

THE APPROXIMATION PROPERTY AND THE CHAIN CONDITION

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1. THE APPROXIMATION PROPERTY

Definition 1.1. Let $\mathbb{P}$ be a poset and κ a cardinal. We say that the poset $\mathbb{P}$ has the κ -approximation property if for every ordinal τ and every $f \in {}^\tau 2^{V^{\mathbb{P}}}$, if $f|_x \in V$ for every $x \in ([\tau]^{<\kappa})^V$, then $f \in V$.

It is known that for an uncountable κ , if $\mathbb{P}$ is an atomless poset of size $< \kappa$ and $\dot{Q}$ is a $\mathbb{P}$ -name for a κ -closed poset, then $\mathbb{P} * \dot{Q}$ has the κ -approximation property (e.g., see Mitchell [1]). In this note, we show that the size assumption for a poset $\mathbb{P}$ can be relaxed to the chain condition assumption.

Definition 1.2. Let κ be a regular uncountable cardinal. A poset $\mathbb{P}$ satisfies the *strong κ -chain condition* (*strong κ -c.c.*, for short) if $\mathbb{P}$ satisfies the κ -c.c. and for every κ -Suslin tree T , $\mathbb{P}$ does not add a cofinal branch of T .

Note 1.3. (1) If there is no κ -Suslin tree, then the κ -c.c. is equivalent to the strong κ -c.c.

(2) For a poset $\mathbb{P}$, if $\mathbb{P} \times \mathbb{P}$ satisfies the κ -c.c., then $\mathbb{P}$ satisfies the strong κ -c.c.

Lemma 1.4. If a poset $\mathbb{P}$ satisfies the μ -c.c. for some $\mu < \kappa$, then $\mathbb{P}$ satisfies the strong κ -c.c. In particular, every poset of size $< \kappa$ satisfies the strong κ -c.c.

Proof. Suppose to the contrary that there is a κ -Suslin tree T such that $\Vdash_{\mathbb{P}} "T \text{ has a cofinal branch } \dot{B}"$. Let $T' = \{t \in T : p \Vdash_{\mathbb{P}} "t \in \dot{B}" \text{ for some } p \in \mathbb{P}\}$. It is easy to check that T' is a downward closed subtree of T of height κ . Since $\mathbb{P}$ satisfies the μ -c.c. and $\mu < \kappa$, each level of T' has size $< \mu$. Now, by Kurepa's theorem, T' has a cofinal branch. Then this branch is a cofinal branch of T , this is a contradiction. $\square$

The following is a main result of this note:

Lemma 1.5. Let κ be a regular uncountable cardinal. Let $\mathbb{P}$ be an atomless poset which satisfies the strong κ -c.c. Let $\dot{Q}$ be a $\mathbb{P}$ -name for a κ -closed poset (trivial poset is possible). Then $\mathbb{P} * \dot{Q}$ has the κ -approximation property.

Proof. Let $\tilde{\mathbb{Q}}$ be a term poset of $\dot{\mathbb{Q}}$, that is, $\tilde{\mathbb{Q}}$ is the set of all $\mathbb{P}$ -names $\dot{q}$ with $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$. For $\dot{q}_0, \dot{q}_1 \in \tilde{\mathbb{Q}}$, define $\dot{q}_0 \leq \dot{q}_1$ if $\Vdash_{\mathbb{P}} \dot{q}_0 \leq \dot{q}_1$ in $\dot{\mathbb{Q}}$. Since $\dot{\mathbb{Q}}$ is a name for a κ -closed poset, $\tilde{\mathbb{Q}}$ is κ -closed.

Let $\dot{x}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$ -name such that $\Vdash \dot{x} \in V$. We say that a condition $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ decides $\dot{x}$ if there is y with $\langle p, \dot{q} \rangle \Vdash \dot{x} = y$.

Claim 1.6. *Let τ be an ordinal and $\dot{f}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$ -name such that $\Vdash \dot{f} : \tau \rightarrow 2$ and $\dot{f}|x \in V$ for every $x \in ([\tau]^{<\kappa})^V$. Let $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ and $x \in [\tau]^{<\kappa}$. Then there are $\dot{q}^* \leq \dot{q}$ and $F \subseteq {}^x 2$ such that:*

- (1) $|F| < \kappa$.
- (2) For every $g \in F$, there is $p' \leq p$ such that $\langle p', \dot{q}^* \rangle \Vdash \dot{f}|x = g$.
- (3) For every $p' \leq p$, there are $p'' \leq p'$ and $g \in F$ such that $\langle p'', \dot{q}^* \rangle \Vdash \dot{f}|x = g$.

Proof. It is easy to check that the set $\{p' \leq p : \exists \dot{q}' (\langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle \text{ and } \langle p', \dot{q}' \rangle \text{ decides } \dot{f}|x)\}$ is predense below p . Take a maximal antichain A which is contained in this set. Since $\mathbb{P}$ satisfies the κ -c.c., we know that $|A| < \kappa$. Then for each $r \in A$, there are $\dot{q}_r$ and g_r such that $\langle r, \dot{q}_r \rangle \leq \langle p, \dot{q} \rangle$ and $\langle r, \dot{q}_r \rangle \Vdash \dot{f}|x = g_r$. Let $F = \{g_r : r \in A\}$ and one can take $\dot{q}^*$ such that $\dot{q}^* \leq \dot{q}$ and $r \Vdash \dot{q}^* = \dot{q}_r$ for every $r \in A$. Then $\dot{q}^*$ and F work. $\square$ [Claim]

In order to show that $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ -approximation property, take $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$, an ordinal τ , and a name $\dot{f}$ such that $\langle p, \dot{q} \rangle \Vdash \dot{f} : \tau \rightarrow 2$ and $\dot{f}|x \in V$ for every $x \in ([\tau]^{<\kappa})^V$. Suppose to the contrary that $\langle p, \dot{q} \rangle \Vdash \dot{f} \notin V$.

By induction on $\alpha < \kappa$, we would find $x_\alpha, \dot{q}_\alpha, F_\alpha$ ($\alpha < \kappa$) such that:

- (1) $x_\alpha \in [\tau]^{<\kappa}$ and $\langle x_\alpha : \alpha < \kappa \rangle$ is $\subseteq$ -increasing.
- (2) $\langle \dot{q}_\alpha : \alpha < \kappa \rangle$ is decreasing in $\tilde{\mathbb{Q}}$ and $\dot{q}_0 \leq \dot{q}$.
- (3) $F_\alpha \subseteq {}^{x_\alpha} 2$ and $|F_\alpha| < \kappa$.
- (4) For every $g \in F_\alpha$, there is $p' \leq p$ such that $\langle p', \dot{q}_\alpha \rangle \Vdash \dot{f}|x_\alpha = g$.
- (5) For every $p' \leq p$ there are $p'' \leq p'$ and $g \in F_\alpha$ such that $\langle p'', \dot{q}_\alpha \rangle \Vdash \dot{f}|x_\alpha = g$, i.e., the set $\{p' \leq p : \langle p', \dot{q}_\alpha \rangle \Vdash \dot{f}|x_\alpha = g \text{ for some } g \in F_\alpha\}$ is predense below p .
- (6) For every $g \in F_\alpha$, there are $g_0, g_1 \in F_{\alpha+1}$ such that $g \subseteq g_0, g_1$ and $g_0 \neq g_1$.

When $\alpha = 0$, pick an arbitrary $x_0 \in [\tau]^{<\kappa}$. Then we can find required $\dot{q}_0 \leq \dot{q}$ and F_0 by Claim 1.6.

Let $\alpha > 0$ and suppose $x_\beta, \dot{q}_\beta, F_\beta$ are defined for all $\beta < \alpha$.

Case 1: α is limit. We can find $x_\alpha \in [\tau]^{<\kappa}$ such that $x_\beta \subseteq x_\alpha$ for $\beta < \alpha$. Since $\tilde{\mathbb{Q}}$ is κ -closed, we can find $\dot{q}^* \leq \dot{q}_\beta$ for every $\beta < \alpha$. Then take $\dot{q}_\alpha \leq \dot{q}^*$ and F_α by Claim 1.6.

Case 2: α is successor, say $\alpha = \beta + 1$. Pick a maximal antichain $A \subseteq \mathbb{P}$ below p such that for every $p' \in A$ there is $g \in F_\beta$ such that $\langle p', \dot{q}_\beta \rangle \Vdash \dot{f}|x_\beta = g$. Note

that $|A| < \kappa$, and, for every $g \in F_\beta$, there is $p' \in A$ with $\langle p', \dot{q}_\beta \rangle \Vdash \dot{f}|x_\beta = g$. Since $|A| < \kappa$ and $\langle p, \dot{q}_\beta \rangle \Vdash \dot{f} \notin V$, we can find $x_\alpha \in [\tau]^{<\kappa}$ such that $x_\beta \subseteq x_\alpha$ for $\beta < \alpha$, but $\langle p', \dot{q}_\beta \rangle$ does not decide $\dot{f}|x_\alpha$ for every $p' \in A$.

Claim 1.7. *For each $p' \in A$, there are $p'_0, p'_1 \leq p'$, $g'_0, g'_1 : x_\alpha \rightarrow 2$, and $\dot{r} \leq \dot{q}_\beta$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{r} \rangle \Vdash \dot{f}|x_\alpha = g'_i$.*

Proof. Since $\langle p', \dot{q}_\beta \rangle$ does not decide $\dot{f}|x_\alpha$, we can take $\langle p'_0, \dot{q}_0 \rangle, \langle p'_1, \dot{q}_1 \rangle \leq \langle p', \dot{q}_\beta \rangle$, and $g'_0, g'_1 : x_\alpha \rightarrow 2$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{q}_i \rangle \Vdash \dot{f}|x_\alpha = g'_i$. We may assume that p'_0 is incompatible with p'_1 ; if p'_0 and p'_1 have a common extension p_2 , take $p''_0, p''_1 \leq p_2$ such that $p''_0 \perp p''_1$ and replace p'_i by p''_i .

Now take $\dot{r} \leq \dot{q}_\beta$ such that $p''_i \Vdash \dot{r} = \dot{q}_i$. Clearly p'_i, g'_i and $\dot{r}$ work. $\square$ [Claim]

For each $p' \in A$, pick $\dot{r}_{p'} \leq \dot{q}_\beta$ such that there are $p'_0, p'_1 \leq p'$, $g'_0, g'_1 : x_\alpha \rightarrow 2$ with $g'_0 \neq g'_1$ and $\langle p'_i, \dot{r}_{p'} \rangle \Vdash \dot{f}|x_\alpha = g'_i$.

Then pick $q^* \leq q_\beta$ such that $p' \Vdash \dot{q}^* = \dot{r}_{p'}$ for every $p' \in A$. Finally, take $\dot{q}_\alpha \leq \dot{q}^*$ and $F_\alpha \subseteq {}^{x_\alpha}2$ as in Claim 1.6. The following claim shows that $x_\alpha, \dot{q}_\alpha$, and F_α work well:

Claim 1.8. *For each $g \in F_\beta$, there are $g_0, g_1 \in F_\alpha$ such that $g_0 \neq g_1$ and $g \subseteq g_0, g_1$.*

Proof. Take $p' \in A$ so that $\langle p', \dot{q}_\beta \rangle \Vdash \dot{f}|x_\beta = g$. Then we can take $p'_0, p'_1 \leq p'$ and $g'_0, g'_1 : x_\alpha \rightarrow 2$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{q}^* \rangle \Vdash \dot{f}|x_\alpha = g'_i$. Clearly $g \subseteq g'_0, g'_1$. By the choice of F_α and $\dot{q}_\alpha$, for each $i < 2$, one can take $p_i \leq p'_i$ and $g_i \in F_\alpha$ such that $\langle p_i, \dot{q}_\alpha \rangle \Vdash \dot{f}|x_\alpha = g_i$. Since $\dot{q}_\alpha \leq \dot{q}^*$, each $\langle p_i, \dot{q}_\alpha \rangle$ is compatible with $\langle p'_i, \dot{q}^* \rangle$. This means that $g'_i = g_i$, so $g_0 \neq g_1$ and $g \subseteq g_0, g_1$. $\square$ [Claim]

Suppose $\dot{q}_\alpha, x_\alpha, F_\alpha$ are defined for $\alpha < \kappa$. Note that, for every $\alpha < \beta < \kappa$ and $g \in F_\beta$, we have $g|x_\alpha \in F_\alpha$; take $p' \leq p$ such that $\langle p', \dot{q}_\beta \rangle \Vdash \dot{f}|x_\beta = g$. Then one can pick $p'' \leq p'$ and $h \in F_\alpha$ such that $\langle p'', \dot{q}_\alpha \rangle \Vdash \dot{f}|x_\alpha = h$. $\langle p', \dot{q}_\beta \rangle$ is compatible with $\langle p'', \dot{q}_\alpha \rangle$. So $h = g|x_\alpha$.

Let $T = \bigcup_{\alpha < \kappa} F_\alpha$. T with the inclusion forms a κ -tree, and each node of T has at least two immediate successors.

Claim 1.9. *T has no antichain of size κ .*

Proof. For each $g \in T$, there are p_g and $\alpha_g < \kappa$ such that $\langle p_g, \dot{q}_{\alpha_g} \rangle \Vdash \dot{f}|x_{\alpha_g} = g$. For g, g' in T , if g and g' are incompatible in T , then p_g is incompatible with $p_{g'}$ in $\mathbb{P}$. This means that if T has an antichain of size κ , then $\mathbb{P}$ also has an antichain of size κ . This is impossible, hence T does not have an antichain of size κ . $\square$ [Claim]

Hence T is a κ -Suslin tree. We finish the proof by showing the following claim, which contradicts the strong κ -c.c. of $\mathbb{P}$:

Claim 1.10. $p \Vdash_{\mathbb{P}} "T \text{ has a cofinal branch}"$.

Proof. Take a $(V, \mathbb{P})$ -generic G with $p \in G$ and work in $V[G]$. Let $\alpha < \kappa$. Since $\{p' \leq p : \langle p', \dot{q}_\alpha \rangle \Vdash "f|_{x_\alpha} = g" \text{ for some } g \in F_\alpha\}$ is predense below p , we can find $p_\alpha \in G$ and $g_\alpha \in F_\alpha \subseteq T$ such that $\langle p_\alpha, \dot{q}_\alpha \rangle \Vdash "f|_{x_\alpha} = g_\alpha"$. Now, for $\alpha < \beta < \kappa$, p_α is compatible with p_β and $\dot{q}_\beta \leq \dot{q}_\alpha$. So $\langle p_\alpha, \dot{q}_\alpha \rangle$ is compatible with $\langle p_\beta, \dot{q}_\beta \rangle$. This means that $g_\alpha \subseteq g_\beta$, so $\{g_\alpha : \alpha < \kappa\}$ is a cofinal branch of T . $\square$ [Claim]

$\square$

Note 1.11. If $\mathbb{P}$ satisfies the κ -c.c. but does not the strong κ -c.c., then $\mathbb{P}$ cannot have the κ -approximation property.

2. APPLICATIONS

We consider some applications of Lemma 1.5.

Definition 2.1. Let κ be a regular uncountable cardinal and $\lambda \geq \kappa$ a cardinal. A set $X \subseteq \mathcal{P}_\kappa \lambda$ has the *strong tree property* if for every $\langle d_x : x \in X \rangle$ with $d_x \subseteq x$, if $|\{d_x \cap a : x \in X\}| < \kappa$ for every $a \in \mathcal{P}_\kappa \lambda$, then there is $D \subseteq \lambda$ such that for every $a \in \mathcal{P}_\kappa \lambda$ the set $\{x \in X : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa \lambda$.

Fact 2.2 (Viale-Weiss [3]). (1) *The following are equivalent:*

- (a) $\mathcal{P}_\kappa \lambda$ has the strong tree property.
- (b) There is some unbounded set $X \subseteq \mathcal{P}_\kappa \lambda$ such that X has the strong tree property.
- (c) Every unbounded subset of $\mathcal{P}_\kappa \lambda$ has the strong tree property.
- (2) κ has the tree property if and only if $\mathcal{P}_\kappa \kappa$ has the strong tree property.
- (3) κ is strongly compact if and only if κ is inaccessible and $\mathcal{P}_\kappa \lambda$ has the strong tree property for every $\lambda \geq \kappa$.
- (4) Suppose Proper Forcing Axiom. Then $\mathcal{P}_{\omega_2} \lambda$ has the strong tree property for every $\lambda \geq \omega_2$.

Viale-Weiss [3] showed that for an inaccessible κ , if a standard κ -stage iteration satisfying the κ -c.c. forces that " $\kappa = \omega_2$ and Proper forcing axiom", then κ must be strongly compact in the ground model. The following is a slight improvement of their result.

Proposition 2.3. *Let κ be a regular uncountable cardinal. Suppose that there is a poset $\mathbb{P}$ which has the strong κ -c.c. and forces that " $\mathcal{P}_\kappa \lambda$ has the strong tree property for every $\lambda \geq \kappa$ ". Then $\mathcal{P}_\kappa \lambda$ has the strong tree property for every $\lambda \geq \kappa$ in the ground model.*

Proof. We check that $\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$. Fix $\lambda \geq \kappa$ and take $\langle d_x : x \in \mathcal{P}_\kappa\lambda \rangle$ such that $d_x \subseteq x$ and $|\{d_x \cap a : x \in \mathcal{P}_\kappa\lambda\}| < \kappa$ for every $a \in \mathcal{P}_\kappa\lambda$. Take a $(V, \mathbb{P})$ -generic G and work in $V[G]$. In $V[G]$, $\mathcal{P}_\kappa^V\lambda$ is unbounded in $\mathcal{P}_\kappa\lambda$ since $\mathbb{P}$ satisfies the κ -c.c. By the strong tree property of $\mathcal{P}_\kappa^V\lambda$ in $V[G]$, we can find $D \subseteq \lambda$ such that $\{x \in \mathcal{P}_\kappa^V\lambda : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa\lambda$ for every $a \in \mathcal{P}_\kappa\lambda$. We see $D \in V$, this completes the proof. For each $a \in \mathcal{P}_\kappa^V\lambda$, there is $x \in \mathcal{P}_\kappa^V\lambda$ with $D \cap a = d_x \cap a \in V$. Thus, by the κ -approximation property of $\mathbb{P}$, we have $D \in V$. $\square$

Next we look at the indestructibility of weak compactness.

Definition 2.4. Let κ be weakly compact. If every κ -directed closed forcing preserves the weak compactness of κ , then κ is said to be *indestructibly weakly compact*.

The existence of an indestructibly weakly compact cardinal is consistent (Laver [2]). The following theorem suggests that the consistency of the existence of an indestructibly weakly compact cardinal might be at least strongly compact cardinal.

Proposition 2.5. *Let κ be a regular uncountable cardinal. If there is a poset which satisfies the strong κ -c.c. and forces that “ κ is indestructibly weakly compact”, then κ is strongly compact.*

Proof. Take $\lambda \geq \kappa$. We see that $\mathcal{P}_\kappa\lambda$ has the strong tree property. Take $\langle d_x : x \in \mathcal{P}_\kappa\lambda \rangle$ with $d_x \subseteq x$ and $|\{d_x \cap a : x \in \mathcal{P}_\kappa\lambda\}| < \kappa$ for every $a \in \mathcal{P}_\kappa\lambda$.

Take a $(V, \mathbb{P})$ -generic G , and a $(V[G], \text{Col}(\kappa, \lambda))$ -generic H . We work in $V[G][H]$. Fix a bijection $\pi : \lambda \rightarrow \kappa$. We know that $\{\pi“x : x \in \mathcal{P}_\kappa^V\lambda\}$ is unbounded in $\mathcal{P}_\kappa\kappa$. Since κ is weakly compact in $V[G][H]$, by the tree property of κ , there is $C \subseteq \kappa$ such that $\{\pi“x \in \mathcal{P}_\kappa\kappa : \pi“(d_x \cap a) = C \cap a\}$ is unbounded for all $a \in \mathcal{P}_\kappa\kappa$. Put $D = \pi^{-1}“C$. Then for every $a \in \mathcal{P}_\kappa\lambda$, the set $\{x \in \mathcal{P}_\kappa^V\lambda : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa\lambda$. We know $D \in V$ since $\mathbb{P} * \text{Col}(\kappa, \lambda)$ has the κ -approximation property by Lemma 1.5. $\square$

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