

Numerical Analysis of Subharmonic Solutions to Duffing's Equation

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1. Preliminaries

The present paper is concerned with subharmonic solutions of Duffing's equation

$$(1.1) \quad \frac{d^2 q}{d\tau^2} + \kappa^2 q (1 + \beta q^2) = P_0 \cos \gamma \tau.$$

By the transformation

$$\kappa \tau = t, \quad \frac{\kappa^2}{P_0} q = x, \quad \frac{\gamma}{\kappa} = \omega, \quad \frac{\beta P_0^2}{\kappa^4} = \epsilon,$$

equation (1.1) can be reduced to the equation

$$(1.2) \quad \frac{d^2 x}{dt^2} + x(1 + \epsilon x^2) = \cos \omega t,$$

which, replacing ωt by t , one can rewrite as follows:

$$(1.3) \quad \frac{d^2 x}{dt^2} + \frac{1}{\Omega} x(1 + \epsilon x^2) = \frac{1}{\Omega} \cos t,$$

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where

$$(1.4) \quad \Omega = \omega^2 .$$

To a subharmonic solution of (1.1), corresponds a solution of (1.3) of the form

$$(1.5) \quad x(t) = a_1 \cos \frac{t}{3} + a_2 \cos t + \dots .$$

Hence replacing t by $3t$ in (1.3) and (1.5), one can reduce the problem to the one to find a solution of the form

$$(1.6) \quad x(t) = a_1 \cos t + a_2 \cos 3t + \dots$$

to the equation

$$(1.7) \quad \frac{d^2 x}{dt^2} + \frac{9}{\Omega} x(1 + \epsilon x^2) = \frac{9}{\Omega} \cos 3t .$$

In the present paper, approximations of solutions of the form (1.6) will be computed by Galerkin's method for various values of the parameters ϵ and ω . The existence of subharmonic solutions corresponding to these approximations will be assured from approximations themselves by the author's method developed in [1]. Error estimates of approximations computed and the stability of the corresponding subharmonic solutions will be given also.

2. Galerkin's method

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Consider a real periodic differential system

$$(2.1) \quad \frac{dx}{dt} = X(x, t),$$

where x and $X(x, t)$ are vectors and $X(x, t)$ is periodic t of period 2π . To get an approximation of a periodic solution of (2.1), we consider a trigonometric polynomial

$$(2.2) \quad x_m(t) = a_0 + \sum_{n=1}^m (a_{2n-1} \sin nt + a_{2n} \cos nt)$$

with unknown coefficients $\alpha = (a_0, a_1, a_2, \dots, a_{2m-1}, a_{2m})$, and we determine these unknown coefficients by the equation

$$(2.3) \quad \begin{aligned} \frac{dx_m(t)}{dt} = & \frac{1}{2\pi} \int_0^{2\pi} X[x_m(t), t] dt \\ & + \frac{1}{\pi} \sum_{n=1}^m \left\{ \sin nt \cdot \int_0^{2\pi} X[x_m(s), s] \sin ns ds \right. \\ & \left. + \cos nt \cdot \int_0^{2\pi} X[x_m(s), s] \cos ns ds \right\}. \end{aligned}$$

Equation (2.3) is evidently equivalent to the equation

$$(2.4) \quad \begin{cases} F_0(\alpha) \triangleq \frac{1}{2\pi} \int_0^{2\pi} X[x_m(t), t] dt = 0, \\ F_{2n-1}(\alpha) \triangleq \frac{1}{\pi} \int_0^{2\pi} X[x_m(t), t] \sin nt dt + na_{2n} = 0, \end{cases}$$

$$\left\{ \begin{aligned} F_{2n}(\alpha) &\triangleq \frac{1}{\pi} \int_0^{2\pi} X[x_m(t), t] \cos nt \, dt - na_{2n-1} = 0 \\ (n=1, 2, \dots, m). \end{aligned} \right.$$

A trigonometric polynomial with coefficients satisfying (2.4) will be called a Galerkin approximation of a periodic solution of (2.1) and the equation (2.4) will be called the determining equation of Galerkin approximations. A method of getting an approximation of a periodic solution by computing a Galerkin approximation is called Galerkin's method.

Galerkin's method is based mathematically on the following theorem due to the author [1].

Theorem 1. Suppose that $X(x, t)$ and its Jacobian matrix $\Psi(x, t)$ with respect to x are continuously differentiable with respect to x and t in the region $D \times L$, where D is a closed bounded region of the x -space and L is the real line. If differential system (2.1) possesses an isolated periodic solution $x = \hat{x}(t)$ lying inside D , then for sufficiently large m_0 , there is a Galerkin approximation $x = \bar{x}_m(t)$ to any order $m \geq m_0$ such that

$$\bar{x}_m(t) \longrightarrow \hat{x}(t), \quad \dot{\bar{x}}_m(t) \longrightarrow \dot{\hat{x}}(t) \quad (\cdot = d/dt)$$

uniformly as $m \longrightarrow \infty$.

By an isolated periodic solution, is meant a periodic solution such that the multipliers of the relative first variation equation are all different from one.

To compute a Galerkin approximation, it is necessary to

solve numerically the determining equation, which is usually nonlinear. Newton's method is well adapted for this purpose, because as will be shown later, the starting approximations can be found usually without difficulty and the iterative process can be practised easily on a computer by means of well-known formulas of Fourier analysis (See[2]).

3. Assurance of the existence of an exact periodic solution and error estimation of approximate solutions

Our method is based on the following theorem due to the author [1].

Theorem 2. Suppose that (2.1) possesses a periodic approximate solution $x=x(t)$ lying inside D and the multipliers of the relative first variation equation

$$(3.1) \quad \frac{dy}{dt} = \Psi[x(t), t]y$$

are all different from one.

Let $\Phi(t)$ be the fundamental matrix of (3.1) satisfying the initial condition $\Phi(0) = E$ (E is the unit matrix) and $H(t, s) = (H_{kl}(t, s))$ be a piecewise continuous matrix such that

$$(3.2) \quad H(t, s) = \begin{cases} \Phi(t)[E - \Phi(2\pi)]^{-1} \Phi^{-1}(s) & \text{for } 0 \leq s \leq t \leq 2\pi, \\ \Phi(t)[E - \Phi(2\pi)]^{-1} \Phi(2\pi) \Phi^{-1}(s) & \text{for } 0 \leq t < s \leq 2\pi, \end{cases}$$

Let M be a positive number such that

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$$(3.3) \quad [2\pi \cdot \max_{0 \leq t \leq 2\pi} \int_0^{2\pi} \sum_{k,l} H_{kl}^2(t,s) ds]^{1/2} \leq M,$$

and r be a non-negative number such that

$$(3.4) \quad \left\| \frac{d\bar{x}(t)}{dt} - X[x(t), t] \right\| \leq r,$$

where the symbol $\|\dots\|$ denotes the Euclidean norm of vectors or the corresponding norm of matrices.

If there are positive constants δ and $\kappa < 1$ such that

$$(3.5) \quad \begin{aligned} (i) \quad D_\delta &= \{x \mid \|x - \bar{x}(t)\| \leq \delta \text{ for some } t\} \subset D, \\ (ii) \quad \|\psi[x, t] - \psi[\bar{x}(t), t]\| &\leq \kappa/M \\ &\text{for all } x, t \text{ such that } \|x - \bar{x}(t)\| \leq \delta, \\ (iii) \quad Me/(1-\kappa) &\leq \delta, \end{aligned}$$

then the given system (2.1) possesses one and only one periodic solution $x = \hat{x}(t)$ in D_δ and this is an isolated periodic solution. Further, for $x = \hat{x}(t)$, it holds that

$$(3.6) \quad \|\bar{x}(t) - \hat{x}(t)\| \leq \frac{Mr}{1-\kappa}.$$

When a Galerkin approximation $\bar{x}_m(t)$ has been computed, one can find the corresponding value of the number M through the numerical integration of (3.1) for $\bar{x}(t) = \bar{x}_m(t)$ and one can find also the corresponding value of the number r by evaluating the Fourier coefficients of the function

$$\frac{d\bar{x}_m(t)}{dt} = X[\bar{x}_m(t), t].$$

Then one can easily check the existence of the constants δ and κ satisfying the condition (3.5). If there exist such constants δ and κ , then by Theorem 2 one can assure the existence of a periodic solution of (2.1) and further by (3.6) one can get an error bound for the Galerkin approximation $\bar{x}_m(t)$.

Remark. In the computation of the number M , one computes the fundamental matrix $\Phi(t)$ of (3.1) satisfying the initial condition $\Phi(0) = E$. If $\bar{x}(t)$ is close to the exact solution $\hat{x}(t)$, then the eigenvalues of $\Phi(2\pi)$ is close to the multipliers of the first variation equation of (2.1) with respect to the exact solution. Hence one can decide the stability of an exact periodic solution by inspecting the module of eigenvalues of the matrix $\Phi(2\pi)$.

4. Numerical computation of subharmonic solutions

1° To get a rough approximation of a periodic solution of (1.7), we consider a trigonometric polynomial of the form

$$(4.1) \quad x = a_1 \cos t + a_2 \cos 3t.$$

Then the determining equation for a polynomial of the above form is

$$(4.2) \quad \begin{cases} a_1 [(9-\Omega) + \frac{27}{4} \varepsilon (a_1^2 + a_1 a_2 + 2a_2^2)] = 0, \\ (1 - \Omega) a_2 + \frac{\varepsilon}{4} (a_1^3 + 6a_1^2 a_2 + 3a_2^3) - 1 = 0 \end{cases}$$

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Since $a_1=0$ does not correspond to the desired subharmonic solution, we assume that

$$(4.3) \quad a_1 \neq 0.$$

Then from (4.2) follows

$$(4.4) \quad \begin{cases} (9-\Omega) + \frac{27}{4} \varepsilon (a_1^2 + a_1 a_2 + 2a_2^2) = 0, \\ (1-\Omega)a_2 + \frac{\varepsilon}{4} (a_1^3 + 6a_1^2 a_2 + 3a_2^3) - 1 = 0. \end{cases}$$

From the first of (4.4) we have

$$(4.5) \quad \Omega = 9 + \frac{27}{4} \varepsilon (a_1^2 + a_1 a_2 + 2a_2^2).$$

Hence substituting (4.5) into the second of (4.4), we have

$$(4.6) \quad 51a_2^3 + 27a_1 a_2^2 + 21a_1^2 a_2 - a_1^3 + \frac{4}{\varepsilon} (8a_2 + 1) = 0.$$

Let us consider the case where

$$(4.7) \quad \varepsilon > 0.$$

Then from (4.5), real solutions of (4.4) can exist only for

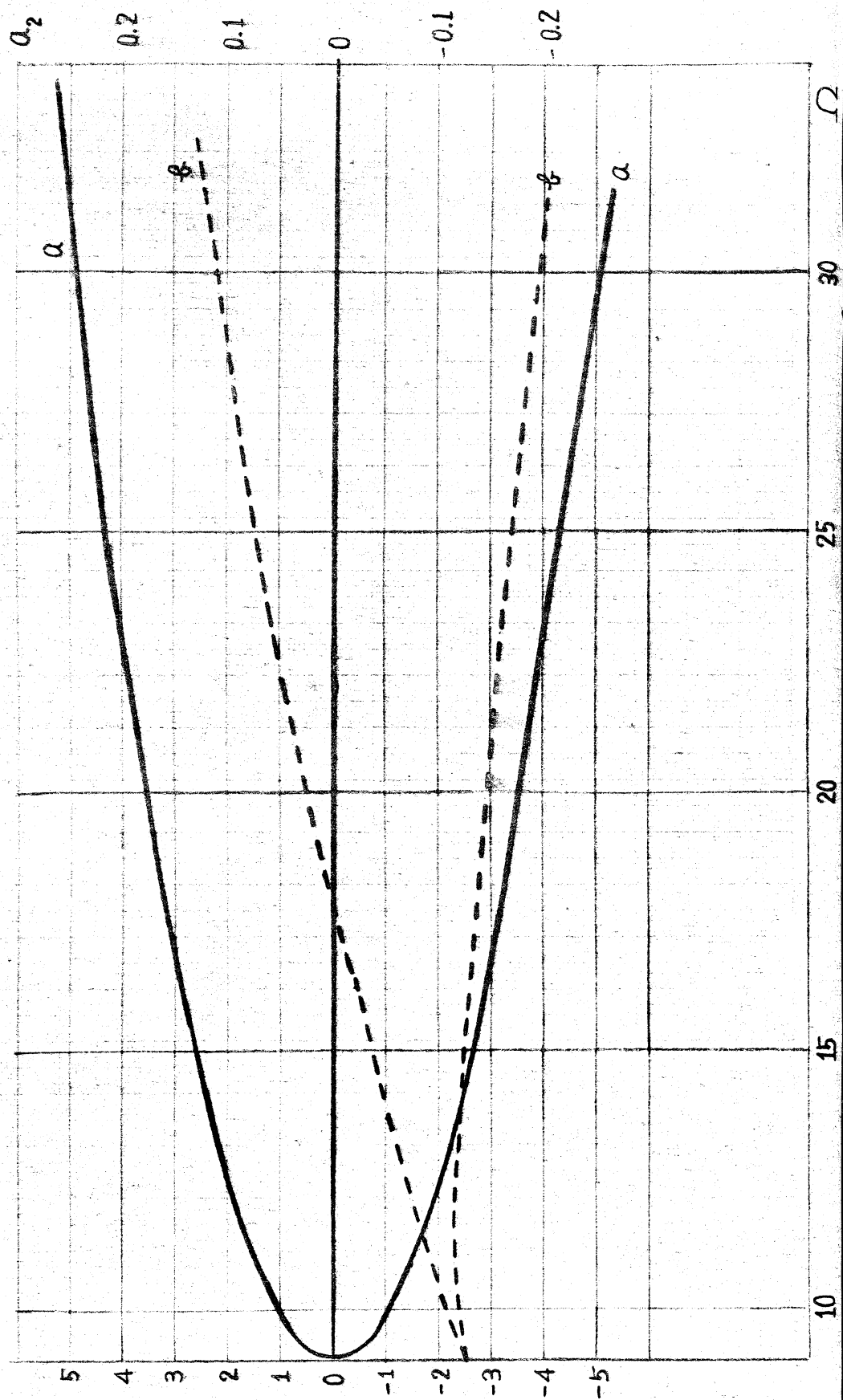
$$(4.8) \quad \Omega > 9.$$

Now the derivative of the left member of (4.6) with respect to a_2 is always positive and hence (4.6) can have only one solution a_2 for any given value of a_1 . Such being the case, for $\varepsilon=1/8$, $1/2$, 1 and $a_1=-5(1)5$, we have computed a_2 satisfying (4.6)

by Newton's method and then we have computed the corresponding values of Ω by (4.5). Drawing the graphs of (Ω, a_1) and (Ω, a_2) from the results obtained, we then have the approximate solutions of (4.4) for given values of Ω . Starting from these approximate solutions, we finally get the solutions of (4.4) by Newton's method for various values of Ω . Figure 1 shows the graphs of (Ω, a_1) and (Ω, a_2) for $\epsilon=1/8$ and Table 1 shows the solutions of (4.4) obtained in the above way for $\epsilon=1/8$.

TABLE 1

ω	Ω	a_1		a_2	
4	16	2.88938	9313	-0.01832	4240
		-2.81281	9167	-0.12545	0320
3.2	10.24	1.25675	9007	-0.10488	5156
		-1.14426	3122	-0.11640	0863
3.1	9.61	0.89420	1639	-0.11557	8278
		-0.77575	5977	-0.11942	2055
3.05	9.3025	0.63741	8236	-0.12059	7225
		-0.51585	2082	-0.12171	4132

Q_1 $\xi = 1/8$ 

2° To get a Galerkin approximation of a periodic solution of (1.7), we consider a trigonometric polynomial of the form

$$(4.9) \quad x(t) = \sum_{n=1}^m a_n \cos (2n-1)t.$$

If $x=x(t)$ is a solution of (1.7), then $x=x(-t)$ and $x=-x(t+\pi)$ are also solutions of (1.7). This suggests the form (4.9) of a periodic solution.

We have computed solutions of the determining equation by Newton's method starting from

$$a_1 = \bar{a}_1, \quad a_2 = \bar{a}_2, \quad a_3 = a_4 = \dots = a_m = 0,$$

where $(a_1, a_2) = (\bar{a}_1, \bar{a}_2)$ is a solution of equation (4.4) obtained in 1°. The Fourier coefficients appearing in the determining equation have been computed by the following formula [2]:

$$\frac{1}{\pi} \int_0^{2\pi} f(t) \cos pt \, dt = \frac{1}{N} \sum_{i=1}^{2N} f(t_i) \cos pt_i$$

$$(p=1, 2, \dots, \nu),$$

where

$$t_i = \frac{2i-1}{2N} \pi \quad (i=1, 2, \dots, 2N)$$

and

$$N \geq \nu+1.$$

Table 2 shows the values of coefficients a_n obtained in the above way with $N=64$.

TABLE 2

 $A_1 : \varepsilon=1/8, \quad \omega=4$

n	a_n		a_n	
1	2.88939	64903	-2.81264	00834
2	-0.01834	34265	-0.12557	78220
3	-0.00034	06838	-0.00232	01890
4	-0.00000	20357	-0.00007	13067
5	0.00000	00117	-0.00000	19406
6	0.00000	00002	-0.00000	00528
7	0.0		-0.00000	00014
8	:		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

 $A_2 : \varepsilon=1/8, \quad \omega=3.2$

n	a_n		a_n	
1	1.25671	76442	-1.14419	82013
2	-0.10489	29084	-0.11640	92077
3	-0.00052	42595	-0.00057	89532
4	0.00002	26175	-0.00002	82413
5	-0.00000	02233	-0.00000	07476
6	-0.00000	00080	-0.00000	00117
7	0.00000	00001	-0.00000	00003
8	0.0		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

$A_3 : \epsilon = 1/8, \omega = 3.1$

n	a_n		a_n	
1	0.89416	37367	-0.77570	72093
2	-0.11558	04311	-0.11942	42151
3	-0.00029	52928	-0.00030	39846
4	0.00002	15640	-0.00002	06942
5	-0.00000	04842	-0.00000	07033
6	-0.00000	00063	-0.00000	00064
7	0.00000	00002	-0.00000	00002
8	0.0		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

$A_4 : \epsilon = 1/8, \omega = 3.05$

n	a_n		a_n	
1	0.63738	94700	-0.51581	55306
2	-0.12059	77217	-0.12171	46120
3	-0.00015	03858	-0.00015	14040
4	0.00001	74607	-0.00001	45577
5	-0.00000	06320	-0.00000	07096
6	-0.00000	00038	-0.00000	00033
7	0.00000	00002	-0.00000	00002
8	0.0		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

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$B_1 : \varepsilon=1/2, \omega=4$

n	a_n		a_n	
1	1.46165	38180	-1.38496	30370
2	-0.04555	49212	-0.09836	10693
3	-0.00084	49916	-0.00180	57494
4	0.00000	59508	-0.00007	68957
5	0.00000	02492	-0.00000	26025
6	-0.00000	00018	-0.00000	00865
7	-0.00000	00001	-0.00000	00030
8	0.0		-0.00000	00001
.	.		0.0	
.	.		.	
.	.		.	
15	0.0		0.0	

$B_2 : \varepsilon=1/2, \omega=3.2$

n	a_n		a_n	
1	0.64303	26531	-0.53088	90496
2	-0.10832	12797	-0.11289	93867
3	-0.00051	46997	-0.00053	17972
4	0.00005	10049	-0.00004	80110
5	-0.00000	13945	-0.00000	23212
6	-0.00000	00368	-0.00000	00375
7	0.00000	00014	-0.00000	00019
8	0.0		-0.00000	00001
9	.		0.0	
.	.		.	
.	.		.	
15	0.0		0.0	

$B_3 : \epsilon=1/2, \omega=3.1$

n	a_n	a_n
1	0.45439 19264	-0.33641 48660
2	-0.11695 64970	-0.11793 31876
3	-0.00026 33429	-0.00026 41626
4	0.00004 53681	-0.00003 46282
5	-0.00000 22006	-0.00000 25224
6	-0.00000 00253	-0.00000 00203
7	0.00000 00017	-0.00000 00015
8	0.0	-0.00000 00001
9		0.0
15	0.0	0.0

$B_4 : \epsilon=1/2, \omega=3.05$

n	a_n	a_n
1	0.31340 53558	-0.19237 42156
2	-0.12113 75427	-0.12103 80304
3	-0.00011 05583	-0.00011 03701
4	0.00003 47781	-0.00002 13758
5	-0.00000 26472	-0.00000 27122
6	-0.00000 00131	-0.00000 00086
7	0.00000 00015	-0.00000 00010
8	-0.00000 00001	-0.00000 00001
9	0.0	0.0
10	0.0	0.0
11	0.0	0.0
12	0.0	0.0
13	0.0	0.0
14	0.0	0.0
15	0.0	0.0

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$C_1 : \varepsilon=1, \omega=4$

n	a_n		a_n	
1	1.04265	32218	-0.96604	99115
2	-0.05366	54105	-0.09024	34248
3	-0.00099	23715	-0.00164	48577
4	0.00001	80707	-0.00008	58323
5	0.00000	04148	-0.00000	33351
6	-0.00000	00123	-0.00000	01232
7	-0.00000	00002	-0.00000	00049
8	0.0		-0.00000	00002
9	:		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

$C_2 : \varepsilon=1, \omega=3.2$

n	a_n		a_n	
1	0.45801	73216	-0.34637	31608
2	-0.10946	46843	-0.11164	50565
3	-0.00048	31870	-0.00048	78993
4	0.00007	49046	-0.00006	06922
5	-0.00000	31416	-0.00000	42355
6	-0.00000	00747	-0.00000	00644
7	0.00000	00045	-0.00000	00045
8	-0.00000	00001	-0.00000	00002
9	0.0		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

$C_3 : \epsilon=1, \omega=3.1$

n	a_n		a_n	
1	0.31600	21647	-0.19866	15617
2	-0.11741	22723	-0.11732	07872
3	-0.00021	64496	-0.00021	57463
4	0.00006	37913	-0.00004	03006
5	-0.00000	45871	-0.00000	48456
6	-0.00000	00459	-0.00000	00309
7	0.00000	00050	-0.00000	00033
8	-0.00000	00002	-0.00000	00002
9	0.0		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

$C_4 : \epsilon=1, \omega=3.05$

n	a_n		a_n	
1	0.19874	95570	-0.07843	84541
2	-0.12123	55889	-0.12075	36311
3	-0.00005	65814	-0.00005	70524
4	0.00004	42107	-0.00001	73127
5	-0.00000	53693	-0.00000	53370
6	-0.00000	00176	-0.00000	00078
7	0.00000	00039	-0.00000	00015
8	-0.00000	00002	-0.00000	00003
9	0.0		0.0	
:	:		:	
:	:		:	
15	0.0		0.0	

5. Existence of subharmonic solutions, error estimation of Galerkin approximations and stability of subharmonic solutions

Equation (1.7) is equivalent to the system

$$(5.1) \quad \begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\frac{9}{\Omega} (x + \epsilon x^3) + \frac{9}{\Omega} \cos 3t. \end{cases}$$

The Jacobian matrix $\Psi(x, y, t)$ of the right member of (5.1) with respect to x and y is

$$(5.2) \quad \Psi(x, y, t) = \begin{bmatrix} 0 & 1 \\ -\frac{9}{\Omega}(1+3\epsilon x^2) & 0 \end{bmatrix}.$$

For an approximate solution $(x, \bar{y}) = (x(t), \bar{y}(t))$, we then have

$$\|\Psi[x, y, t] - \Psi[\bar{x}(t), \bar{y}(t), t]\| = \frac{27}{\Omega} \epsilon |x^2 - \bar{x}^2(t)|.$$

Hence, if

$$|x - \bar{x}(t)| \leq \delta,$$

then we have

$$(5.3) \quad \|\Psi[x, y, t] - \Psi[\bar{x}(t), \bar{y}(t), t]\| \leq \frac{27}{\Omega} \epsilon \delta (\delta + 2|\bar{x}(t)|).$$

Thus for system (5.1), condition (3.5) is satisfied if

$$(5.4) \quad \begin{cases} \frac{27}{\Omega} \epsilon \delta (\delta + 2 \max_t |x(t)|) \leq \frac{\kappa}{M}, \\ \frac{Mr}{1-\kappa} \leq \delta. \end{cases}$$

The method of verification of the above inequalities will be illustrated with an example.

Consider the case where $\varepsilon=1/8$, $\omega=3.05$ ($\Omega=9.3025$) and

$$(5.5) \quad \bar{x}(t) = 0.63738 \ 94700 \cos t - 0.12059 \ 77217 \cos 3t \\ - 0.00015 \ 03858 \cos 5t + 0.00001 \ 74607 \cos 7t \\ - 0.00000 \ 06320 \cos 9t - 0.00000 \ 00038 \cos 11t \\ + 0.00000 \ 00002 \cos 13t.$$

First, putting

$$\frac{d^2 \bar{x}(t)}{dt^2} + \frac{9}{\Omega} [\bar{x}(t) + \varepsilon \bar{x}^3(t)] - \frac{9}{\Omega} \cos 3t = \sum_{k=1}^{\infty} d_k \cos(2k-1)t,$$

we compute

$$\sum_{k=1}^{25} |d_k|,$$

which is found to be $0.86207 \ 96094 \times 10^{-8}$ for (5.5).

Hence we may take r so that

$$(5.6) \quad r = 8.63 \times 10^{-9}.$$

Next, by the author's method developed in [3,4], we compute the fundamental matrix $\Phi(t)$ of the first variation equation of (5.1) with respect to $(x, y) = [\bar{x}(t), d\bar{x}(t)/dt]$ in the form

$$\Phi(t) = \frac{1}{2} A_0 + \sum_{n=1}^{30} A_n T_n\left(\frac{t}{\pi} - 1\right),$$

where $T_n(t)$ ($n=1, 2, \dots, 30$) are Chebyshev polynomials.

By (3.2), we then compute

$$H(t_p, t_q) \quad (p, q=0, 1, 2, \dots, 256),$$

where

$$t_p = p\pi/128 \quad (p=0, 1, 2, \dots, 256).$$

Making use of the values of $H(t_p, t_q)$ obtained, we compute the integrals

$$\int_0^{2\pi} \sum_{k, \ell} H_{k\ell}^2(t_p, s) ds \quad (p=0, 2, 4, \dots, 256)$$

by Simpson's rule with mesh $h=1/128$. By (3.3), we then compute

$$[2\pi \cdot \max_p \int_0^{2\pi} \sum_{k\ell} H_{k\ell}^2(t_p, s) ds]^{1/2},$$

which is found to be $0.10143\ 69424 \times 10^3$ for (5.5). Hence for (5.5), we may take M so that

$$(5.7) \quad M = 101.5$$

Now for (5.5),

$$(5.8) \quad \max_t |\bar{x}(t)| = 0.75815\ 56742.$$

Hence by (5.6), (5.7) and (5.8), the inequalities of (5.4) can be written as follows:

$$(5.9) \left\{ \begin{array}{l} \frac{27}{9.3025} \times 0.125 \times (+1.51631 \ 13484) \delta < \frac{\kappa}{101.5} , \\ \frac{101.5 \times 8.63 \times 10^{-9}}{1 - \kappa} < \delta \end{array} \right.$$

Since

$$101.5 \times 8.3 \times 10^{-9} = 8.75945 \times 10^{-7},$$

we assume tentatively that

$$(5.10) \quad \delta < 9 \times 10^{-7}.$$

Then the first inequality of (5.9) is satisfied if

$$\frac{27}{9.3025} \times 0.125 \times 1.51631 \ 22484 \times \delta < \frac{\kappa}{101.5},$$

that is,

$$\frac{9.3025}{27 \times 0.125 \times 1.51631 \ 22484 \times 101.5} \kappa$$

$$= 0.01790 \ 8374 \dots \times \kappa.$$

Then both inequalities of (5.9) are satisfied if

$$(5.11) \quad \frac{8.75945 \times 10^{-7}}{1 - \kappa} < \delta < 0.01790 .$$

Consider the inequality

$$(5.12) \quad 8.75945 \times 10^{-7} < 0.01790 \kappa (1 - \kappa).$$

Since

$$\frac{8.75945 \times 10^{-7}}{0.01790} = 4.893547 \dots \times 10^{-5},$$

we readily see that inequality (5.12) is satisfied by

$$(5.13) \quad \kappa = 5 \times 10^{-5}.$$

For this value of κ ,

$$\frac{8.75945 \times 10^{-7}}{1-\kappa} = 8.75988 \ 79943 \ \dots \times 10^{-7},$$

$$0.01790\kappa = 8.950 \ \dots \times 10^{-7}.$$

Hence taking into account the inequality (5.10), from (5.11) we have

$$(5.14) \quad 8.75989 \times 10^{-7} < \delta < 8.950 \times 10^{-7},$$

which shows the existence of the positive numbers δ and $\kappa < 1$ satisfying (5.4). Thus by Theorem 2 we see the existence of an isolated periodic solution $x = \hat{x}(t)$ of (1.7) such that

$$(5.15) \quad |\hat{x}(t) - \bar{x}(t)| \leq 8.760 \times 10^{-7}.$$

This proves that (1.7) really possesses a subharmonic solution close to $x = \bar{x}(t)$ given by (5.5) and further that the approximation (5.5) of a subharmonic solution is within the error bound 8.760×10^{-7} .

Now, if $x = \hat{x}(t)$ is a periodic solution of (1.7), then $x = \hat{x}(-t)$ and $x = -\hat{x}(t+\pi)$ are also periodic solutions of (1.7). However by (5.14) and (5.15),

$$\begin{cases} |\hat{x}(-t) - \bar{x}(t)| = |\hat{x}(-t) - \bar{x}(-t)| \leq \delta \\ |[-\hat{x}(t+\pi)] - \bar{x}(t)| = |[-\hat{x}(t+\pi)] - [-\bar{x}(t+\pi)]| \leq \delta. \end{cases}$$

Since there exists only one periodic solution in D_δ by Theorem 2, we then see that

$$\hat{x}(-t) = -\hat{x}(t+\pi) = \hat{x}(t),$$

which shows that $x=\hat{x}(t)$ is of the form

$$(5.16) \quad \hat{x}(t) = \sum_{n=1}^{\infty} a_n \cos(2n-1)t.$$

From the value of $\phi(2\pi)$, we readily see that the eigenvalues λ_1, λ_2 of $\phi(2\pi)$ are

$$\lambda_1, \lambda_2 = 0.99321 \ 02880 \pm 0.11633 \ 28121i,$$

which implies

$$|\lambda_1| = |\lambda_2| = 0.99999 \ 99996.$$

This shows that the stability of $x=\hat{x}(t)$ will be neutral.

For the case where $\epsilon=1/8$, $\omega=3.05$ ($\Omega=9.3025$) and

$$(5.17) \quad \begin{aligned} \bar{x}(t) = & -0.51581 \ 55306 \cos t - 0.12171 \ 46120 \cos 3t \\ & -0.00015 \ 14040 \cos 5t - 0.00001 \ 45577 \cos 7t \\ & -0.00000 \ 07096 \cos 9t - 0.00000 \ 00033 \cos 11t \\ & -0.00000 \ 00002 \cos 13t, \end{aligned}$$

we have

$$r = 1.31 \times 10^{-8}, \quad M=123.7.$$

Hence we again get the assurance for the existence of a subharmonic solution, and for (5.17), we get the error bound 1.6207×10^{-6} .

However in this case, the eigenvalues of $\Phi(2\pi)$ are found to be

$$1.09964 \ 4524 \quad \text{and} \quad 0.90938 \ 47852,$$

which show that the subharmonic solution close to (5.17) is unstable.

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