

Self-similar blow-up for a chemotaxis system in higher dimensional domains

By

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§ 1. Introduction and statement of main results

We consider solutions of a parabolic-elliptic system

$$(1.1) \quad \begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\ 0 = \Delta v + u & \text{in } \Omega \times (0, T), \end{cases}$$

where either $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$ or $\Omega = \mathbf{R}^N$ with $N \geq 3$. In the former case, we assume $\partial u / \partial \nu - u \partial v / \partial \nu = 0$ and $v = 0$ on $\partial \Omega$, where ν denotes the outer unit normal vector. This system arises in the study of the motion of bacteria by chemotaxis as a simplification of the Keller-Segel model (see [16], [22]). Here, u and v represent the density of the bacteria and the concentration of the chemo-attractant, respectively. This system also has been used as a model for the evolution of self-attracting clusters (see [27], [28], [2]).

In this note we consider the blow-up rate of solutions to the system

$$(1.2) \quad \begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\ 0 = \Delta v + u & \text{in } \Omega \times (0, T) \\ \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} = 0 \text{ and } v = 0 & \text{on } \partial \Omega \times (0, T) \\ u(x, 0) = u_0(|x|) & \text{in } \Omega, \end{cases}$$

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where $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$ with $N \geq 3$ and $0 < L < \infty$, and $u_0(r)$ is a nonnegative continuous function on $[0, L]$. We restrict ourselves to the study of radially symmetric solutions. It is known by [2] that the system (1.2) has a unique local classical solution (u, v) . It is easy to see that u and v are positive for $0 < t < T$, and that the conservation of the initial mass of u holds, that is,

$$(1.3) \quad \|u(\cdot, t)\|_1 = \|u_0(\cdot)\|_1 \quad \text{for } 0 < t < T,$$

where $\|\cdot\|_p$ denotes the standard $L^p(\Omega)$ norm for $1 \leq p \leq \infty$. A solution (u, v) is said to blow-up at $t = T < \infty$ if (u, v) is classical in $\Omega \times (0, T)$ and satisfies $\limsup_{t \rightarrow T} \|u(\cdot, t)\|_\infty = \infty$. A simple argument shows that if u blows up at a finite time $t = T$ then

$$\liminf_{t \rightarrow T} (T - t) \|u(\cdot, t)\|_\infty > 0.$$

We say that the blow-up is of type I if u satisfies

$$\limsup_{t \rightarrow T} (T - t) \|u(\cdot, t)\|_\infty < \infty.$$

The blow-up is called type II if it is not type I. We note that self-similar solutions, given by (2.1) below, blow up in type I rate.

We briefly review some known results concerning blow-up behavior for (1.1) and related systems. In the case $N = 2$, Herrero and Velázquez [15] considered the system

$$(1.4) \quad \begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T) \\ \tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0, T), \end{cases}$$

together with initial conditions

$$u(x, 0) = u_0(|x|) \quad \text{and} \quad v(x, 0) = v_0(|x|) \quad \text{for } x \in \Omega,$$

where $\Omega = \{x \in \mathbf{R}^2 : |x| < L\}$ and $\tau > 0$. It was shown in [15] that (1.4) has radially symmetric solutions such that u develops a Dirac delta-type singularity at the origin in a finite time, and that u blows up in type II rate. See also [13], [14]. Senba and Suzuki [25] considered the system (1.4) in the case where Ω is a bounded smooth domain in $\mathbf{R}^2$ and $\tau = 0$ together with the initial condition $u(x, 0) = u_0(x)$ for $x \in \Omega$. Denote by T the maximal existence time of the solution to (1.4). It was shown in [25] that, if $T < \infty$, then the solution (u, v) of (1.4) satisfies

$$(1.5) \quad u(x, t) \rightharpoonup \sum_{q \in \mathcal{B}} m(q) \delta_q + f$$

in the sense of measures as $t \rightarrow T$, where $\mathcal{B}$ is the set of blow-up points, δ_q is the delta function whose support is the point $q \in \overline{\Omega}$, $m(q)$ is the constant satisfying $m(q) \geq 8\pi$ if $q \in \Omega$, and $m(q) \geq 4\pi$ if $q \in \partial\Omega$, and f is a nonnegative function in $L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{B})$. Furthermore, Senba [24] showed that, if $\Omega = \{x \in \mathbf{R}^2 : |x| < L\}$, and if a radial solution (u, v) blows up at $t = T < \infty$, then (1.5) holds with $\mathcal{B} = \{0\}$ and $m(0) = 8\pi$, and u blows up in type II rate. For nonradial case, see [26].

In the case $N = 3$, Herrero et al [11], [12] have investigated the blow-up behavior of solutions by using matched asymptotic expansions. In [12] they showed that (1.2) has a sequence of self-similar blow-up solutions, and they in [11] showed the existence of Burgers like blow-up solutions which are not self-similar. These solutions consist of an imploding smoothed out shock wave that collapses into a Dirac mass when the singularity is formed, and blow up in type II rate. Later, Brenner et al [4] investigated the problem in the case $3 \leq N \leq 9$ by a numerical approach, and showed the existence and stability of both self-similar blow-up solutions and Burgers like blow-up solutions.

For a solution (u, v) to (1.2), putting

$$n(x, t) = \Theta u(x, \Theta t) \quad \text{and} \quad \phi(x, t) = -\Theta v(x, \Theta t)$$

with $\Theta = 1/\|u_0\|_1$, we find that (n, ϕ) solves the problem

$$(1.6) \quad \begin{cases} n_t = \nabla \cdot (\Theta \nabla n + n \nabla \phi) & \text{in } \Omega \times (0, T) \\ \Delta \phi = n & \text{in } \Omega \times (0, T) \\ \Theta \frac{\partial n}{\partial \nu} + n \frac{\partial \phi}{\partial \nu} = 0 \text{ and } v = 0 & \text{on } \partial\Omega \times (0, T) \\ n(x, 0) = n_0(x) & \text{in } \Omega, \end{cases}$$

where $n_0(x) = \Theta u_0(|x|)$ for $x \in \Omega$. Note that n_0 satisfies $\|n_0\|_1 = 1$. Guerra and Peletier [10] considered the problem (1.6) in the case $3 \leq N \leq 9$. They in [10] characterize the blow-up behavior of solutions in terms of initial data, and showed that the solution behaves like a self-similar solution near the blow-up point.

In this note, we consider the system (1.2) in the case $3 \leq N \leq 9$, and derive criteria of the blow-up rate of solutions. In particular, we will identify an explicit class of initial data for which the blow-up is of type I rate.

To state our results, define U_0 and V_0 , respectively, by

$$(1.7) \quad U_0(r) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} u_0(s) ds \quad \text{for } 0 \leq r \leq L$$

and

$$(1.8) \quad V_0(r) = \frac{U_0(r)}{r^2} = \frac{1}{r^N} \int_0^r s^{N-1} u_0(s) ds \quad \text{for } 0 \leq r \leq L.$$

For the initial condition, we assume that

$$(1.9) \quad V'_0(r) \leq 0 \quad \text{for } 0 \leq r \leq L,$$

where $' = d/dr$. It is easy to see that (1.9) holds if u_0 satisfies

$$(1.10) \quad u_0 \in C^1[0, L] \quad \text{and} \quad u'_0(r) \leq 0 \quad \text{for } 0 \leq r \leq L.$$

Our first result is the following.

Theorem 1.1. *Let $3 \leq N \leq 9$, and assume that (1.9) holds.*

- (i) *Suppose U_0 satisfies $U_0(r) \leq 2$ for $0 \leq r \leq L$. Then a solution (u, v) of (1.2) does not blow up in finite time.*
- (ii) *Suppose that $U_0(r) - 2$ has exactly one zero for $0 \leq r < L$ and $U_0(L) > 2$. If a solution (u, v) of (1.2) blows up in finite time, then the blow-up is of type I.*

It should be mentioned that more general criteria will be given in Theorem 3.1 below.

As a consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let $3 \leq N \leq 9$, and assume that (1.9) holds. Suppose that $U_0(r)$ is increasing for $0 < r < L$. If a solution (u, v) of (1.2) blows up in finite time, then the blow-up is of type I.*

Note that U_0 satisfies

$$(r^{N-1}U'_0(r))' = r^{N-1}(2u_0(r) + ru'_0(r)) \quad \text{for } 0 < r < L.$$

Assume that u_0 satisfies (1.10) and

$$(1.11) \quad ru'_0(r) + 2u_0(r) \text{ has at most one zero for } 0 \leq r \leq L.$$

Then one easily see that $U'_0(r)$ has at most one zero for $(0, L]$, and that $U'_0(r) > 0$ for $0 < r < r_0$ and $U'_0(r) < 0$ for $r_0 < r \leq L$ if $U'_0(r)$ has a zero at $r_0 \in (0, L)$. Assume, in addition, that

$$(1.12) \quad \int_0^L s^{N-1}u_0(s)ds > 2L^{N-2}.$$

Then $U_0(L) > 2$ and $U_0(r) - 2$ has exactly one zero for $0 \leq r < L$. By Theorem 1.1 (ii) we obtain the following:

Corollary 1.3. *Let $3 \leq N \leq 9$, and assume that u_0 satisfies (1.10), (1.11) and (1.12). If a solution of (1.2) blows up in finite time, then the blow-up is of type I.*

We recall here some sufficient conditions for blow-up in finite time by [2], [21].

Proposition 1.4. *Let $N \geq 3$. Assume that one of the following (i)-(iii) holds:*

(i) $U_0(L) > 2N$;

(ii) $U_0(L) \geq 4$ and U_0 satisfies, with some $T_0 > 0$,

$$(1.13) \quad U_0(r) \geq \frac{4r^2}{2(N-2)T_0 + r^2} \quad \text{for } 0 \leq r \leq L;$$

(iii) u_0 satisfies

$$(1.14) \quad \int_{|x| \leq L} |x|^N u_0(|x|) dx < \left(\frac{\|u_0\|_1^{(2N-2)/N}}{4(N-1)\omega_N} \right)^{N/(N-2)},$$

where ω_N is the surface area of the unit sphere in $\mathbf{R}^N$.

Then a solution (u, v) of (1.2) blows up in finite time $t = T < \infty$. Furthermore, in the case (ii), the solution blows up at time T with $T \leq T_0$.

The blow-up of solutions was shown in the case (i) by Biler [3, Theorem 3]. We can show the blow-up of solutions in the case (ii) by the comparison argument, and in the case (iii) by following the argument due to Nagai [21, Theorem 3.1]. For the proof of Proposition 1.4, see [20].

As a consequence of Corollaries 1.2 and 1.3 and Proposition 1.4, we can show the existence of solutions which blow up with type I rate. As a simple example, let $u_0(r) \equiv \ell$ with $\ell > 2N/L$. Then a solution (u, v) of (1.2) blows up in finite time with type I rate by Corollary 1.2 and Proposition 1.4 (i). (See [10, Corollary 1.2].) For another example, let $u_0(r) = \ell G(r, \tau)$ with $\tau > 0$ and $\ell > 0$, where $G(r, t) = (4\pi t)^{-N/2} e^{-r^2/4t}$ is the heat kernel. Then (1.10) and (1.11) hold, and it is easy to see that

$$\|u_0\|_1 = \omega_N \int_0^L s^{N-1} u_0(s) ds \rightarrow \ell \quad \text{and} \quad \int_{|x| \leq L} |x|^N u_0(|x|) dx \rightarrow 0$$

as $\tau \rightarrow 0$. Combining Corollary 1.3 and Proposition 1.4 (iii), we obtain the following:

Corollary 1.5. *Let $3 \leq N \leq 9$, and let $u_0(r) = \ell G(r, \tau)$ with $\tau > 0$ and $\ell > 2L^{N-2}\omega_N$. Then there exists $\tau_0 > 0$ such that, if $\tau \in (0, \tau_0]$, then a solution (u, v) of (1.2) blows up in finite time with type I rate.*

We note that, in Corollary 1.5, the initial function u_0 converges to a Dirac delta function in the sense of measure as $\tau \rightarrow 0$. Thus this corollary suggests that self-similar blow-up may be seen even if initial function is close to a Dirac delta function.

Next, we consider the local blow-up profile of solutions to (1.2). Assume that V_0 , defined by (1.8), satisfies

$$(1.15) \quad (V_0)_{rr} + \frac{N+1}{r}(V_0)_r + N(V_0)^2 + rV_0(V_0)_r \geq 0 \quad \text{for } 0 \leq r \leq L.$$

Guerra and Peletier [10] showed that, when $N \geq 3$ and (1.9) and (1.15) hold, any type I blow-up solution behaves like a self-similar solution near the singularity $x = 0$.

Our result is the following.

Theorem 1.6. *Let $3 \leq N \leq 9$, and assume that (1.9) and (1.15) hold. If a solution (u, v) of (1.2) blows up in finite time, then the blow-up is of type I.*

Remark. Define the average density function V by

$$(1.16) \quad V(r, t) = \frac{1}{r^N} \int_0^r s^{N-1} u(s, t) ds \quad \text{for } 0 \leq r \leq L, \ 0 \leq t < T.$$

It was shown by Guerra and Peletier [10, Theorem 2.3] that when $N \geq 3$ and (1.9) and (1.15) hold, if a solution (u, v) blows up in finite time $t = T$ with type I rate, then V satisfies

$$(1.17) \quad \lim_{t \rightarrow T} (T - t) V(\rho \sqrt{T - t}, t) = \Phi(\rho) / \rho^2$$

uniformly on compact set $|\rho| \leq C$ for every $C > 0$, where Φ is a certain positive function. Combining with Theorem 1.6, we find that when $3 \leq N \leq 9$ and (1.9) and (1.15) hold, if a solution (u, v) blows up in finite time, then (1.17) holds. Note here that the condition (1.15) ensures that $V_t \geq 0$ for all $0 < t < T$. (See (3.6) below.) It is still an open problem whether (1.17) holds for type I blow-up solutions without the condition (1.15).

We note that similar results hold for the well-studied problem for semilinear heat equation

$$(1.18) \quad \begin{cases} u_t - \Delta u = u^p & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(|x|) & \text{in } \Omega, \end{cases}$$

where $p > 1$, $\Omega = \{x \in \mathbf{R}^N : |x| < L\}$, and $u_0(r)$ is nonnegative and nonincreasing for $0 \leq r \leq L$. A simple comparison argument shows that any blow-up solution satisfies

$$\liminf_{t \rightarrow T} (T - t)^{1/(p-1)} \|u(\cdot, t)\|_\infty > 0.$$

Assume that $u_0 = u_0(|x|)$ satisfies

$$(1.19) \quad \Delta u_0 + u_0^p \geq 0 \quad \text{in } \Omega.$$

By the maximum principle, the condition (1.19) implies that $u_t \geq 0$ for all $0 < t < T$. Friedman and McLeod [6] showed that, if (1.19) holds, then any blow-up solution satisfies

$$(1.20) \quad \limsup_{t \rightarrow T} (T - t)^{1/(p-1)} \|u(\cdot, t)\|_\infty < \infty.$$

Bebernes and Eberly showed in [1] that, under the condition (1.19), finite time blow-up solutions are asymptotically self-similar. Precisely, any solution u of (1.18) which blows up in finite time $t = T$ satisfies

$$\lim_{t \rightarrow T} (T - t)^{1/(p-1)} u((T - t)^{1/2} y, t) = \kappa$$

uniformly on compact set $|y| \leq C$ for every $C > 0$ with $\kappa = (p-1)^{-1/(p-1)}$. It should be mentioned that Matos [19] later showed that any blow-up solution which satisfies (1.20) is asymptotically self-similar in the supercritical case without the condition (1.19). For the precise characterization of the behavior of blow-up solutions to (1.18), we refer to Giga and Kohn [7, 8, 9] in the subcritical case and Matano and Merle [17, 18] in the supercritical case.

This note is organized as follows: In Section 2, we will show the existence of a sequence of self-similar solutions to (1.1) with $\Omega = \mathbf{R}^N$. In Section 3, we derive criteria of the blow-up rate of solutions, and give the proof of Theorems 1.1 and 1.6.

§ 2. Backward self-similar solutions

The proof of Theorems 1.1 and 1.6 are based on the study of the properties of backward self-similar solutions to the system (1.1) with $\Omega = \mathbf{R}^N$. The system (1.1) with $\Omega = \mathbf{R}^N$ is invariant under the scaling

$$(u, v) \mapsto (u_\lambda, v_\lambda) = (\lambda^2 u(\lambda x, \lambda^2 t), v(\lambda x, \lambda^2 t))$$

for $\lambda > 0$. A solution (u, v) is called *self-similar* if $(u, v) = (u_\lambda, v_\lambda)$ for each $\lambda > 0$, and is called *backward* if (u, v) is defined for all $t < 0$. By the transformation in the time, a backward self-similar solution has the form

$$(2.1) \quad u(x, t) = \frac{1}{T - t} \phi(x/\sqrt{T - t}) \quad \text{and} \quad v(x, t) = \psi(x/\sqrt{T - t})$$

for $x \in \mathbf{R}^N$ and $t < T$, where (ϕ, ψ) satisfies

$$(2.2) \quad \begin{cases} \Delta \phi - \nabla \cdot \left(\phi \left(\frac{x}{2} + \nabla \psi \right) \right) + \frac{N-2}{2} \phi = 0, & x \in \mathbf{R}^N \\ 0 = \Delta \psi + \phi, & x \in \mathbf{R}^N. \end{cases}$$

We will obtain the existence of a sequence of self-similar solutions to (2.2) together with the properties of solutions. For the proof, see [20].

Theorem 2.1. *Let $3 \leq N \leq 9$. Then the system (2.2) has radially symmetric solutions $\{(\phi_j, \psi_j)\}_{j=1}^\infty$ such that $\phi_j(r) > 0$ for $r \geq 0$ and $\phi_j(0) \rightarrow \infty$ as $j \rightarrow \infty$. For each $j = 1, 2, \dots$, define*

$$(2.3) \quad \Phi_j(r) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} \phi_j(s) ds, \quad r > 0.$$

Then $\Phi_j(r) - 2$ has exactly $2j$ zeros on $(0, \infty)$ and no zeros on (R_0, ∞) with $R_0 = 2\sqrt{N-1}$. Furthermore, there exists a sequence $\{\alpha_j\}_{j=1}^\infty$ satisfying $0 < \dots < \alpha_{j+1} < \alpha_j < \dots < \alpha_1$ and $\alpha_j \rightarrow 0$ as $j \rightarrow \infty$ such that the following (i) and (ii) hold.

(i) *For any constant $c > 0$,*

$$\inf_{0 < r < c\alpha_j} \frac{\Phi_j(r)}{r^2} \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

(ii) *For any $\varepsilon > 0$, there exist a constant $c_0 = c_0(\varepsilon) > 0$ and an integer $j_0 = j_0(\varepsilon) \in \mathbf{N}$ such that if $j \geq j_0$ then*

$$\sup_{r \geq c_0 \alpha_j} |\Phi_j(r) - 2| < \varepsilon \quad \text{and} \quad \sup_{r \geq c_0 \alpha_j} |r\Phi_j'(r)| < \varepsilon.$$

Remark. For each fixed $r > 0$, we have

$$(2.4) \quad r^2 \phi_j(r) \rightarrow 2(N-2) \quad \text{as } j \rightarrow \infty.$$

In fact, it follows from (2.3) that

$$r^2 \phi_j(r) = (N-2)\Phi_j(r) + r\Phi_j'(r) \quad \text{for } r > 0.$$

Since (ii) holds and $\alpha_j \rightarrow 0$ as $j \rightarrow \infty$, we obtain (2.4).

The existence of self-similar solutions was already shown by [11, 4, 23]. It seems, however, that the properties on the location of zeros and properties (i) and (ii) are new, and these properties play an important role in the proof of the theorems.

§ 3. Proof of Theorems 1.1 and 1.6 (sketch)

We restrict our attention to radially symmetric solutions to (1.2) of the form $u = u(r, t)$ and $v = v(r, t)$, $r = |x|$, and consider the system

$$(3.1) \quad \begin{cases} r^{N-1}u_t = (r^{N-1}u_r)_r - (r^{N-1}uv_r)_r, & 0 < r < L, \ 0 < t < T, \\ 0 = (r^{N-1}v_r)_r + r^{N-1}u, & 0 < r < L, \ 0 < t < T, \\ u_r(0, t) = u_r(L, t) - u(L, t)v_r(L, t) = 0, & 0 < t < T, \\ v_r(0, t) = v(L, t) = 0, & 0 < t < T, \\ u(r, 0) = u_0(r), & 0 \leq r \leq L. \end{cases}$$

Put

$$M = \int_0^L r^{N-1} u_0(r) dr.$$

Then, by the third formula in (3.1), it follows that

$$(3.2) \quad \int_0^L r^{N-1} u(r, t) dr = M \quad \text{for } 0 \leq t < T.$$

Put $\hat{u}$ by

$$\hat{u}(r, t) = \int_0^r s^{N-1} u(s, t) ds = -r^{N-1} v_r(r, t).$$

Here we have used the second formula in (3.1). Then the system (3.1) can be reduced to a single equation

$$\hat{u}_t = r^{N-1} (r^{1-N} \hat{u}_r)_r + r^{1-N} \hat{u} \hat{u}_r.$$

Define

$$(3.3) \quad U(r, t) = r^{2-N} \hat{u}(r, t) = \frac{1}{r^{N-2}} \int_0^r s^{N-1} u(s, t) ds.$$

Then U satisfies

$$(3.4) \quad U_t = U_{rr} + \frac{N-3}{r} U_r - \frac{2(N-2)}{r^2} U + \frac{(N-2)U^2 + rUU_r}{r^2}$$

for $0 < r < L, 0 < t < T$ and

$$U(0, t) = \lim_{r \rightarrow 0} U(r, t) = 0 \quad \text{and} \quad U(L, t) = ML^{2-N} \quad \text{for } 0 \leq t < T.$$

Note here that, by using l'Hospital's rule, we obtain

$$(3.5) \quad \lim_{r \rightarrow 0} \frac{U(r, t)}{r^2} = \lim_{r \rightarrow 0} \frac{\int_0^r s^{N-1} u(s, t) ds}{r^N} = \frac{u(0, t)}{N} \quad \text{for } 0 < t < T.$$

Put $V(r, t) = U(r, t)/r^2$. Then V satisfies

$$(3.6) \quad V_t = V_{rr} + \frac{N+1}{r} V_r + NV^2 + rVV_r \quad \text{for } 0 < r < L, 0 < t < T$$

and $V(L, t) = ML^{-N}$ for $0 < t < T$. We will show here that

$$(3.7) \quad V_r(0, t) = \lim_{r \rightarrow 0} V_r(r, t) = 0 \quad \text{for } 0 < t < T.$$

In fact,

$$\begin{aligned} V_r(r, t) &= \frac{1}{r} \left(u(r, t) - Nr^{-N} \int_0^r s^{N-1} u(s, t) ds \right) \\ &= \frac{u(r, t) - u(0, t)}{r} - N \frac{\int_0^r s^{N-1} (u(s, t) - u(0, t)) ds}{r^{N+1}}. \end{aligned}$$

By using l'Hospital's rule, we obtain

$$\lim_{r \rightarrow 0} \frac{\int_0^r s^{N-1} (u(s, t) - u(0, t)) ds}{r^{N+1}} = \frac{1}{N+1} \lim_{r \rightarrow 0} \frac{u(r, t) - u(0, t)}{r} = \frac{u_r(0, t)}{N+1}.$$

Since $u_r(0, t) = 0$ for $0 \leq t < T$, we obtain (3.7).

Let (ϕ_j, ψ_j) be a radially symmetric solutions of (2.2) obtained in Theorem 2.1, and put Φ_j by (2.3). Take $T > 0$, and put

$$(3.8) \quad U_j(r, t) = \Phi_j(r/\sqrt{T-t}) = \frac{1}{r^{N-2}(T-t)} \int_0^r s^{N-1} \phi_j(s/\sqrt{T-t}) ds$$

for $0 \leq r \leq L$, $0 \leq t < T$. Then $U = U_j$ solves (3.4) and

$$U_j(0, t) = 0 \quad \text{and} \quad U_j(L, t) = \Phi_j(L/\sqrt{T-t}) \quad \text{for } 0 < t < T.$$

By the similar argument as in (3.5) we obtain

$$(3.9) \quad \lim_{r \rightarrow 0} \frac{U_j(r, t)}{r^2} = \frac{\phi_j(0)}{N(T-t)} \quad \text{for } 0 < t < T.$$

Put $V_j(r, t) = U_j(r, t)/r^2$. Then $V = V_j$ solves (3.6) and

$$(V_j)_r(0, t) = 0 \quad \text{and} \quad V_j(L, t) = \Phi_j(L/\sqrt{T-t})L^{-2} \quad \text{for } 0 < t < T.$$

By using the zero number properties of solutions for linear parabolic equations [5], we will derive criteria of the blow-up rate of solutions to (1.2) in terms of the function U defined by (3.3). We obtain Theorems 1.1 and 1.6 as a consequence of the following result.

Theorem 3.1. *Let $3 \leq N \leq 9$, and assume that (1.9) holds. Let (u, v) be a radially symmetric solution of (1.2) for $0 \leq t < T$, and define U by (3.3).*

(i) *Assume that there exist $t_0 \in [0, T)$ and $r_0 \in (0, L]$ such that*

$$U(r, t_0) \leq 2 \quad \text{for } 0 \leq r \leq r_0 \quad \text{and} \quad U(r_0, t) \leq 2 \quad \text{for } t_0 \leq t < T.$$

Then the solution (u, v) does not blow up at $t = T$.

(ii) *Assume that there exist $t_0 \in [0, T)$ and $r_0 \in (0, L]$ such that $U(r, t_0) - 2$ has exactly one zero for $0 \leq r \leq r_0$ and $U(r_0, t) > 2$ for $t_0 \leq t < T$. If the solution (u, v) blows up at $t = T < \infty$ then the blow-up is of type I.*

It is clear that U_0 , defined by (1.7), satisfies $U_0(r) = U(r, 0)$ for $0 \leq r \leq L$. By the property (3.2) we see that $U(L, t) = U_0(L)$ for $0 \leq t < T$. Then, by applying Theorem 3.1 with $r_0 = L$ and $t_0 = 0$, we obtain Theorem 1.1 immediately.

Proof of Theorem 1.6. Assume that (u, v) blows up at $t = T < \infty$. By Theorem 3.1 (i) we have $\{r \in (0, L) : U(r, t) > 2\} \neq \emptyset$ for any $0 \leq t < T$. It is easy to see that there exist $t_0 \in (0, T)$ and $r_0 \in (0, L)$ such that $U(r_0, t_0) > 2$ and $U(r, t_0) - 2$ has exactly one zero for $0 < r < r_0$. Note that the condition (1.15) ensures that $V_t \geq 0$ for all $t \in (0, T)$. Then $U(r_0, t)$ is nondecreasing in $t \in (t_0, T)$, and hence $U(r_0, t) > 2$ for $t_0 \leq t < T$. By Theorem 3.1 (ii), the blow-up is of type I. $\square$

We will give a sketch of the proof of Theorem 3.1. For the detail, see [20].

Proof of Theorem 3.1. (Sketch). (i) By using the comparison argument, we may assume that $U(r, t) < 2$ for $0 \leq r < r_0$ and $t_0 < t \leq T$. Take $\hat{T} > T$, and define $\hat{U}_j$ by

$$\hat{U}_j(r, t) = \Phi_j \left(\frac{r}{\sqrt{\hat{T} - t}} \right) \quad \text{for } 0 \leq r \leq L, t_0 \leq t < \hat{T},$$

where $\{\Phi_j\}_{j=1}^\infty$ is a sequence of function obtained in Theorem 2.1. By using the properties in Theorem 2.1, we will find that there exists $j_0 \in \mathbf{N}$ such that, if $j = j_0$, then

$$(3.10) \quad \hat{U}_j(r, t_0) > U(r, t_0) \quad \text{for } 0 < r \leq r_0,$$

and

$$(3.11) \quad \hat{U}_j(r_0, t) > U(r_0, t) \quad \text{for } t_0 \leq t \leq T.$$

Put V and $\hat{V}_j$ by

$$V(r, t) = \frac{U(r, t)}{r^2} \quad \text{and} \quad \hat{V}_j(r, t) = \frac{\hat{U}_j(r, t)}{r^2},$$

respectively. Then V and $\hat{V}$ solve (3.6) and satisfy $V_r(0, t) = \hat{V}_r(0, t) = 0$ for $t_0 \leq t < T$. It follows from (3.10) and (3.11) that

$$\hat{V}_j(r, t_0) > V(r, t_0) \quad \text{for } 0 \leq r \leq r_0 \quad \text{and} \quad \hat{V}_j(r_0, t) > V(r_0, t) \quad \text{for } t_0 \leq t < T.$$

Then, by the maximum principle, we obtain $V(r, t) < \hat{V}_j(r, t)$ for $0 \leq r \leq r_0$, $t_0 \leq t < T$. From (3.5) and (3.9) we see that

$$\lim_{r \rightarrow 0} V(r, t) = \lim_{r \rightarrow 0} \frac{U(r, t)}{r^2} = \frac{u(0, t)}{N} \quad \text{and} \quad \lim_{r \rightarrow 0} \hat{V}_j(r, t) = \lim_{r \rightarrow 0} \frac{\hat{U}_j(r, t)}{r^2} = \frac{\phi_j(0)}{N(\hat{T} - t)}.$$

This implies that $u(0, t) < \phi_j(0)/(\hat{T} - t)$ for $t_0 \leq t < T < \hat{T}$. Note that (1.9) implies $u(0, t) = \|u(\cdot, t)\|_\infty$. Then $\sup_{0 \leq t < T} \|u(\cdot, t)\|_\infty < \infty$, and hence (u, v) does not blow up at $t = T$.

(ii) For a continuous function ψ defined on an interval J , we define the zero number of the function ψ on J by $\mathcal{Z}_J[\psi] = \#\{r \in J : \psi(r) = 0\}$. We will find that $\mathcal{Z}_{[0, r_0]}[U(\cdot, t) - 2] = 1$ for $t_0 \leq t < T$, and we may assume that $T - t_0 > 0$ is small enough so that

$$(3.12) \quad \frac{r_0}{\sqrt{T - t_0}} > R_0,$$

where R_0 is the constant which appears in Theorem 2.1.

Define U_j by (3.8) for $j = 1, 2, \dots$. First we show that, for each $j = 1, 2, \dots$,

$$(3.13) \quad U_j(r_0, t) < U(r_0, t) \quad \text{for } t_0 \leq t < T.$$

Since $\Phi_j(r) - 2$ has exactly $2j$ zeros on $(0, R_0]$ and no zeros on (R_0, ∞) by Theorem 2.1, we see that $\Phi_j(r) < 2$ for $r > R_0$. From (3.12) we obtain $U_j(r_0, t) < 2$ for $t_0 \leq t < T$. Since $U(r_0, t) > 2$ for $t_0 \leq t < T$, we obtain (3.13).

By using the properties in Theorem 2.1, we will find that there exists $j_0 \in \mathbf{N}$ such that, if $j \geq j_0$, then $U_j(r, t_0) - U(r, t_0)$ has exactly one zero for $0 \leq r < r_0$. Put V and V_j by $V(r, t) = U(r, t)/r^2$ and $V_j(r, t) = U_j(r, t)/r^2$, respectively. Then $\mathcal{Z}_{[0, r_0]}[V_j(\cdot, t_0) - V(\cdot, t_0)] = 1$. By the zero number property, we obtain $\mathcal{Z}_{[0, r_0]}[V_j(\cdot, t) - V(\cdot, t)] = 1$ for $t_0 < t < T$. We denote by $\tilde{r}(t)$ a unique zero of $V_j(r, t) - V(r, t)$. Then $\tilde{r}(t) \in (0, r_0)$ for $t_0 \leq t < T$, and $V(r, t) < V_j(r, t)$ for $0 \leq r < \tilde{r}(t)$. From (3.5) and (3.9) we obtain, for each $t \in [t_0, T)$,

$$\lim_{r \rightarrow 0} V(r, t) = \lim_{r \rightarrow 0} \frac{U(r, t)}{r^2} = \frac{u(0, t)}{N} \quad \text{and} \quad \lim_{r \rightarrow 0} V_j(r, t) = \lim_{r \rightarrow 0} \frac{U_j(r, t)}{r^2} = \frac{\phi_j(0)}{N(T - t)}.$$

Then it follows that $(T - t)u(0, t) \leq \phi_j(0)$ for $t_0 \leq t < T$. This implies that the blow-up is of type I. $\square$

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