

ON THE HERZ-TYPE SPACES WITH POWER WEIGHTS AND THE BOUNDEDNESS OF SOME SUBLINEAR OPERATORS

日本大学・経済学部 松岡 勝 男 (KATSUO MATSUOKA)
COLLEGE OF ECONOMICS OF NIHON UNIVERSITY

1. INTRODUCTION

First, we state the notation which is used throughout this paper. For a measurable set $E \subset \mathbb{R}^n$, we denote the Lebesgue measure of E by $|E|$ and the characteristic function of the set E by χ_E . Also, let for $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $P_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{P_k}$. And let for $k \in \mathbb{N}$, $\tilde{P}_k = P_k$, $\tilde{\chi}_k = \chi_{\tilde{P}_k}$ and $\tilde{P}_0 = B_0$, $\tilde{\chi}_0 = \chi_{\tilde{P}_0}$. Further, we denote the open ball in $\mathbb{R}^n$, having center 0 and radius $R > 0$, by $B(0, R)$.

Now, we define the homogeneous and non-homogeneous Herz spaces (see [LiY]).

Definition 1. Let $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$.

(a) The homogeneous Herz space $\dot{K}_{p,r}^\alpha(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,

$$\dot{K}_{p,r}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p,r}^\alpha} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha r} \|f\chi_k\|_{L^p}^r \right)^{1/r} < \infty \right\};$$

$$\dot{K}_{p,\infty}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p,\infty}^\alpha} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|f\chi_k\|_{L^p} < \infty \right\}.$$

(b) The non-homogeneous Herz space $K_{p,r}^\alpha(\mathbb{R}^n)$ is defined by, for $0 < r < \infty$,

$$K_{p,r}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{K_{p,r}^\alpha} = \left(\sum_{k=0}^{\infty} 2^{k\alpha r} \|f\tilde{\chi}_k\|_{L^p}^r \right)^{1/r} < \infty \right\};$$

$$K_{p,\infty}^\alpha(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{K_{p,\infty}^\alpha} = \sup_{k \geq 0} 2^{k\alpha} \|f\tilde{\chi}_k\|_{L^p} < \infty \right\}.$$

Here, throughout this talk, there are similar definitions and results for the non-homogeneous case as those for the homogeneous case. But, for simplicity, we only state the definitions and results for the homogeneous case.

Next, we recall the definition of the Hardy-Littlewood maximal operator M : that is, for any measurable function f on $\mathbb{R}^n$,

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \quad (x \in \mathbb{R}^n),$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^n$ containing x .

Moreover, we define the standard singular integral operator T .

Definition 2. We say that T is a standard singular integral operator, if there exists a function K which satisfies the following conditions:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

exists almost everywhere, where $f \in L^2(\mathbb{R}^n)$;

$$|K(x)| \leq \frac{C_K}{|x|^n} \quad \text{and} \quad |\nabla K(x)| \leq \frac{C_K}{|x|^{n+1}}, \quad x \neq 0;$$

$$\int_{\epsilon < |x| < N} K(x)dx = 0 \quad \text{for all} \quad 0 < \epsilon < N.$$

Then, the following strong-type estimates of the boundedness of the Hardy-Littlewood maximal operator M and a standard singular integral operator T on $L^p(\mathbb{R}^n)$ are well-known:

$$M : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n),$$

where $1 < p \leq \infty$;

$$T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n),$$

where $1 < p < \infty$.

Furthermore, let S be a sublinear operator satisfying for any integrable function f with a compact support,

$$(*) \quad |Sf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \text{supp } f,$$

where $c > 0$ is independent of f and x .

We remark that $(*)$ is satisfied by several operators in harmonic analysis, including the Hardy-Littlewood maximal operator M and a standard singular integral operator T .

Then, the following theorem was shown.

Theorem 3 ([LiY]). *Let $1 < p < \infty$, $0 < r \leq \infty$ and $-n/p < \alpha < n/p'$, where $1/p + 1/p' = 1$, and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then*

$$T : \dot{K}_{p,r}^\alpha(\mathbb{R}^n) \rightarrow \dot{K}_{p,r}^\alpha(\mathbb{R}^n).$$

Second, we define the weighted Herz spaces $\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ (see [K], [LuY] and [LYY]).

Now, for a nonnegative locally integrable function on $\mathbb{R}^n$, i.e. a weight (or a weight function), w , we write $w(E) = \int_E w(x)dx$ ($E \subset \mathbb{R}^n$) and define

$$L^p(w)(\mathbb{R}^n) = \left\{ f : \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty \right\}.$$

Definition 4. For $0 < \alpha < \infty$, $1 \leq p < \infty$, $0 < r \leq \infty$ and the weights w_1 and w_2 ,

$$\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(w_2)(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p,r}^\alpha(w_1, w_2)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_{p,r}^\alpha(w_1, w_2)} = \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha r/n} \|f \chi_k\|_{L^p(w_2)}^r \right\}^{1/r}.$$

In particular, when $w_1 = w_2 = w$, we put

$$\dot{K}_{p,r}^\alpha(w)(\mathbb{R}^n) = \dot{K}_{p,r}^\alpha(w, w)(\mathbb{R}^n).$$

Also, the following theorem was proved.

Theorem 5 ([LiY]). Let $1 < p < \infty$, $0 < r < \infty$, $0 < \alpha < n/p'$, where $1/p + 1/p' = 1$, $w_1(x) = 1$, $w_2(x) = |x|^{-a}$ ($0 \leq a < n$), and let T be a sublinear operator satisfying (*). If T is bounded on $L^p(\mathbb{R}^n)$, then

$$T : \dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n) \rightarrow \dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n).$$

In this talk, we will introduce some weighted Herz-type space, $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which is a weighted Herz space $\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ with the critical index $\alpha = n/p'$, where $1/p + 1/p' = 1$, and show the boundedness of the sublinear operator T satisfying (*) at the critical index $\alpha = n/p'$.

2. THE BOUNDEDNESS ON SOME WEIGHTED HERZ-TYPE SPACES

First, we define the particular cases of the Herz spaces $\dot{K}_{p,r}^\alpha(\mathbb{R}^n)$ and the weighted Herz spaces $\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ (see [CL], [FW], [G], [GH], [LS₁], [LS₂] and [M]).

Definition 6. For $1 \leq p < \infty$

$$\begin{aligned} \dot{A}^p(\mathbb{R}^n) &= \dot{K}_{p,1}^{n/p'}(\mathbb{R}^n) \\ &= \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{A}^p} = \sum_{k=-\infty}^{\infty} 2^{kn/p'} \|f \chi_k\|_p < \infty \right\}, \end{aligned}$$

where $1/p + 1/p' = 1$.

Definition 7. Let w_1 and w_2 be the weights. For $1 \leq p < \infty$

$$\begin{aligned} \dot{A}^p(w_1, w_2)(\mathbb{R}^n) &= \dot{K}_{p,1}^{n/p'}(w_1, w_2)(\mathbb{R}^n) \\ &= \left\{ f \in L_{loc}^p(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{A}^p(w_1, w_2)} < \infty \right\}, \end{aligned}$$

where $1/p + 1/p' = 1$ and

$$\|f\|_{\dot{A}^p(w_1, w_2)} = \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1/p'} \|f \chi_k\|_{L^p(w_2)}.$$

In particular, when $w_1 = w_2 = w$, we put

$$\dot{A}^p(w)(\mathbb{R}^n) = \dot{A}^p(w, w)(\mathbb{R}^n).$$

Next, we define the central $(\alpha, p; w_1, w_2)$ -block, and observe the block decomposition of $\dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$ (see [LS₁], [LS₂] and [LuY]).

Definition 8. Let $0 < \alpha < \infty$ and $1 \leq p < \infty$, and let w_1, w_2 be a weights. Then, we state that a measurable function $b(x)$ is a central $(\alpha, p; w_1, w_2)$ -block, if the support of b is contained in a ball $B = B(0, R)$ ($R > 0$), and so that

$$\|b\|_{L^p(w_2)} \leq [w_1(B)]^{-\alpha/n}.$$

Theorem 9. Let $0 < \alpha < \infty$, $1 \leq p < \infty$, and $0 < r < \infty$, and let $w_1 \in A_1$ and w_2 be a weight. Then, the following are equivalent:

- (i) $f \in \dot{K}_{p,r}^\alpha(w_1, w_2)(\mathbb{R}^n)$;
- (ii) $f = \sum_{k=-\infty}^{\infty} \lambda_k b_k$ where the b_k 's are central $(\alpha, p; w_1, w_2)$ -blocks and $\sum_{k=-\infty}^{\infty} |\lambda_k|^r < \infty$.

Besides,

$$\|f\|_{\dot{K}_{p,r}^\alpha} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^r \right)^{1/r},$$

where the infimum is taken over all such decompositions.

Then, using the block decomposition of $\dot{A}^p(w)(\mathbb{R}^n)$, the boundedness of the sublinear operator satisfying $(*)$ on $\dot{A}^p(w)(\mathbb{R}^n)$ was shown.

Theorem 10 ([LS₁] and [LS₂]). Let $1 < p < \infty$, $w(x) = |x|^{-a}$ ($0 < a < n$), and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then

$$T : \dot{K}_{p,1}^{n/p'}(w)(\mathbb{R}^n) \rightarrow \dot{K}_{p,1}^{n/p'}(w)(\mathbb{R}^n),$$

where $1/p + 1/p' = 1$, i.e.

$$T : \dot{A}^p(w)(\mathbb{R}^n) \rightarrow \dot{A}^p(w)(\mathbb{R}^n).$$

Now, we are in a position to show the result of our purpose, i.e. the boundedness of the sublinear operator satisfying $(*)$ on $\dot{A}^p(w_1, w_2)(\mathbb{R}^n)$, which extends the above results.

Theorem 11. Let $1 < p < \infty$, $w_i(x) = |x|^{-a_i}$ such that $0 < a_i < n$ ($i = 1, 2$), and let T be a sublinear operator satisfying $(*)$. If T is bounded on $L^p(\mathbb{R}^n)$, then

$$T : \dot{K}_{p,1}^{n/p'}(w_1, w_2)(\mathbb{R}^n) \rightarrow \dot{K}_{p,1}^{n/p'}(w_1, w_2)(\mathbb{R}^n),$$

where $1/p + 1/p' = 1$, i.e.

$$T : \dot{A}^p(w_1, w_2)(\mathbb{R}^n) \rightarrow \dot{A}^p(w_1, w_2)(\mathbb{R}^n).$$

Proof. The proof of this theorem is similar to that of Theorem 2 of [LS₂].

By Theorem 9, it suffices to show that for any central $(n/p', p; w_1, w_2)$ -block b ,

$$\|Tb\|_{\dot{A}^p(w_1, w_2)} \leq C,$$

where C is independent of b .

Now, let $B = B(0, R)$ be the supporting ball of b . Then, since we can choose a $j \in \mathbb{N}$ such that $2^{j-2} < R \leq 2^{j-1}$. Therefore,

$$\begin{aligned} \|Tb\|_{\dot{A}^p(w_1, w_2)} &= \left(\sum_{k \leq j} + \sum_{k > j} \right) [w_1(B_k)]^{1/p'} \|(Tb)\chi_k\|_{L^p(w_2)} \\ &= S_1 + S_2, \text{ say.} \end{aligned}$$

First, we estimate S_1 . By the assumption, it follows that T maps $L^p(w_2)(\mathbb{R}^n)$ into $L^p(w_2)(\mathbb{R}^n)$ (see [SW]). Consequently,

$$\begin{aligned} \|(Tb)\chi_k\|_{L^p(w_2)} &\leq C \left(\int_B |b(x)|^p w_2(x) dx \right)^{1/p} \\ &\leq C [w_1(B_j)]^{1/p'}. \end{aligned}$$

Thus,

$$S_1 \leq C \sum_{k \leq j} \left[\frac{w_1(B_k)}{w_1(B_j)} \right]^{1/p'} \leq C \sum_{k \leq j} 2^{(k-j)(n-a_1)/p'} < \infty.$$

Next, in order to estimate S_2 , note that if $x \in P_k$, $y \in B$ and $j < k$, then $|x - y| \sim |x|$. Hence, using the size condition of T , it follows that

$$\begin{aligned} \|(Tb)\chi_k\|_{L^p(w_2)}^p &\leq C \int_{P_k} \left(\int_B \frac{|b(y)|}{|x - y|^n} dy \right)^p w_2(x) dx \\ &\leq C \int_{P_k} \frac{1}{|x|^{np}} \left(\int_B |b(y)|^p dy \right) |B|^{p-1} w_2(x) dx \\ &\leq C \int_{P_k} \frac{1}{|x|^{np}} \frac{1}{\text{essinf}_{y \in B} w_2(y)} \left(\int_B |b(y)|^p w_2(y) dy \right) |B|^{p-1} w_2(x) dx. \end{aligned}$$

Since $w_2 \in A_1$,

$$\frac{w_2(B)}{|B|} \leq C \text{essinf}_{y \in B} w_2(y),$$

and therefore we have

$$\|(Tb)\chi_k\|_{L^p(w_2)} \leq C [w_1(B)]^{-1/p'} [w_2(B)]^{-1/p} |B| \left(\int_{P_k} \frac{1}{|x|^{np}} w_2(x) dx \right)^{1/p}.$$

Thus, by the assumption,

$$\begin{aligned} S_2 &\leq C \sum_{k > j} \left[\frac{w_1(B_k)}{w_1(B)} \right]^{1/p'} \left[\frac{w_2(B_k)}{w_2(B)} \right]^{1/p} |B| 2^{-kn} \\ &\leq C \sum_{k > j} 2^{(k-j)(n-a_1)/p'} 2^{(k-j)(n-a_2)/p} 2^{(j-k)n} \\ &= C \sum_{k > j} 2^{(j-k)(a_1/p' + a_2/p)} \\ &< \infty. \end{aligned}$$

□

REFERENCES

- [CL] Y. Chen and K. Lau, Some new classes of Hardy spaces, *J. Func. Anal.*, **84** (1989), 255–278.
- [FW] H. G. Feichtinger and F. Weisz, Herz spaces and summability of Fourier transforms, *Math. Nachr.*, **281** (2008), 309–324.
- [G] J. García-Cuerva, Hardy spaces and Beurling algebras, *J. London Math. Soc. (2)*, **39** (1989), 499–513.
- [GH] J. García-Cuerva and M.J. L. Herrero, A theory of Hardy spaces associated to the Herz spaces, *Proc. London Math. Soc.*, **69** (1994), 605–628.
- [K] Y. Komori, Weak type estimates for Calderon-Zygmund operators on Herz spaces at critical indexes, *Math. Nachr.*, **259** (2003), 42–50.
- [LiY] S. Z. Li and D. C. Yang, Boundedness of some sublinear operators on Herz spaces, *Illinois J. Math.*, **40** (1996), 484–501.
- [LS₁] S. Z. Lu and F. Soria, Sublinear operators on the Beurling algebras with power weights, *J. of Beijing Normal Univ. (Natur. Sci.)*, **30** (1994), 170–175.
- [LS₂] S. Z. Lu and F. Soria, On the Herz spaces with power weights, in *Fourier Analysis and Partial Differential Equations* (CRC Press 1995), 227–236.
- [LuY] S. Z. Lu and D. C. Yang, The decomposition of weighted Herz space on $\mathbb{R}^n$ and its applications, *Science in China (Series A)*, **38** (1995), 147–158.
- [LYY] S. Z. Lu, K. Yabuta and D. C. Yang, Boundedness of some sublinear operators in weighted Herz-type spaces, *Kodai Math. J.*, **23** (2000), 391–410.
- [M] K. Matsuoka, On some weighted Herz spaces and the Hardy-Littlewood maximal operator, *Proceedings of International Symposium on Banach and Function Spaces*, Kitakyushu, Japan, September 14–17, 2006.
- [SW] F. Soria and G. Weiss, A remark on singular integrals and power weights, *Indiana Univ. Math. J.*, **43** (1994), 187–204.