

SMALLESTNESS AND MINIMALITY OF PAIRWISE SUFFICIENT SUBFIELDS

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This is a continuation of the preceding article[1] by J.K. Ghosh. The same definitions and notations as in that article will be used here, except that the basic space Ω is now replaced by $X = \{x\}$.

1. The smallest subfield with pairwise sufficiency and containment of carriers.

We begin with a simple example which, however, retains all the essential features of the discrete case in general.

[Example 1] Let X be an uncountable space, $\underline{A}$ the sigma-field of all the subsets of X and $\underline{P}$ be the family of all one-point probability measures on X . Define $\underline{C}$ to be the family of all the countable and cocountable sets and for each x in X define $\underline{C}(x)$ to be the family of all the countable sets which do not contain x and all the cocountable sets which contain x . Clearly both of these families are sub-sigma-fields (simply, "subfields") of $\underline{A}$. It follows from Theorem 5 of [1] that $\underline{C}(x)$ is minimal pairwise sufficient (MPS, in short) and that $\underline{C}$ is pairwise minimum sufficient (PSS, as I would rather call it pairwise smallest sufficient).

I would like to point out that $\underline{C}(x)$ is also PSS. In fact, all the subfields which share the same partition as a PSS subfield are also PSS, because in the discrete case, if two subfields have a same partition they are equivalent in terms

of the partial order (II) defined in [1]. Hence there are great many PSS in this case: All the separating subfields are PSS. Here, of course, a subfield $\underline{B}$ is called separating if for any two points in X there is a set in $\underline{B}$ which contains one and only one of them. (Here the example ends).

To single out one "smallest" subfield, one other concept seems to me more convenient: The smallest subfield with pairwise sufficiency and containment of supports (SPSC). This is defined as the smallest (minimum, in terms of [1]) one wrt. the partial order (I), among all the pairwise sufficient subfields which contain supports of all P in $\underline{P}$ relative to $\underline{P}$ itself, according to Definition 1 of [1]. In the foregoing example, $\underline{C}$ is SPSC, as well as the particular PSS written $\underline{A}_{pms}$ constructed in Theorem 1 of [1]. This coincidence is not an accident, as the following theorem shows.

[Theorem 1] Suppose for each P in $\underline{P}$ the supports relative to $\underline{P}$ exists. Then $\underline{A}_{pms}$ is SPSC.

[Outline of the proof] Assume that $\underline{B}$ is pairwise sufficient. then the functions $\Psi_{P_1 P_2}$, defined in Theorem 1 of [1] must be $\underline{B}$ -measurable. If we assume that $\underline{B}$ contains the supports of all P in $\underline{P}$, then the functions I_P which appear in the same Theorem are $\underline{B}$ -measurable. Hence $\underline{B}$ includes $\underline{A}_{pms}$. (End)

Thus the existence of SPSC and its being PSS is proved under the same generality as the existence of the latter has thus far been proved. A possible criticism of this concept might

be that while pairwise sufficiency is a "pairwise concept", that is, one preserved by the equivalence in terms of (II), the concept of support is not, and the definition of SPSC has a sort of inconsistency in its combining these two concepts belonging to two different categories. On the other hand, SPSC emerges quite naturally from the following theorem which holds under a slightly more general conditions than what is called weak domination.

[Generalized Neyman Factorization Theorem] (Yamada and Morimoto) A subfield $\underline{B}$ is pairwise sufficient and contains the supports of all P in $\underline{P}$ if and only if every P in $\underline{P}$ has a $\underline{B}$ -measurable density wrt. a pivotal measure.

Under the same generality, the existence of SPSC immediately follows: The subfield generated by all the versions of the densities of all P in $\underline{P}$ wrt. a pivotal measure is SPSC.

I would not state explicitly the conditions for the theorem or the definition of a pivotal measure here, because Neyman factorization is not the main subject here, and the existence of SPSC has been proved in [1] under a more general condition, that is, the existence of supports.

2. Characterization of minimal pairwise sufficient subfields in the discrete case.

I state in this section recent results by Namba[2]. I again take up Example 1, although the results are easily rephrased for the discrete case in general. Theorem 4 of [1] is now specialized to: A subfield is pairwise sufficient if and only if it is

separating. Thus our problem is to decide whether a given separating subfield is a minimal one of that kind or not.

Suppose that $\underline{B}$ is separating and let $\underline{F} = \{F_i; i \in I\}$ be a family of sets which generates $\underline{B}$. Define 2^I to be the space of all functions on I to $\{0,1\}$. Here I is the set of indices attached to the sets in $\underline{F}$. Points in this space are written $y = (y(i); i \in I)$, $z = (z(i); i \in I)$ etc.

We define a mapping f on X onto a subset Y of 2^I as

follows: A point x in X is mapped to a point $y = f(x)$

which satisfies $y(i) = 1$ if x belongs to F_i and $y(i) = 0$

otherwise. By f , $\underline{F}$ is carried to the family of all such sets that are written as $\{y; y(i) = 1\}$ for some i in I .

And $\underline{B}$ is carried by f to the sigma-field generated by it.

We conveniently denote them by $\underline{F}$ and $\underline{B}$ again. A neighbour-

hood of y in Y is defined to be a set $N(y; K)$, where K is any

countable subset of I , which is the totality of those points

z in Y which satisfies $z(i) = y(i)$ for all i in K . The

neighbourhoods, when y ranges over Y and K assumes to be all

countable subsets of I , give rise to a topology on Y . Y is

called ω_1 -compact wrt. this topology if the following condition

is satisfied: Assume that to each y in Y there corresponds

a neighbourhood $N(y; K(y))$. Then one can choose a countable

number of points $y_0, y_1, \dots, y_k, \dots$ such that $\bigcup_{k=0}^{\infty} N(y; K(y_k)) = Y$.

We now state a theorem of Namba[2] which gives a complete characterization of minimality of a separating subfield.

[Theorem 2] $\underline{B}$ is minimal separating if and only if $\underline{Y}$ is ω_1 -compact.

Let us see how this theorem works with the subfields given in Example 1.

[Example 2] Under the framework of Example 1, take the following generators $\underline{F}$ and $\underline{F}(x)$ of $\underline{C}$ and $\underline{C}(x)$, respectively:

$\underline{F}$ = all the singletons in X .

$\underline{F}(x)$ = all the singletons except x .

Corresponding sets of indices I for these generators are X and $X - \{x\}$, respectively. By the correspondence f , the space X is mapped to one of the following two spaces, depending on cases:

Y = all y such that $y(i) = 1$ for one single i in $I = X$.

$Y(x)$ = all y such that $y(i) = 1$ for one single i in $I = X - \{x\}$, and 0.

Here, 0 denotes the point of 2^I such that $0(i) = 0$ for all i in I . Notice that $f(x) = 0$.

The ω_1 -compactness of $Y(x)$ is proved as follows: Suppose that $N(y; K(y))$ corresponds to y , for each y in Y . If a point y does not belong to $N(0, K(0))$, the neighbourhood corresponding to 0, then there exists i in $K(0)$ such that $y(i) = 1$. As $K(0)$ is countable and as each y can assume the value 1 for at most one i , there are a countable number of points which do not belong to $N(0, K(0))$. Take them as $y_1, y_2, \dots, y_k, \dots$ and 0 as y_0 . Then it is clear that the neighbourhoods corresponding these points collectively cover Y .

This proof does not work for Y , because it does not contain 0. On the other hand it is easy to disprove ω_1 -compactness.

There exists an example of a minimal separating subfield which does not contain any singletons:

[Example 3] Let I be an uncountable set of indices i and for each non-negative integer n let Y_n be the set of all functions I to $\{0,1\}$ which assume the value 1 for at most n indices in I . Put $Y = \bigcup_{n=1}^{\infty} Y_n$, the set of all functions assuming the value 1 for a finite number of indices i in I . Let $\underline{F}$ be the family of all sets of the form: $\{y ; y(i) = 1\}$ for some i in I . Define $\underline{B}$ as the subfield generated by $\underline{F}$, which is equal to the totality of the sets B for which there exists a countable subset $K(B)$ of I such that $y \in B$ and $Y(i)=z(i)$ for all i in $K(B)$ imply $y \in B$. Then it is clear that $\underline{B}$ is separating and $\underline{B}$ does not contain any singletons. To prove that it is minimal, we are sufficed to prove the ω_1 -compactness of Y .

The proof is similar in nature to that given in the previous example, except only that we need induction over n .

References

- [1] Ghosh, J.K., Minimality of pairwise sufficient σ -fields, in this volume.
- [2] Namba, K., Representation theorem for minimal σ -algebras, to appear in the Proceedings of the Conference on Logic and Set Theorey in Belgrade, August-September, 1977.