in bounded weakly pseudoconvex domains in $\mathbb{C}^n$.

In the case of weakly pseudoconvex domains there are several results in $\mathbb{C}^2$.

Theorem 8. (Range[21]) Let $D \subset \mathbb{C}^2$ be a bounded convex domain with real analytic boundary. Then there are positive constants α and K such that for every bounded $\bar{\partial}$ -closed $f \in C_{0,1}^1(D)$ there is $u \in C^1(D)$ such that $\bar{\partial}u = f$ and

$$|u(z) - u(z')| \leq K \|f\|_{L^\infty(D)} |z - z'|^\alpha, \quad z, z' \in D.$$

Theorem 9. (Show[26]) Let D be a pseudoconvex domain in $\mathbb{C}^2$ of uniform strict type m . Let f be a continuous $(0,1)$ -form on $\bar{D}$ and $\bar{\partial}f = 0$, then there exists a function $u \in \Lambda_{1/m}(\bar{D})$ such that $\bar{\partial}u = f$ and u satisfies the following estimates:

- (i) $\|u\|_{L^1(D)} \leq c(\|f\|_{L^1(D)} + \|f\|_{L^1(\partial D)})$,
- (ii) if $p = 1$, then $\|u\|_{L^{(m+2)/(m+1)-\varepsilon}(\partial D)} \leq c\|f\|_{L^1(\partial D)}$ for every small $\varepsilon > 0$,
- (iii) if $1 < p < m + 2$, then $\|u\|_{L^q(\partial D)} \leq c_p \|f\|_{L^p(\partial D)}$ where $\frac{1}{q} = \frac{1}{p} - \frac{1}{m+2}$,
- (iv) if $p = m + 2$, then $\|u\|_{L^q(\partial D)} \leq c_p \|f\|_{L^p(\partial D)}$ for all $q < \infty$,
- (v) if $m + 2 < p \leq \infty$, then $\|u\|_{\Lambda_{1/m-(m+2)/mp}(\partial D)} \leq c_p \|f\|_{L^p(\partial D)}$,
- (vi) $\|u\|_{\Lambda_{1/m}^p(\partial D)} \leq c_p \|f\|_{L^p(\partial D)}$ for every $1 \leq p \leq \infty$.

Theorem 10. (Range[23]) Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m , and let $f \in C_{0,1}^1(\bar{D})$ be $\bar{\partial}$ -closed. Then for every $\eta > 0$ there is a solution $u^{(\eta)}$ of $\bar{\partial}u = f$ on D which satisfies

$$|u^{(\eta)}(z) - u^{(\eta)}(w)| \leq C_\eta \|f\|_{L^\infty} |z - w|^{(1/m)-\eta}$$

for $z, w \in D$.

Theorem 11. (Polking[18]) *Let $D \Subset \mathbb{C}^2$ be convex with C^2 boundary. Then there is an integral solution operator T for $\bar{\partial}$ on D such that $\|Tf\|_{L^p(D)} \leq C_p \|f\|_{L^p(D)}$ for all $1 < p < \infty$.*

Theorem 12. (Range[24]) *Let $D \Subset \mathbb{C}^2$ be convex with C^2 boundary. Then there is an integral solution operator T for $\bar{\partial}$ on D such that*

- (i) $\|Tf\|_{\Lambda_\alpha(D)} \leq C_\alpha \|f\|_{\Lambda_\alpha(D)}$ for all f with $\bar{\partial}f = 0$ and all $\alpha > 0$.
- (ii) $\|Tf\|_{BMO(D)} \leq C \|f\|_{L^\infty(D)}$.

Now we study the uniform and L^p estimates for solutions of the $\bar{\partial}$ -problem in pseudoconvex domains which may be of infinite type.

Let $\Psi \in C^2([0, 1])$ be a real valued function satisfying

- (A) $\Psi(0) = 0$ and $\Psi(1) = 1$.
- (B) $\Psi'(t) > 0$, $0 < t \leq 1$.
- (C) $\Psi'(t) + t\Psi''(t) > 0$, $0 < t \leq 1$.
- (D) There exists $\tau \in (0, 1)$ such that $\Psi''(t) > 0$, $0 < t < \tau$.

Define

$$D_\Psi = \{z \in \mathbb{C}^n : |z_j| < 1, j = 1, \dots, n, \sum_{j=1}^{n-1} |z_j|^2 + \Psi(|z_n|^2) < 1\}.$$

For $\alpha > 0$, define $\Psi_\alpha(t) = e \exp(-1/t^\alpha)$. Then Ψ_α satisfies all conditions (A)-(D). In this case the domain D_{Ψ_α} is not of finite type.

Theorem 13. (Adachi-Chig[2]) *Let $f \in L^p_{0,q}(D_\Psi)$, $1 \leq p \leq \infty$, be $\bar{\partial}$ -closed. If $\int_0^1 |\log \Psi(s)| s^{-\frac{1}{2}} ds < \infty$, then there is a solution u of $\bar{\partial}u = f$ on D_Ψ such that*

$$\|u\|_{L^p(D_\Psi)} \leq c(p) \|f\|_{L^p(D_\Psi)}.$$

where the constant $c(p)$ is independent of f .

Remark. In case $n = 2$ and $p = \infty$, Theorem 13 was obtained by Verdera[28].

4. $\bar{\partial}$ - problems in ellipsoids.

Define

$$D_1 = \{z \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^{m_i} < 1\},$$

$$D_2 = \{z = (z_1, \dots, z_n) : \sum_{i=1}^n (x_i^{l_i} + y_i^{m_i}) < 1, z_j = x_j + iy_j\},$$

where m_i, l_i are positive even integers. We set

$$k_1 = \sup_{1 \leq i \leq n} \{m_i\}, \quad k_2 = \sup_{1 \leq i \leq n} \{\inf\{l_i, m_i\}\}.$$

We have the following:

Theorem 14. (Range[20]) *For each $\alpha < 1/k_1$, there exists a constant C_α such that for every bounded, $\bar{\partial}$ -closed $(0,1)$ -form f on D_1 , there exists a α -Hölder continuous function u on D_1 such that $\bar{\partial}u = f$ and $\|u\|_{\Lambda_\alpha(D_1)} \leq C_\alpha \|f\|_{L^\infty(D_1)}$.*

Theorem 15. (Diederich-Fornaess-Wiegerinck[5]) *There exists a constant C such that for every bounded, $\bar{\partial}$ -closed $(0,1)$ -form f on D_2 , there exists a $\alpha = 1/k_2$ -Hölder continuous function u on D_2 such that $\bar{\partial}u = f$ and $\|u\|_{\Lambda_\alpha(D_2)} \leq C \|f\|_{L^\infty(D_2)}$.*

Remark. *Diederich, Fornaess and Wiegerinck pointed out in their paper that Theorem 14 is also true in case $\alpha = 1/k_1$.*

Theorem 16. (Chen-Krantz-Ma[4]) *Let D_1 be the complex ellipsoid defined above. Then for every $\bar{\partial}$ -closed $(0,1)$ -form f with coefficients in $L^p(D_1)$, there exists a function u on D_1 such that $\bar{\partial}u = f$, and u satisfies the following estimates:*

- (i) *if $p = 1$, then $\mu\{|u| > t\} \leq C\{\|f\|_{L^1(D_1)}^{\frac{1}{t}}\}^\lambda$ for all $t > 0$, where $\lambda = \frac{k_1n+2}{k_1n+1}$,*
- (ii) *if $1 < p < k_1n + 2$, then $\|u\|_{L^q(D_1)} \leq C_p \|f\|_{L^p(D_1)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{k_1n+2}$,*
- (iii) *if $p = k_1n + 2$, then $\|u\|_{L^q(D_1)} \leq C_p \|f\|_{L^p(D_1)}$ for all $q < \infty$.*
- (iv) *if $p > k_1n + 2$, then $\|u\|_{\Lambda_\alpha(D_1)} \leq C_p \|f\|_{L^p(D_1)}$, where $\alpha = \frac{1}{k_1} - (n + \frac{2}{k_1})\frac{1}{p}$.*

We give the results obtained by Fleron[8]. Ho[12] obtained similar results in the case where D is a complex ellipsoid.

Theorem 17. (Fleron[8]) *Let $1 \leq q \leq n - 1$. Let D be a real or complex ellipsoid. Suppose that Δ^q is the maximal order of contact of the boundary of the ellipsoid D with q -dimensional complex linear spaces. Then there are linear operators $T_q : C_{(0,q)}(\bar{D}) \rightarrow C_{(0,q-1)}(D)$ satisfying the following:*

- (i) *if $f \in C_{(0,q)}^1(\bar{D})$ and $\bar{\partial}f = 0$, then $\bar{\partial}(T_q f) = f$ on D ,*
- (ii) *there is a constant $c > 0$ such that $|T_q f(z) - T_q f(z')| \leq c \|f\|_{L^\infty(D)} |z - z'|^{\frac{1}{\Delta^q}}$ for $z, z' \in D$.*

Now we give the following optimal L^p estimates for solutions of the $\bar{\partial}$ -problem in ellipsoids.

Theorem 18. (Adachi[1]) *Let m be the maximal order of contact of the boundary of the complex ellipsoid D with q -dimensional complex linear subspaces. Let $p \geq 1$. Then for every $\bar{\partial}$ -closed $(0,q)$ -form f with coefficients in $L^p(D)$, there exists a function u on D such that $\bar{\partial}u = f$, and u satisfies the following estimates:*

- (i) *if $p = 1$, then $\mu\{|u| > t\} \leq C\{\|f\|_{L^1(D)}^{\frac{1}{t}}\}^\lambda$ for all $t > 0$, where $\lambda = \frac{mn+2}{mn+1}$,*
- (ii) *if $1 < p < mn + 2$, then $\|u\|_{L^s(D)} \leq C_p \|f\|_{L^p(D)}$, where $\frac{1}{s} = \frac{1}{p} - \frac{1}{mn+2}$,*
- (iii) *if $p = mn + 2$, then $\|u\|_{L^s(D)} \leq C_p \|f\|_{L^p(D)}$ for all $s < \infty$,*
- (iv) *if $p > mn + 2$, then $\|u\|_{\Lambda_\alpha(D)} \leq C_p \|f\|_{L^p(D)}$, where $\alpha = \frac{1}{m} - (n + \frac{2}{m})\frac{1}{p}$.*

Theorem 19. (Adachi[1]) *Let m be the maximal order of contact of the boundary of the real ellipsoid D with q -dimensional complex linear subspaces. Let $p \geq 1$. Then for every $\bar{\partial}$ -closed $(0, q)$ -form f with coefficients in $L^p(D)$, there exists a function u on D such that $\bar{\partial}u = f$ and u satisfies the following estimates:*

- (i) *if $p = 1$, then $\|u\|_{L^{\gamma-\varepsilon}(D)} \leq c\|f\|_{L^1(D)}$, where $\gamma = \frac{mn+2}{mn+1}$ and ε is any small number,*
- (ii) *if $1 < p < mn+2$, then $\|u\|_{L^s(D)} \leq c\|f\|_{L^p(D)}$, where $s < q_0$ and q_0 satisfies $\frac{1}{q_0} = \frac{1}{p} - \frac{1}{mn+2}$,*
- (iii) *if $p = mn+2$, then $\|u\|_{L^s(D)} \leq C_p\|f\|_{L^p(D)}$ for all $s < \infty$,*
- (iv) *if $p > mn+2$, then $\|u\|_{\Lambda_\alpha(D)} \leq C_p\|f\|_{L^p(D)}$, where $\alpha = \frac{1}{m} - (n + \frac{2}{m})\frac{1}{p}$.*

5. Applications of the $\bar{\partial}$ -problem.

A. Uniform approximation of holomorphic functions.

Theorem 20. (Kerzman[13]) *Let $D \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. There exists an open set $\hat{D} \Subset \mathbb{C}^n$, $D \subset \bar{D} \Subset \hat{D}$, which has the following properties:*

- (a) *Any continuous function $u : \bar{D} \rightarrow \mathbb{C}$ which is holomorphic in D can be uniformly approximated on $\bar{D}$ by holomorphic functions $\hat{u}$ defined on $\hat{D}$.*
- (b) *Let $u : D \rightarrow \mathbb{C}$ be holomorphic and assume $u \in L^p(D)$, $1 \leq p \leq \infty$. Then there exists a sequence of holomorphic functions $\hat{u}_n : \hat{D} \rightarrow \mathbb{C}$ such that (b_1) , (b_2) and (b_3) below hold:*
 - (b₁) $\hat{u}_n \rightarrow u$ uniformly on compact subsets of D when $n \rightarrow \infty$,
 - (b₂) $\|\hat{u}_n\|_{L^p(D)} \leq K\|u\|_{L^p(D)}$, $1 \leq p \leq \infty$,
 - (b₃) if $p < \infty$, then $\|\hat{u}_n - u\|_{L^p(D)} \rightarrow 0$ when $n \rightarrow \infty$, where K is independent of n, p and u .

B. Vanishing cohomology theorems.

Theorem 21. (Kerzman[13], Lieb[16]) *Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary. Let $\mathcal{F}$ and $\mathcal{B}$ be the sheaf of germs of holomorphic functions in D which are continuous on $\bar{D}$ and the sheaf of germs of holomorphic functions in D which are bounded in D , respectively. Then we have*

$$H^q(\bar{D}, \mathcal{F}) = H^q(\bar{D}, \mathcal{B}) = 0 \quad \text{for } q \geq 1.$$

C. The Poincaré-Lelong equation.

Theorem 22. (Show[27]) *Let D be a real ellipsoid in $\mathbb{C}^n$. Given any analytic variety of complex dimension $(n-1)$ such that M is the zero sets of an analytic function on D of finite area, there exists a function h belonging to the Nevanlinna class such that M is the zero sets of h .*

Theorem 23. (Arlebrink[3]) *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with C^3 boundary. If X is a positive divisor of D with finite area and the canonical cohomology class of X in $H^2(D, \mathbb{Z})$ is zero, then there exists a bounded holomorphic function that defines X .*

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