

# Identification problems for nonlinear perturbed sine-Gordon equations

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## 1. Introduction

In Ha and Nakagiri [9] we studied the identification problems of the damped sine-Gordon equation

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y = \delta f, \quad (1.1)$$

where  $\alpha, \beta, \gamma, \delta$  are unknown constant parameters. In [9] the existence and the necessary conditions of optimality for the optimal parameter  $q^* = (\alpha, \beta^*, \gamma^*, \delta^*)$  is established for the appropriate cost without including the cost of parameters  $q = (\alpha, \beta, \gamma, \delta)$ .

Several types of perturbed sine-Gordon equations differently from (1.1) are proposed to describe the dynamics of the phase difference in the Josephson junctions in various situations. We refer to, e.g. [1], [3]-[6], [11]. In Kivshar and Malomed [5] the perturbed equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \sin y = \epsilon \frac{\partial^2}{\partial x^2} \left( \frac{\partial y}{\partial t} \right) \quad (1.2)$$

is proposed by taking into account of losses or dissipation due to the current along a dielectric barrier in Josephson junctions. The nonlinear perturbation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \sin y = \epsilon \sin 2y \quad (1.3)$$

is also proposed by Kivshar and Malomed [4] to determine the inelastic interaction of a fast kink and a weakly bounded breather. The additional nonlinear perturbations  $\sum_{i=1}^L \epsilon_i \sin \kappa_i y$  are possible in (1.3).

Recently in Ramos [10] the numerical analysis of perturbed sine-Gordon equation of the generalized form

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \sin y = \epsilon_1 \frac{\partial y}{\partial t} + \epsilon_2 y + \epsilon_3 \sin 2y + \epsilon_4 \frac{\partial^2}{\partial x^2} \left( \frac{\partial y}{\partial t} \right) \quad (1.4)$$

subject to homogeneous Neumann boundary conditions in the finite line is studied rather completely based on the implicit finite difference methods. There are various interesting observations of solutions in [10] according to the differences of perturbations for  $\epsilon_i$  terms. It is an important physical problem to identify such constant parameters  $\epsilon_i$ .

In this paper we study the problems of identification of a general equation described by

$$\frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial \Delta y}{\partial t} - \beta \Delta y + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = \nu f \quad (1.5)$$

in  $R^n$ , where  $\alpha, \beta, \gamma_i, \delta, \kappa_i$  and  $\nu$  are constants and  $f$  is a prescribed source function. In our identification problems all parameters  $\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu$  are assumed to be unknown but the number  $L$  is prescribed. The objective of this paper is to extend the results in [9] to the equations (1.5) under the homogeneous Neumann boundary conditions in  $n$ -dimensions.

## 2. Perturbed sine-Gordon equations

Let  $\Omega$  be an open bounded set of  $R^n$  with a piecewise smooth boundary  $\Gamma = \partial\Omega$ . Let  $Q = (0, T) \times \Omega$  and  $\Sigma = (0, T) \times \Gamma$ . We consider the Kivshar-Malomed type perturbed sine-Gordon equations described by

$$\frac{\partial^2 y}{\partial t^2} - \alpha \frac{\partial \Delta y}{\partial t} - \beta \Delta y + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = f \quad \text{in } Q, \quad (2.1)$$

where  $\alpha, \beta > 0$ ,  $\delta, \gamma_i, \kappa_i \in R, i = 1, \dots, L, \Delta$  is a Laplacian in  $R^n$  and  $f$  is a given function. The boundary condition is the homogeneous Neumann condition

$$\frac{\partial y}{\partial n} = 0 \quad \text{on } \Sigma. \quad (2.2)$$

The initial values are given by

$$y(0, x) = y_0(x) \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial y}{\partial t}(0, x) = y_1(x) \quad \text{in } \Omega. \quad (2.3)$$

First we introduce two Hilbert spaces  $H$  and  $V$  by  $H = L^2(\Omega)$  and  $V = H^1(\Omega)$ , respectively. We endow the space  $H = L^2(\Omega)$  with the inner product and norm

$$(\psi, \phi) = \int_{\Omega} \psi(x) \phi(x) dx, \quad |\psi| = (\psi, \psi)^{1/2}, \quad \forall \phi, \psi \in L^2(\Omega). \quad (2.4)$$

For  $\phi, \psi \in V = H^1(\Omega)$  we define

$$(\psi, \phi) = \sum_{i=1}^n \int_{\Omega} \frac{\partial}{\partial x_i} \psi(x) \frac{\partial}{\partial x_i} \phi(x) dx. \quad (2.5)$$

The duality pairing between  $V$  and  $V'$  is denoted by  $\langle \cdot, \cdot \rangle$ . The inner product and norm of  $V = H^1(\Omega)$  are defined by

$$(\psi, \phi)_1 = (\psi, \phi) + (\psi, \phi), \quad \|\psi\| = (\psi, \psi)_1^{1/2}, \quad \forall \phi, \psi \in H^1(\Omega). \quad (2.6)$$

Then the pair  $(V, H)$  is a Gelfand triple space with a notation,  $V \hookrightarrow H \equiv H' \hookrightarrow V'$ , which means that embeddings  $V \subset H$  and  $H \subset V'$  are continuous, dense and compact. The norm of the dual space  $V'$  is denoted by  $\|\cdot\|_*$ .

Now we introduce the bilinear form

$$a(\phi, \varphi) = \int_{\Omega} \nabla \phi \cdot \nabla \varphi dx = \langle \phi, \varphi \rangle, \quad \forall \phi, \varphi \in H^1(\Omega). \quad (2.7)$$

Then we can define the bounded operator  $A \in \mathcal{L}(V, V')$  through (2.7). The operator  $A$  is an isomorphism from  $V$  onto  $V'$  and it is also considered as a self-adjoint operator in  $H = L^2(\Omega)$  with dense domain  $\mathcal{D}(A)$  in  $V$  and in  $H$ ,

$$\mathcal{D}(A) = \{\phi \in V : A\phi \in H\} = \{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma\}.$$

Also we define the sine function for  $z \in H = L^2(\Omega)$  by

$$(\sin z)(x) = \sin z(x) \text{ for a.e. } x \in \Omega.$$

Using the operator  $A$  and the sine function  $\sin y$ , the problem (2.1), (2.2), (2.3), is converted to the following Cauchy problem in  $H$ :

$$\begin{cases} \frac{d^2 y(t)}{dt^2} + \alpha A \frac{dy(t)}{dt} + \beta A y(t) + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = f(t), & t \in (0, T), \\ y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1. \end{cases} \quad (2.8)$$

The solution space should be introduced in this perturbed case is defined by

$$W_V(0, T) = \{g | g \in L^2(0, T; V), g' \in L^2(0, T; V), g'' \in L^2(0, T; V')\}$$

with inner product

$$(f, g)_{W_V(0, T)} = \int_0^T (\langle f(t), g(t) \rangle + \langle f'(t), g'(t) \rangle + \langle f''(t), g''(t) \rangle_{V'}) dt,$$

where  $(\cdot, \cdot)_{V'}$  is the inner product of  $V'$ . We denote by  $\mathcal{D}'(0, T)$  the space of distributions on  $(0, T)$ . The definition of weak solutions of the problem (2.8) is as follows.

**Defintion 2.1.** A function  $y$  is said to be a weak solution of (2.8) if  $y \in W_V(0, T)$  and  $y$  satisfies

$$\begin{aligned} \langle y''(\cdot), \phi \rangle + \langle \alpha y'(\cdot), \phi \rangle + \langle \beta y(\cdot), \phi \rangle + \sum_{i=1}^L \langle \gamma_i \sin \kappa_i y(\cdot), \phi \rangle + \langle \delta y(\cdot), \phi \rangle &= \langle f(\cdot), \phi \rangle \\ \text{for all } \phi \in V \text{ in the sense of } \mathcal{D}'(0, T), \\ y(0) = y_0, \quad y'(0) = y_1. \end{aligned}$$

For the existence, uniqueness and regularity of weak solutions for (2.8), we can prove the following theorem. For a proof, see Ha and Nakagiri [8].

**Theorem 2.1.** Let  $\alpha, \beta > 0$ ,  $\delta, \gamma_i, \kappa_i \in \mathbf{R}, i = 1, \dots, L$  and  $f, y_0, y_1$  be given satisfying

$$f \in L^2(0, T; V'), \quad y_0 \in H^1(\Omega), \quad y_1 \in L^2(\Omega). \quad (2.9)$$

Then the problem (2.8) has a unique weak solution  $y$  in  $W_V(0, T)$ . The solution  $y$  has the regularity

$$y \in C([0, T]; H^1(\Omega)), \quad y' \in C([0, T]; L^2(\Omega)). \quad (2.10)$$

### 3. Identification of constant parameters

In this section we study the identification problems for perturbed sine-Gordon equations described by

$$\begin{cases} y'' + (\alpha_0 + \alpha^2)Ay' + (\beta_0 + \beta^2)Ay + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = \nu f & \text{in } (0, T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \quad (3.1)$$

where  $\alpha_0 > 0$  and  $\beta_0 > 0$  are fixed. In (3.1) we multiply the constant  $\delta$  to the forcing term  $f$  and replace the diffusion parameters  $\alpha$  to  $\alpha_0 + \alpha^2$  and  $\beta$  to  $\beta_0 + \beta^2$  to obtain the linear space of parameters  $\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu$ . Hence the diffusion terms in (3.1) never disappear and are uniformly coercive for all  $\alpha, \beta \in \mathbf{R}$ .

For the setting of the identification problems for (3.1), we assume that the parameters  $\alpha, \beta, \gamma_i, \delta, \kappa_i$  and  $\nu$  appeared in (3.1) are unknown and we take  $\mathcal{P} = \mathbf{R}^{2L+4}$  as the set of parameters  $q = (\alpha, \beta, \gamma_1, \dots, \gamma_L, \delta, \kappa_1, \dots, \kappa_L, \nu)$ . The Euclidean norm and the inner product of  $\mathcal{P}$  are denoted simply by  $|\cdot|$  and  $(\cdot, \cdot)$ , respectively. For simplicity of notations we write  $q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu) \in \mathcal{P}$ .

By Theorem 2.2, for each  $q \in \mathcal{P}$  there exists a unique weak solution  $y = y(q) \in W_V(0, T)$  of (3.1). Then we can uniquely define the solution map  $q \rightarrow y(q)$  of  $\mathcal{P}$  into  $W_V(0, T)$ .

Let  $K$  be a Hilbert space of observations and let  $\|\cdot\|_K$  be its norm. The observation of  $y(q)$  is assumed to be given by

$$z(q) = Cy(q) \in K, \quad (3.2)$$

where  $C$  is a bounded linear observation operator of  $W_V(0, T)$  into  $K$ .

The cost functional attached to (3.1) with (3.2) is given by

$$J(q) = \|Cy(q) - z_d\|_K^2 + (Mq, q) \quad \text{for } q \in \mathcal{P}, \quad (3.3)$$

where  $z_d \in K$  is a desired value of  $y(q)$  and  $M$  is a symmetric and non-negative  $(2L+4) \times (2L+4)$  matrix on  $\mathcal{P} = \mathbf{R}^{2L+4}$ .

Assume that an admissible subset  $\mathcal{P}_{ad}$  of  $\mathcal{P}$  is convex and closed. As in [9] we study the existence and characterization problems for the perturbed sine-Gordon equations. That is, the following two problems:

(i) Find an element  $q^* \in \mathcal{P}_{ad}$  such that

$$\inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*); \quad (3.4)$$

(ii) Give a characterization to such the  $q^*$ .

As usual we call  $q^*$  the optimal parameter and  $y(q^*)$  the optimal state. In order to solve (ii), we shall derive the necessary conditions on  $q^*$ . If  $J(q)$  is Gâteaux differentiable at  $q^*$  in the direction  $q - q^*$ , then  $q^*$  has to satisfy

$$DJ(q^*)(q - q^*) \geq 0 \quad \text{for all } q \in \mathcal{P}_{ad}, \quad (3.5)$$

where  $DJ(q^*)$  denotes the Gâteaux derivative of  $J(q)$  at  $q = q^*$  in the direction  $q - q^*$ .

### 3.1. Existence of optimal parameters

The following theorem shows the continuity of solution map  $q \rightarrow y(q)$ , which is crucial to solve the problems (i) and (ii).

**Theorem 3.1.** *The map  $q \rightarrow y(q) : \mathcal{P} \rightarrow W_V(0, T)$  is weakly continuous. That is,  $y(q_n) \rightarrow y(q)$  weakly in  $W_V(0, T)$  as  $q_n \rightarrow q$  in  $\mathbf{R}^{2L+4}$ .*

The following theorem follows immediately from Theorem 3.1 and the lower semi-continuity of norms.

**Theorem 3.2.** *If  $\mathcal{P}_{ad} \subset \mathcal{P} = \mathbf{R}^{2L+4}$  is compact or  $M$  is a positive and symmetric on  $\mathbf{R}^{2L+4}$ , then there exists at least one optimal parameter  $q^* \in \mathcal{P}_{ad}$  for the cost (3.3).*

### 3.2. Necessary conditions

For proving that  $J(q)$  is Gâteaux differentiable at  $q^*$  in the space of parameters, we have to estimate the quotients  $z_\lambda = (y(q_\lambda) - y(q^*))/\lambda$  in the space  $W_V(0, T)$ , where  $q_\lambda = q^* + \lambda(q - q^*)$ ,  $\lambda \in (0, 1]$  and  $q, q^* \in \mathcal{P}$ . We set  $y_\lambda = y(q_\lambda)$  and  $y^* = y(q^*)$  for simplicity.

Let us begin to prove the weak Gâteaux differentiability of the solution map  $q \rightarrow y(q)$  of  $\mathcal{P}$  into  $W_V(0, T)$ .

**Theorem 3.3.** *The map  $q \rightarrow y(q)$  of  $\mathcal{P}$  into  $W_V(0, T)$  is weakly Gâteaux differentiable. That is, for fixed  $q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu)$  and  $q^* = (\alpha^*, \beta^*, \gamma_i^*, \delta^*, \kappa_i^*, \nu^*)$  in  $\mathcal{P}$  the weak Gâteaux derivative  $z = Dy(q^*)(q - q^*)$  of  $y(q)$  at  $q = q^*$  in the direction  $q - q^*$  exists in  $W_V(0, T)$  and it is a unique weak solution of the evolution equation*

$$\left\{ \begin{array}{l} z'' + (\alpha^{*2} + \alpha_0)Az' + (\beta^{*2} + \beta_0)Az + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)z + \delta^* z \\ = 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* \\ + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f \text{ in } (0, T), \\ z(0) = z'(0) = 0, \end{array} \right. \quad (3.6)$$

where  $y^* = y(q^*)$ .

Since the map  $q \rightarrow y(q) : \mathcal{P} \rightarrow W_V(0, T)$  is Gâteaux differentiable at  $q^*$  in the direction  $q - q^*$ , the inequality (3.5) is equivalent to

$$\langle Cy(q^*) - z_d, Cz \rangle_{K', K} \geq 0, \quad \forall q \in \mathcal{P}_{ad}, \quad (3.7)$$

where  $z$  is the solution of (3.6). To avoid the identification problem to be complicated we study the problem according to two types of simple observations as follows:

1. Observe the distributed state  $Cy(q) = y(q) \in L^2(0, T; H)$  and take  $K = L^2(0, T; H)$ ;
2. Observe the time terminal state  $Cy(q) = y(q, T) \in H$  and take  $K = H$ .

1. Case of  $Cy(q) = y(q) \in L^2(0, T; H)$

In this case we give the cost functional by

$$J(q) = r\|y(q) - z_d\|_{L^2(0,T;H)}^2 + (Mq, q), \quad (3.8)$$

where  $z_d \in L^2(0, T; H)$  and  $r > 0$ . Then the necessary condition (3.7) with respect to (3.8) is written by

$$r(y(q^*) - z_d, z)_{L^2(0,T;H)} + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \quad (3.9)$$

Hence by standard arguments we have the following theorem.

**Theorem 3.4.** *The optimal parameter  $q^*$  for the cost (3.8) is characterized by the two states  $y = y(q^*), p = p(q^*)$  of equations*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^L \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f & \text{in } (0, T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \quad (3.10)$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)p + \delta^* p = r(y(q^*) - z_d) & \text{in } (0, T), \\ p(T) = p'(T) = 0. \end{cases} \quad (3.11)$$

and one inequality

$$\begin{aligned} & \int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f \rangle dt \\ & + (Mq^*, q^* - q) \geq 0 \quad \text{for all } q \in \mathcal{P}_{ad}. \end{aligned} \quad (3.12)$$

## 2. Case of $Cy(q) = y(q; T) \in H$

In this case the cost functional is given by

$$J(q) = r|y(q; T) - z_d|^2 + (Mq, q), \quad (3.13)$$

where  $z_d \in H$  and  $r > 0$ . Then the necessary condition (3.7) with respect to (3.13) is written by

$$r(y(q^*; T) - z_d, z(T)) + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \quad (3.14)$$

Thus we have the following theorem.

**Theorem 3.5.** *The optimal parameter  $q^*$  for the cost (3.13) is characterized by the two states  $y = y(q^*), p = p(q^*)$  of equations*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^L \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f & \text{in } (0, T), \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \quad (3.15)$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)p + \delta^* p = 0 & \text{in } (0, T), \\ p(T) = 0, \quad p'(T) = -r(y(q^*; T) - z_d). \end{cases} \quad (3.16)$$

and one inequality

$$\begin{aligned} & \int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f \rangle dt \\ & + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \end{aligned} \quad (3.17)$$

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