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Infinite Dimensional Combinatorial Topology

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A <u>simplicial complex</u> K means a geometric one, that is, K is a collection of (closed) simplexes in a real vector space, and the <u>underlying space</u> $|K|$ is the union of all simplexes of K with the weak topology determined by the Euclidian topology on simplexes. A <u>subdivision</u> K' of K is a simplicial complex such that each simplex of K' is contained in a simplex of K and every simplex of K is a finite union of simplexes of K' (hence $|K'| = |K|$). Simplicial complexes K and L are said to be <u>combinatorially equivalent</u> if they admit simplicially isomorphic subdivisions. We say that a space X <u>is triangulated by</u> a simplicial complex K (or K <u>triangulates</u> X) if X is homeomorphic to $|K|$.

An $\mathbb{R}^\infty$-<u>manifold</u> is a separable paracompact (topological) manifold modeled on $\mathbb{R}^\infty = \operatorname{dir\,lim} \mathbb{R}^n$. The real line $\mathbb{R}$ is naturally triangulated by the complex

$$R = \{n, [n,n+1] \mid n = 0, \pm 1, \pm 2, \ldots\} .$$

For $n > 1$, $\mathbb{R}^n$ is triangulated by the product complex R^n . Such triangulations give the natural triangulation R^∞ of $\mathbb{R}^\infty$.

By the Triangulation Theorem [2], each $\mathbb{R}^\infty$-manifold M is homeomorphic to $|K| \times \mathbb{R}^\infty$ for some countable (locally finite) simplicial complex K. Thus each $\mathbb{R}^\infty$-manifold is triangulated by a countable simplicial complex. Then the following two problems arise:

(A) What simplicial compelex is an $\mathbb{R}^\infty$-manifold?

(B) Are two simplicial complexes triangulating same $\mathbb{R}^\infty$-manifold combinatorially equivalent?

We can easily answer to (A) as follows : A countable simplicial complex is an $\mathbb{R}^\infty$-manifold if and only if the star of each vertex is homeomorphic to $\mathbb{R}^\infty$. In fact, the "if" part is obvious and the "only if" part follows from [6, Proposition 6-3] and the Classification Theorem [2].

By Δ^∞, we denote the countably infinite full complex, namely the countably infinite simplicial complex each whose finite vertices span a simplex of Δ^∞. From [4, Proposition] (cf [5]) Δ^∞ triangulates $\mathbb{R}^\infty$. Furthermore, using the following characterization, it is easily seen that Δ^∞ is combinatorially equivalent to the natural triangulation R^∞ of $\mathbb{R}^\infty$.

CHARACTERIZATION of Δ^∞ [8, 3-1] : A countable simplicial complex K is combinatorially equivalent to Δ^∞ if and only if each p.l. embedding $f : P \to |K|$ from a closed subpolyhedron of a compact (Euclidian) polyhedron Q extends to a p.l. embedding $\tilde{f} : Q \to |K|$.

A combinatorial ∞-manifold is a countable simplicial complex such that the star of each vertex is combinatorially equivalent to Δ^∞. In this definition, the term "star" can be replaced by "link" [8, Proposition 4-1]. Using the standard pseudo-radial projection srguments, it can be shown that a simplicial complex is a combinatorial ∞-manifold if and only if so is its subdivision. Using the above characterization of Δ^∞, we can prove that if $K = \cup_{n=1}^{\infty} K_n$ where each K_n is a combinatorial k_n-submanifold of K_{n+1} with $k_n < k_{n+1}$ then K is a combinatorial ∞-manifold [8, Corollary 3-2]. Combining this with the Open Embedding Theorem, it follows

COMBINATORIAL TRIANGULATION THEOREM for $\mathbb{R}^\infty$-manifolds [8, 3-3] : Each $\mathbb{R}^\infty$-manifold is triangulated by a combinatorial ∞-manifold.

For combinatorial ∞-manifolds, we can answer affirmatively to (B), that is, we can prove the following:

HAUPTVERMUTUNG for combinatorial ∞-manifolds [8, 2-4] : Any two homeomorphic combinatorial ∞-manifolds are combinatorially equivalent.

We have a characterization of combinatorial ∞-manifolds [8, 3-4] similar to one of $\mathbb{R}^\infty$-manifolds [5, Theorem 1-3]. Using this characterization, we can see that for each countable simplicial complex K, the product complex $K \times \Delta^\infty$ is a combinatorial ∞-manifold [8, Theorem 3-6]. Thus the following is proved:

STABLE HAUPTVERMUTUNG for simplicial complexes [8, 3-8] : <u>For any two homeomorphic countable simplicial complexes</u> K <u>and</u> L , <u>the product complexes</u> $K \times \Delta^{\infty}$ <u>and</u> $L \times \Delta^{\infty}$ <u>are combinatorially equivalent</u>.

The next problem remaines open:

(C) <u>Can an</u> $\mathbb{R}^{\infty}$-<u>manifold be triangulated by a simplicial complex which is not a combinatorial</u> ∞-<u>manifold</u>? <u>Can</u> $\mathbb{R}^{\infty}$ <u>be triangulated by a simplicial complex which is not combinatorially equivalent to</u> Δ^{∞} ?

In finite dimensional case, R.D. Edwards [1] showed that the 5-sphere has non-combinatorial triangulation K as a corollary of his double suspension theorem. For this complex K , each n-fold suspension $\Sigma^n K$ is a non-combinatorial triangulation of the (n+5)-sphere S^{n+5} . However the infinite suspension $\Sigma^{\infty} K$ = dir lim $\Sigma^n K$ is a combinatorial ∞-manifold by [8, Corollary 3-11].

R.E. Heisey [3] defined $\mathbb{R}^{\infty}$-<u>piecewise linear</u> ($\mathbb{R}^{\infty}$-p.l.) <u>maps</u> between open subsets of $\mathbb{R}^{\infty}$ and introduced the notion of <u>piecewise linear</u> $\mathbb{R}^{\infty}$-<u>structure</u> (p.l. $\mathbb{R}^{\infty}$-<u>structure</u>) as in Differential Topology. A <u>piecewise linear</u> $\mathbb{R}^{\infty}$-<u>manifold</u> (p.l. $\mathbb{R}^{\infty}$-<u>manifold</u>) is a separable paracompact space together with a p.l. $\mathbb{R}^{\infty}$-structure (His p.l. $\mathbb{R}^{\infty}$-manifolds are essentially same as infinite poly-manifolds defined in [9, Ch.2, p.10].) Then $\mathbb{R}^{\infty}$-p.l. <u>maps</u> and $\mathbb{R}^{\infty}$-p.l. <u>isomorphisms</u> between two p.l. $\mathbb{R}^{\infty}$-manifolds are defined similarly as differential maps and diffeomorphisms between two differential manifolds. Since $\mathbb{R}^{\infty} \times \mathbb{R}^{\infty}$ is identified with $\mathbb{R}^{\infty}$

by the natural linear homeomorphism, p.l. $\mathbb{R}^\infty$-submanifolds (of infinite codimensions) can be defined as locally flat submanifolds. He established the following:

$\mathbb{R}^\infty$-p.l. EMBEDDING THEOREM [3] : For each p.l. $\mathbb{R}^\infty$-manifold M , there exsists an $\mathbb{R}^\infty$-p.l. isomorphism $f : M \to N$, N a closed $\mathbb{R}^\infty$-submanifold of $\mathbb{R}^\infty$.

He also showed in [3] that a p.l. $\mathbb{R}^\infty$-submanifold N of $\mathbb{R}^\infty$ is an $\mathbb{R}^\infty$-polyhedron, that is, for each compact polyhedron C in $\mathbb{R}^\infty$, $C \cap N$ is a polyhedron in the usual sense.

The author [7] proved the following:

$\mathbb{R}^\infty$-p.l. HAUPTVERMUTUNG : Two p.l. $\mathbb{R}^\infty$-manifolds are $\mathbb{R}^\infty$-p.l. isomorphic if and only if they are homeomorphic.

By the Open Embedding Theorem [2], each $\mathbb{R}^\infty$-manifold can be embedded as an open subset of $\mathbb{R}^\infty$. Since open subsets of $\mathbb{R}^\infty$ have the p.l. $\mathbb{R}^\infty$-structure inherited from $\mathbb{R}^\infty$, we have the following:

UNQUENESS of p.l. $\mathbb{R}^\infty$-structure : Each $\mathbb{R}^\infty$-manifold has a unique p.l. $\mathbb{R}^\infty$-structure.

Let K be a combinatorial ∞-manifold. For each vertex v of K , there is a homeomorphism $f_v : |St(v,K)| \to \mathbb{R}^\infty = |R^\infty|$ which is a simplicial isomorphism with respect to subdivisions of $St(v,K)$ and R^∞ . Put $U_v = \operatorname{int} |St(v,K)|$ and $\phi_v = f_v|U_v$. Then $\{(U_v,\phi_v) \mid v \in K^0\}$ is clearly a p.l. $\mathbb{R}^\infty$-structure of $|K|$. Thus any combinatorial ∞-manifold has the natural p.l. $\mathbb{R}^\infty$-structure. Conversely, any p.l. $\mathbb{R}^\infty$-manifold is $\mathbb{R}^\infty$-p.l. isomorphic to a combinatorial ∞-manifold with the natural p.l. $\mathbb{R}^\infty$-structure because it is $\mathbb{R}^\infty$-p.l. isomorphic to an open subset of $\mathbb{R}^\infty$ by the Open Embedding Theorem and the above $\mathbb{R}^\infty$-p.l. Hauptvermutung. Let $f : |K| \to |L|$ be a homeomorphism (or a map) between combinatorial ∞-manifolds. Then it follows that f is an $\mathbb{R}^\infty$-p.l. isomorphism (or an $\mathbb{R}^\infty$-p.l. map) with respect to the natural p.l. $\mathbb{R}^\infty$-structures if and only if it is p.l. on each finite subcomplex. Especially if f is simplicial with respect to subdivisions then f is $\mathbb{R}^\infty$-p.l. However even if f is an $\mathbb{R}^\infty$-p.l. isomorphism with respect to the natural p.l. $\mathbb{R}^\infty$-structures, f need not be simplicial with respect to any subdivisions of the wholes.

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