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Mixed Duality for Nonsmooth Multiobjective Fractional Programming without a Constraint Qualification

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1 Introduction and Preliminaries

For a nonempty subset C of $\mathbb{R}^n$, we denote C^*, the dual cone of C and defined by

$$C^* = \{u \in \mathbb{R}^n | u^T x \geqq 0, \ \ \forall \ \ x \in C\}.$$

Further, for $x^* \in C$, $N_C(x^*)$ denotes the normal cone to C at x^* defined by

$$N_C(x^*) = \{d \in \mathbb{R}^n | < d, x - x^* > \leqq 0, \ \forall \ x \in C\},$$

clearly, $(C - x^*)^* = -N_C(x^*)$.

We consider the following multiobjective nonsmooth fractional programming problem,

$$\begin{array}{lll} \text{(FP)} & \text{Minimize} & \dfrac{f(x) + s(x|D)}{g(x)} \\ & \text{subject to} & h(x) \leqq 0, \\ & & x \in C, \end{array}$$

where C is a convex set and $D_i, i = 1, \cdots, p$ are compact convex sets of $\mathbb{R}^n$ and f_i, g_i, h_j $i = 1, \cdots, p$, $j = 1, \cdots, m$ are real valued locally Lipschitz functions defined on C. The index sets are $P = \{1, 2, \cdots, p\}$, $M =$

$\{1, 2, \cdots, m\}$. We denote the feasible set $\{x \in C | h_j(x) \leqq 0,\ j = 1, \cdots, m\}$ by F. Let $I(x^*) = \{j \in M | h_j(x^*) = 0\}$ denote the index set of active constraints at x^*. We assume that for each $i \in P$, $f_i(x) + s(x|D_i) \geqq 0,\ g_i(x) > 0$.

The minimal index set of active constraints for F is denoted by

$$I^= = \{j \in M | x \in F \rightarrow h_j(x) = 0\}.$$

We also denote

$$I^<(x^*) = I(x^*) \setminus I^= = \{j \in I(x^*) | x \in F \text{ such that } h_j(x) < 0\}.$$

Definition 1.1 *A feasible solution x^* for (FP) is efficient for (FP) if and only if there is no other feasible x for (FP) such that*

$$\frac{f_{i_0}(x) + s(x|D_{i_0})}{g_{i_0}(x)} < \frac{f_{i_0}(x^*) + s(x^*|D_{i_0})}{g_{i_0}(x^*)}, \text{ for some } i_0 \in P,$$

and

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leqq \frac{f_i(x^*) + s(x^*|D_i)}{g_i(x^*)}, \ \forall\, i \in P.$$

Definition 1.2 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function. Then*
(i) *it is said to be generalized convex at x if for any y*

$$f(y) - f(x) \geqq < \xi, y - x >, \ \forall\, \xi \in \partial_c f(x),$$

(ii) *it is said to be generalized quasiconvex at x if for any y such that $f(y) \leqq f(x)$,*

$$< \xi, y - x > \leqq 0, \ \forall\, \xi \in \partial_c f(x),$$

(iii) *it is said to be generalized strictly quasiconvex at x if for any y such that $f(y) \leq f(x)$,*

$$< \xi, y - x > < 0, \ \forall\, \xi \in \partial_c f(x),$$

2 Optimality Conditions

We will establish necessary and sufficient optimality conditions for the fractional problem (FP).

We consider the following nonlinear programming problem:

$$\begin{aligned}(\mathrm{F}P_u)\quad \text{Minimize}\quad & (f_1(x)+s(x|D_1)-u_1g_1(x),\\ & \cdots, f_p(x)+s(x|D_p)-u_pg_p(x))\\ \text{subject to}\quad & h_j(x)\leqq 0,\ j\in M,\ x\in C,\end{aligned}$$

where for each $u=(u_1,\cdots,u_p)\in \mathbb{R}^n_+$.

Lemma 2.1 *If x^* is an efficient solution for (FP), then x^* is an efficient solution for (FP_{u^*}) where $u^*=\frac{f_i(x^*)+s(x^*|D_i)}{g_i(x^*)}$.*

We denote $\phi_i(x^*)=\frac{f_i(x^*)+s(x^*|D_i)}{g_i(x^*)}$.

Theorem 2.1 *If x^* be an efficient solution of (FP) and $h_j(\cdot), j\in M$ are generalized strictly quasiconvex at x^*, then there exist $\tau^*\in\mathbb{R}^p, \mu^*\in\mathbb{R}^m$, and $z_i^*\in\mathbb{R}^n, i\in P$ such that*

$$\begin{aligned}0\in \textstyle\sum_{i=1}^p \tau_i^*[\partial_c(f_i(x^*)+(x^*)^Tz_i)-\phi_i(x^*)\partial_c g_i(x^*)]&\\ +\textstyle\sum_{j\in I^<(x^*)}\mu_j^*\partial_c h_j(x^*)+N_c(x^*),&\qquad (*)\\ \tau_i^*>0,\ i\in P,\ \mu_j^*\geqq 0,\ j\in M,\ \textstyle\sum_{i=1}^p\tau_i^*=1.&\end{aligned}$$

Corollary 2.1 *Let x^* be an efficient solution for (FP), and $h_j, j\in M$ are generalized strictly quasiconvex at x^*, then there exist $\tau^*\in\mathbb{R}^p, \mu^*\in\mathbb{R}^m$ and $z_i^*\in D_i, i\in P$ such that*

$$0 \in \sum_{i=1}^{p} \tau_i^* [\partial_c (f_i(x^*) + (x^*)^T z_i) - \phi_i(x^*) \partial_c g_i(x^*)] + \sum_{j=1}^{m} \mu_j^* \partial_c h_j(x^*) + N_c(x^*),$$

$$(z_i^*)^T x^* = s(x^* | D_i), \ z_i^* \in D_i, \ i \in P,$$

$$\mu_j^* h_j(x^*) = 0, \ j \in M,$$

$$\tau_i^* > 0, \ i \in P, \ \mu_j^* \geqq 0, \ j \in M, \ \sum_{i=1}^{p} \tau_i^* = 1.$$

Theorem 2.2 *Let x^* be a feasible solution of* (FP) *and assume that the conditions* $(*)$ *hold at x^*. If all of the functions $f_i(\cdot), i = 1, \cdots, p$, $-g_i(\cdot)$ and $h_j(\cdot), j = 1, \cdots, m$ are generalized convex at x^*, then x^* is an efficient solution of* (FP).

3 Mixed Duality

We introduce a mixed type dual fractional programming problem and establish weak, strong and converse duality theorems.

$$\text{(FD) Maximize} \quad \left(\frac{f_1(y) + y^T z_1 + \sum_{j \in M_1} \mu_j h_j(y)}{g_1(y)}, \cdots, \frac{f_p(y) + y^T z_p + \sum_{j \in M_1} \mu_j h_j(y)}{g_p(y)}\right)$$

$$\text{subject to} \quad 0 \in \sum_{i \in P} g_i(y)(\partial_c(\tau_i(f_i(y) + y^T z_i)) + \tau_i \sum_{j \in M_1} \partial_c \mu_j h_j(y)) - \sum_{i \in P} (f_i(y)$$

$$+ y^T z_i + \sum_{j \in M_1} \mu_j h_j(y)) \partial_c \tau_i g_i(y) + \sum_{j \in M_2} \partial_c \mu_j h_j(y) + N_c(y),$$

$$\mu_j h_j(y) \geqq 0, j \in M_2, \ z_i \in D_i, i \in P,$$

$$\mu_j \geqq 0, j \in M, \ \tau_i > 0, \sum_{i \in M} \lambda_i = 1, y \in C.$$

where M_1 is a subset of $M = \{1, \cdots, m\}$, $M_2 = M \setminus M_1$. Assume that for each $i \in P$, $f_i(y) + y^T z_i + \sum_{j \in M_1} \mu_j h_j(y) \geqq 0, g_i(y) > 0$.

Theorem 3.1 (Weak Duality) *Let x be a feasible solution for problem* (FP) *and (y, z, τ, μ) be a feasible solution for problem* (FD). *If all of the functions $f_i(\cdot)$, $g_i(\cdot)$, $i \in P$, and $h_{M_1}(\cdot)$ are generalized convex at y, and $\mu_j h_j(\cdot)$, $j \in M_2$ are generalized quasiconvex at y, then the following cannot hold:*

$$\frac{f_{i_0}(x) + s(x|D_{i_0})}{g_{i_0}(x)} < \frac{f_{i_0}(y) + y^T z_{i_0} + \sum_{j \in M_1} \mu_j h_j(y)}{g_{i_0}(y)}, \text{ for some } i_0 \in P$$

and

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leqq \frac{f_i(y) + y^T z_i + \sum_{j \in M_1} \mu_j h_j(y)}{g_i(y)}, \ \forall\, i \in P.$$

Theorem 3.2 (Strong Duality) *Let x^* be an efficient solution of problem* (FP). *If the assumptions of Theorem 3.1 hold and $h_j(\cdot), j = 1, \cdots, m$ are generalized strictly quasiconvex, then there exist $\tau^* \in \mathbb{R}^p_+$, $z^* \in D_i$, $i \in P$ and $\mu^* \in \mathbb{R}^m_+$ such that $(x^*, z^*, \tau^*, \mu^*)$ is an efficient solution of* (FD), *$(x^*)^T z_i^* = s(x^*|D_i), i \in P$ and the optimal values of* (FP) *and* (FD) *are equal.*

Theorem 3.3 (Strict Converse Duality) *Let x^* and $(y^*, z^*, \tau^*, \mu^*)$ be efficient solutions of* (FP) *and* (FD), *respectively. If in addition the hypotheses of Theorem 3.2 hold, then $y^* = x^*$, that is, y^* solves* (FP) *and the objective values of* (FP) *and* (FD) *are equal.*

4 Special Cases

We give some special cases of our dual programming.

(1) If $D_i = \{0\}$, $i = 1, \cdots, p$ and $I_1 = \emptyset$, then our mixed dual problem (FD) reduce to the Mond-Weir type problem (D_1) in Weir and Mond [13].

(2) If $D_i = \{0\}$, $i = 1, \cdots, p$ and $I_2 = \emptyset$, then our mixed dual problem (FD) reduce to the Wolfe type problem (D_2) in Weir and Mond [13].

(3) If $D_i = \{0\}$, $i = 1, \cdots, p$, then our mixed dual problem (FD) reduce to the mixed dual problem (MD) in Nobakhtian [10].

References

[1] C. R. Bector, S. Chandra and I. Husain, *Optimality condition and duality in subdifferentiable multiobjective fractional programming*, Journal of Optimization Theory and Application, Vol. 79, pp. 105-125, 1993.

[2] X. Chen, *Optimality and duality for the multiobjective fractional programming with the generalized (F, ρ) convexity*, Journal of Mathematical Analysis and Applications, Vol. 273, pp. 190-205, 2002.

[3] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley & Sons, New York, 1983.

[4] B. Jimenez and V. Novo, *Optimality conditions in directional differentiable Pareto problems with a set constraint via tangent cones*, Numerical Functional Analysis and Optimization, Vol. 24, pp. 557-574, 2003.

[5] D. S. Kim and K. D. Bae, *Optimality conditions and duality for a class of nondifferentiable multiobjective programming problems*, Taiwanese Journal of Mathematics, Vol.13, no.2B, pp.789-804, 2009.

[6] D. S. Kim and S. Schaible, *Optimality and duality for invex nonsmooth multiobjective programming problems*, Optimization, Vol. 53, pp. 165-176, 2004.

[7] J. C. Liu, *Optimality and duality for multiobjective fractional programming involving nonsmooth pseudoinvex functions*, Optimization, Vol. 37, pp. 27-39, 1996.

[8] D. T. Luc, *A multiplier rule for multiobjective programming problems with continuous data*, SIAM Journal on Optimization, Vol. 13, pp. 168-178, 2002.

[9] S. K. Mishra, S. Y. Wang and K. K. Lai, *Nondifferentiable multiobjective programming under generalized d-invexity*, European Journal of Operational Research, Vol. 160, pp. 218-222, 2005.

[10] S. Nobakhtian, *Duality without constraint qualification in nonsmooth optimization*, International Journal of Mathematics and Mathematical Sciences, Vol. 2006, pp. 1-11, 2006.

[11] S. Nobakhtian, *Optimality criteria and duality in multiobjective programming involving nonsmooth invex functions*, Journal of Global Optimization, Vol. 35, pp. 595-606, 2006.

[12] V. Preda, *Optimality and duality in fractional multiple objective programming involving semilocally preinvex and related functions*, Journal of Mathematical Analysis and Applications, Vol. 288, pp. 362-382, 2003.

[13] T. WEIR AND B. MOND, Generalized convexity and duality in multiobjective programming, Bulletin of the Australian Mathematical Society, Vol. 39, pp. 287-299, 1989.

[14] P. WOLFE, *A duality theorem for non-linear programming*, Quarterly of Applied Mathematics, Vol. 19, pp. 239-244, 1961.

[15] H. ZHOU AND W. SUN, *Mixed duality without a constraint qualification for minimax fractional programming*, Optimization, Vol. 52, pp. 617-627, 2003.