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A NOTE ON ADJOINT SEMIGROUPS ASSOCIATED WITH SOME LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Set $\mathbb{E} = \mathbb{R}^n$ or $\mathbb{C}^n$,

$$C_\gamma = \{\varphi : (-\infty, 0] \to \mathbb{E} : \varphi \text{ is continuous and } a_\gamma(\varphi) = \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists}\},$$

where $\gamma \in \mathbb{R}$, and set

$$|\varphi|_\gamma = \sup\{e^{\gamma\theta}|\varphi(\theta)| : -\infty < \theta \le 0\} \quad \varphi \in C_\gamma.$$

Then C_γ is a Banach space with respect to the norm $|\cdot|_\gamma$, and it is isomorphic to the Banach space $C = C([-1,0],\mathbb{E})$: an isometric isomorphism ι_γ from C_γ onto C is given by

$$(\iota_\gamma\varphi)(s) = \begin{cases} e^{\gamma s/1+s}\varphi(s/1+s) & -1 < s \le 0 \\ a_\gamma(\varphi) & s = -1. \end{cases}$$

The t-segment x_t of a function $x : (-\infty, t] \to \mathbb{E}$ is a function, on $(-\infty, 0]$ into $\mathbb{E}$, defined by $x_t(\theta) = x(t+\theta)$ $\theta \le 0$. Consider a linear functional differential equation with the phase space C_γ:

(1) $$\dot{x} = L(x_t),$$

where L is a continuous linear operator on C_γ into $\mathbb{E}$. It is well known [3,4] that this equation has a unique solution $x(\varphi)$ satisfying the initial condition $x_0 = \varphi$ for any φ in C_γ, and that the one parameter family of operators $T(t)$, $t \geq 0$, defined by

$$T(t)\varphi = x_t(\varphi) \quad \text{for} \quad \varphi \in C_\gamma$$

is a strongly continuous semigroup of bounded linear operators on C_γ. This semigroup is called the solution semigroup of Equation (1). We consider the representation of the adjoint semigroup of $T(t)$.

For linear functional differential equations with finite delays on the phase space C, it is shown [2] that C is sun-reflexive with respect to the solution semigroups. On the other hand, for the case $\gamma = 0$, the space C_0 is not sun-reflexive with respect to $T(t)$, [5]. The space $C_0^{\odot\odot}$ defined below is isomorphic to the space $\mathbb{E} \times BU$, where BU is the space of bounded, uniformly continuous functions on $(-\infty,0]$ into $\mathbb{E}$ with the supremum norm; C_0 is imbedded onto the subspace $\{(a_0(\varphi),\varphi) : \varphi \in C_0\} \subset \mathbb{E} \times BU$. Diekmann and Greiner suggested that, for any γ, the space C_γ would not be sun-reflexive with respect to $T(t)$ from the imformation about the spectrum of the resolvent $(\lambda I - A)^{-1}$, where A is the infinitesimal generator of $T(t)$. What is $C_\gamma^{\odot\odot}$? This note is an unaccomplised approach to this question.

Before proceeding, we give the definition of sun-reflexivity. For a moment, denote by $T(t)$ a strongly continuous semigroup of bounded linear operators on a Banach space X. Then the adjoins semigroup $T^*(t)$ is not necessarily a strongly continuous on X^*. The maximal closed subspace of X^* on which $T^*(t)$ becomes a strongly contious semigroup is the one defined by

$$X^{\odot} = \{x^{\odot} \in X^*: \lim_{t\to 0+} |T^*(t)x^{\odot} - x^{\odot}| = 0\}.$$

It is known that this space coincides with the closure of the $\mathcal{D}(A^*)$, the domain of the adjoint operator of the infinitesimal generator A of $T(t)$. Denote by $T(t)$ the restriction of $T^*(t)$ on $X^{\odot}$. Repeating the above process for $T^{\odot}(t)$, we have a strongly continuous semigroup $T^{\odot\odot}(t)$ which is the restriction of $T^{\odot *}(t)$ on the space $X^{\odot\odot} = \{x^{\odot\odot} \in X^{\odot *}: \lim_{t\to 0+} |T^{\odot *}(t)x^{\odot\odot} - x^{\odot\odot}| = 0\}$. The original space X is isomorphically imbedded to a subspace of $X^{\odot\odot}$, and $T^{\odot\odot}(t)$ is an extension of $T(t)$. If X is isomorphic to the whole space $X^{\odot\odot}$, we say that X is sun-reflexive with respect to the semigroup $T(t)$. This occurs if and only if the resolvent $(\lambda I - A)^{-1}$ is weakly compact for any λ in the resolvent set $\rho(A)$, see [6]. Refer the book [1] for the theory of adjoint semegroups. The space $X^{\odot\odot}$ depends on $T(t)$; we may obtain another space as $X^{\odot\odot}$ from a different semigroup. But for the solution semigroup of Equation (1), the spaces $C_\gamma^{\odot}$, $C_\gamma^{\odot\odot}$ are independent of the choice of the linear operator L. This fact holds for solution semigroups of linear functional differential equations with infinite delays on the very general phase space, [5].

Now we return to the solution semigroup $T(t)$ of Equation (1). It is easy to show the following result, cf [5].

Theorem 1. A function φ in C_γ is in $\mathcal{D}(A)$ if and only if it is continuously differentiable, $\dot{\varphi}$ is in C_γ and $\dot{\varphi}(0) = L(\varphi)$; and $A\varphi = \dot{\varphi}$ for φ in $\mathcal{D}(A)$.

To obtain the information about the space $C_\gamma^{\odot\odot}$, we take $L = 0$ in Equation (1); that is, we consider the equation

$$\dot{x} = 0. \tag{2}$$

Denote by $S(t)$ the solution semigroup of this equation, and by B its infinitesimal generator. As claimed in the above, $\mathcal{D}(A)$ and $\mathcal{D}(B)$ have the same closure in C_γ^{*}, which we denote by $C_\gamma^{\odot}$. Since the case that $\gamma = 0$ is already treated in [5], we assume that $\gamma \neq 0$ hereafter.

If φ is in C_γ, the function u defined by $u(\theta) = e^{-\gamma\theta}\varphi(\theta)$ for θ in $(-\infty,0]$ is in C_0. Set $j_\gamma(\varphi) = u$. Then j_γ is an isometric isomorphism between C_γ and C_0. We characterize $\mathcal{D}(B)$ by the condition on u.

Theorem 2. Assume that $\gamma \neq 0$. Then φ in C_γ is in $\mathscr{D}(B)$ if and only if $u = j_\gamma(\varphi)$ is represented as

$$(3) \qquad u(\theta) = \frac{1}{\gamma} v(0) - \int_\theta^0 v(s)\, ds \quad \text{for } \theta \text{ in } (-\infty, 0]$$

and for some function v in C_0 such that

$$(4) \qquad \lim_{\theta \to -\infty} v(\theta) = 0 \quad \text{and} \quad \lim_{r \to -\infty} \int_r^0 v(s)\, ds \quad \text{converges.}$$

Proof. A function φ in C_γ is continuously differentiable if and only is u is so, and $\dot{\varphi}(\theta) = e^{-\gamma\theta}\{-\gamma u(\theta) + \dot{u}(\theta)\}$. Thus $a_\gamma(\dot{\varphi})$ exists if and lnly if $\lim_{\theta \to -\infty} \{-\gamma u(\theta) + \dot{u}(\theta)\}$ exists; and $\dot{\varphi}(0) = 0$ if and only if $-\gamma u(0) + \dot{u}(0) = 0$. Set $v(\theta) = \dot{u}(\theta)$. Then the last condition is equivalent that u is represented as in the above form (3). The condition that u is in C_0 is equivalent that v has the second dondition in (4). If $u(\theta)$ and $-\gamma u(\theta) + \dot{u}(\theta)$ converge as $\theta \to -\infty$, then $u(\dot{\theta}) \to 0$ as $\theta \to -\infty$. Thus we have the first dondition for v in (4).

If η is a function of bounded variation on $[-1,0]$ and φ is continuous on $[-1,0]$, we can write

$$\int_{-1}^0 d\eta(t)\varphi(t) = [\eta(-1+) - \eta(-1)]\varphi(-1) + \lim_{r \to -1+} \int_r^0 d\eta(t)\varphi(t),$$

where $\eta(-1+) = \lim_{t \to -1+} \eta(t)$. Since $\iota_0 : C_0 \to C$ is an isomorphism, $\iota_0{}^* ; C^* \to C_0{}^*$ is also an isomorphism. Hence, from Riesz's representation theorem, for any u^* in $C_0{}^*$ there exist a vector a in $\mathbb{E}^*$, and a $\mathbb{E}^*$-valued function f, of bounded variation on $(-\infty,0]$, such that

$$\langle u^*,u\rangle = au(-\infty) + \int_{-\infty}^{0} df(\theta)u(\theta) \qquad \text{for } u \text{ in } C_0,$$

where $u(-\infty) = \lim_{\theta\to-\infty} u(\theta)$ and $\int_{-\infty}^{0} = \lim_{r\to-\infty} \int_{r}^{0}$. Of course, the norm of u^* is given by

$$|u^*| = |a| + \mathrm{Var}(f,(-\infty,0]) = |a| + \lim_{r\to-\infty} \mathrm{Var}(f,[r,0]).$$

If f is normalized in the sense that $f(0) = 0$ and f is left continuous on $(-\infty,0)$, the pair (a,f) is uniquely determined by u^*. Denote by NBV the class of those normalized functions. We may identify the space $C_0{}^*$ with $\mathbb{E}^* \times \mathrm{NBV}$, and regard the isomorphism $j_\gamma{}^*$ as the one from $\mathbb{E}^* \times \mathrm{NBV}$ to $C_\gamma{}^*$: $\langle j_\gamma{}^*(a,f),\varphi\rangle = \langle (a,f),u\rangle$, where $u = j_\gamma(\varphi)$. Namely the space $\mathbb{E}^* \times \mathrm{NBV}$ is the coordinate space of $C_\gamma{}^*$.

Theorem 3. Assume that $\gamma \neq 0$. Then an element (a,f) in $\mathbb{E}^* \times \mathrm{NBV}$ is a coordinate of an element of $\mathcal{D}(B)$ if and only if a is arbitrary, $f(\theta)$ is absolutely continuous on $(-\infty,0)$, and the equivalence class of $\dot{f}(\theta)$ in $L^1((-\infty,0),\mathbb{E}^*)$ contains a function which is in NBV and converges to 0 as $\theta \to -\infty$.

For such a (a,f) the coordinate (b,g) of $B^*(j_\gamma{}^*(a,f))$ is given by

$$\text{(5)} \quad b = -\gamma a$$

$$\text{(6)} \quad g(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ -\gamma[f(\theta) - f(0-)] - \dot{f}(\theta) & \text{for } \theta < 0, \end{cases}$$

where we read that $\dot{f}$ stands for the function in NBV mentioned in the above.

Proof. Consider the condition

$$\text{(7)} \quad j_\gamma{}^*(a,f) \text{ is in } \mathcal{D}(B^*) \text{ and } B^*(j_\gamma{}^*(a,f)) = j_\gamma{}^*(b,g),$$

for (a,f) and (b,g) in $E^* \times \text{NBV}$. By the definition of B^*, this means that, for every φ in $\mathcal{D}(B)$, $\langle j_\gamma{}^*(a,f),B\varphi\rangle = \langle j_\gamma{}^*(b,g),\varphi\rangle$, or $\langle (a,f),-\gamma u+\dot{u}\rangle = \langle (b,g),u\rangle$, where $u = j_\gamma(\varphi)$. The condition $f, g \in \text{NBV}$ implies that $f(0) = g(0) = 0$; and the condition $\varphi \in \mathcal{D}(B)$ implies that $\dot{u}(-\infty) = 0$. From integration by parts we then have that

$$\langle (a,f),-\gamma u+\dot{u}\rangle = -[a-f(-\infty)]\gamma u(-\infty) + \int_{-\infty}^{0} [\gamma f(\theta)d\theta+df(\theta)]\dot{u}(\theta)$$

$$\langle (b,g),u\rangle = [b-g(-\infty)]u(-\infty) - \int_{-\infty}^{0} dg(\theta)\dot{u}(\theta).$$

Since u is represented as in (3), it holds that

$$u(-\infty) = \frac{1}{\gamma}v(0) - \int_{-\infty}^{0} v(\theta)d\theta.$$

Hence we can write

$$\int_{-\infty}^{0} [df(\theta)+\gamma f(\theta)d\theta]v(\theta) = -c\{\frac{1}{\gamma}v(0) - \int_{-\infty}^{0} v(\theta)d\theta\} - \int_{-\infty}^{0} dg(\theta)v(\theta),$$

where

$$c = \gamma(f(-\infty) - a) + g(-\infty) - b.$$

Furthermore, if we define a function h in NBV by

$$h(\theta) = \begin{cases} 0 & \theta = 0 \\ \frac{c}{\gamma} & \theta < 0, \end{cases}$$

we have that

$$\text{(8)} \qquad \int_{-\infty}^{0} [df(\theta)+\gamma f(\theta)]d\theta]v(\theta) = \int_{-\infty}^{0} [dh(\theta)+cd\theta-dg(\theta)]v(\theta).$$

Consequently Condition (7) is equivalent that Relation (8) holds for every v in C_0 having Property (4). Since every function with compact support has this property for v, it follows that $df(\theta)+ \gamma f(\theta) = dh(\theta)+cd\theta-dg(\theta)$, or

(9) $$f(\theta) - \gamma\int_\theta^0 f(s)\,ds = h(\theta) + c\theta + \int_\theta^0 g(s)\,ds \quad \text{for} \quad \theta \le 0.$$

Notice that the functions in both sides are normalized.

Suppose that Equation (9) holds for f and g in NBV. Then f is locally absolutely continuous on $(-\infty,0)$, which implies that

$$\mathrm{Var}(f,[s,t]) = \int_s^t |\dot{f}(\theta)|\,d\theta \quad \text{for} \quad -\infty < s < t < 0.$$

Since f is of bounded variation on $(-\infty,0]$, it follows that $|\dot{f}(\theta)|$ is integrable on $(-\infty,0)$: that is, f is absolutely continuous on $(-\infty,0)$. Furthermore, from Equation (9) we have that

(10) $$\dot{f}(\theta) + \gamma f(\theta) = c - g(\theta) \quad \text{a.e. in} \quad (-\infty,0).$$

Since $f(0-) = h(0-) = \frac{c}{\gamma}$ from Equation (9), we obtain Relation (6); and the solution f of Equation (10) is

(11) $$f(\theta) = \frac{c}{\gamma} + \int_\theta^0 e^{-\gamma(\theta-s)}g(s)\,ds \quad \text{for} \quad \theta < 0.$$

Suppose that $\gamma > 0$. Sincce $f(-\infty)$ exists, we have that $\lim_{\theta\to-\infty} e^{\gamma\theta}[f(\theta) - c/\gamma] = 0$, which implies that

$$\int_{-\infty}^{0} e^{\gamma s} g(s)\, ds = 0.$$

Hence f is rewritten as

$$f(\theta) = \frac{c}{\gamma} - \int_{-\infty}^{\theta} e^{-\gamma(\theta-s)} g(s)\, ds = \frac{c}{\gamma} - \int_{-\infty}^{0} e^{\gamma t} g(t+\theta)\, dt,$$

from which it follows that

$$f(-\infty) = \frac{c}{\gamma} - \frac{g(-\infty)}{\gamma}. \tag{12}$$

Suppose $\gamma < 0$. Writing the integral in (11) as

$$\int_{\theta}^{0} e^{-\gamma(\theta-s)} g(s)\, ds = \left[\int_{\theta}^{N} + \int_{N}^{0} \right] e^{-\gamma(\theta-s)} g(s)\, ds \qquad \theta < N < 0,$$

we have directly Relation (12), or

$$c = \gamma f(-\infty) + g(-\infty).$$

This relation and the definition of c imply Relation (5); Equation (10) then becomes

$$\dot{f}(\theta) = -\gamma[f(\theta) - f(-\infty)] + g(\theta) - g(-\infty) \qquad \text{a.e. in } (-\infty,0).$$

This means that the equivalence class of $\dot{f}$ in $L^1((-\infty,0),\mathbb{E}^*)$ contain a function which is in NBV and converges to 0 as θ

$\to -\infty$. Notice that such a function is unique for the equivalence class of $\dot{f}$ since it is left continuous on $(-\infty,0)$.

Conversely, suppose f has this property, and let $k(\theta)$ be the function, in $NBV \cap L^1((-\infty,0),\mathbb{E}^*)$, such that $\dot{f}(\theta) = k(\theta)$ a.e. in $(-\infty,0)$ and that $k(-\infty) = 0$. Define $g(\theta)$ by $g(0) = 0$ and $g(\theta) = -\gamma[f(\theta) - f(0-)] - k(\theta)$ for $\theta < 0$. Then g is in NBV and $\dot{f}(\theta) = -\gamma[f(\theta) - f(0-)] - g(\theta)$ a.e. in $(-\infty,0)$, which implies that

$$(13) \qquad f(\theta) - f(0-) = -\gamma\int_0^\theta f(s)\,ds + \gamma\theta f(0-) - \int_0^\theta g(s)\,ds$$

for $\theta < 0$. Since $g(-\infty) = -\gamma[f(-\infty) - f(0-)] - k(-\infty) = -\gamma[f(-\infty) - f(0-)]$, we have that $\gamma f(0-) = \gamma f(-\infty) + g(-\infty)$. If a and b in $\mathbb{E}^*$ satisfy Relation (5), it then follows that $\gamma f(0-) = \gamma[f(-\infty) - a] + g(-\infty) - b$. Therefore relation (13) becomes Relation (9), as required.

Theorem 3 says that $j_\gamma^*(a,f)$ is in $\mathcal{D}(B^*)$ if and only if f is represented as

$$(14) \qquad f(\theta) = d + \int_\theta^0 k(s)\,ds \quad \text{for} \quad \theta < 0,$$

and for some d in $\mathbb{E}^*$ and for some k in $NBV \cap L^1((-\infty,0),\mathbb{E}^*)$ with $k(-\infty) = 0$. In this case the C_γ^* norm of $j_\gamma^*(a,f)$ is given by

$$|j_\gamma^*(a,f)| = |a| + \mathrm{Var}(f,(-\infty,0)) = |a| + |d| + \int_{-\infty}^{0} |k(s)|\, ds.$$

Thus we have the following result, which is also valid in the case that $\gamma = 0$, [5].

Corollary4 4.

$$C_\gamma^\odot \simeq \mathbb{E}^* \times \mathbb{E}^* \times L^1((-\infty,0),\mathbb{E}^*).$$

REFERENCES

[1] P.L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation", Springer-Verlag, Berlin, Heidelberg, New York 1967.

[2] O. Diekmann, Perturbed dual semigroups and delay equations, Centrum voor Wiskunde en Informatica, Report AM-R8604, 1986.

[3] T. Hagemann and T. Naito, Functional differential equations with infinite delay on the space C_γ, Lecture Notes in Math., 1017, 1983, 207-214.

[4] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcialaj Ekvacioj, 21(1978), 11-41.

[5] Y. Hino, S. Murakami and T. Naito, "Functional Differential Equations with Infinite Delay", to appear.

[6] B. de Pagter, Characterizations of $\odot$-reflexivity, preprint.