

On N-Fractional Calculus of the Function $((z - b)^2 - c)^{-4}$

Tsuyako Miyakoda

Abstract

We discuss the N-fractional calculus of $f(z) = ((z - b)^2 - c)^{-4}$. By applying the fractional calculus, we have three kinds of the representation for γ -th differintegral of the function $((z - b)^2 - c)^{-4}$ from the different way. And some identities are reported.

1 Introduction

We adopt the following definition of the fractional calculus.

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{\nu+1}} \quad (\nu \notin Z^-), \end{aligned} \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in Z^+), \quad (2)$$

where

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C_-, \quad 0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C_+,$$

$\zeta \neq z$, $z \in C$, $\nu \in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{\nu+1}} \right) \quad (\nu \notin Z^-), \quad (\text{Refer to [1]}) \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in Z^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\alpha (N^\beta f) \quad (\alpha, \beta \in R), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [4]

(III)

In some previous papers, the following result are known as elementary properties.

Lemma. We have [1]

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii),

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

Moreover in the previous works we refer to the next theorem [6].

Theorem D. We have

(i)

$$\begin{aligned} (((z-b)^\beta - c)^\alpha)_\gamma &= e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k \\ &\quad (|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}| < \infty), \end{aligned} \tag{9}$$

and

(ii)

$$\begin{aligned} (((z-b)^\beta - c)^\alpha)_n &= (-1)^n (z-b)^{\alpha\beta-n} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \\ &\quad (n \in \mathbb{Z}_0^+, \quad |\frac{c}{(z-b)^\beta}| < 1), \end{aligned} \tag{10}$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \text{with } [\lambda]_0 = 1,$$

(Pochhammer's Notation).

2 N-Fractional Calculus of the Functions $f(z) = ((z-b)^2 - c)^{-4}$

In order to have a representation of N-fractional calculus with γ -order, we directly apply the theorem to the function at the beginning.

Theorem 1. Let

$$f = f(z) = ((z-b)^2 - c)^{-4} \quad \left(((z-b)^2 - c)^4 \neq 0 \right) \quad (1)$$

we have

$$(f)_\gamma = e^{-i\pi\gamma}(z-b)^{-8-\gamma} \sum_{k=0}^{\infty} \frac{[4]_k \Gamma(2k+8+\gamma)}{k! \Gamma(2k+8)} \left(\frac{c}{(z-b)^2} \right)^k \quad (2)$$

Proof. According to Theorem D, we have the equation (2) directly.

Secondly, we consider the function as a product of two functions like as

$$f(z) = ((z-b)^2 - c)^{-5} \cdot ((z-b)^2 - c),$$

and we have the new representation for $(f)_\gamma$ as follows.

Theorem 2. We set $f = f(z)$, and S, H, G as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \quad (3)$$

$$H(k, \gamma, m) = \frac{[5]_k \Gamma(2k+10+\gamma-m)}{k! \Gamma(2k+10)} S^k, \quad (4)$$

$$G(\gamma, m) = \sum_{k=0}^{\infty} H(k, \gamma, m). \quad (5)$$

We have

$$(f)_\gamma = e^{-i\pi\gamma}(z-b)^{-8-\gamma} \{ (1-S)G(\gamma, 0) - 2\gamma G(\gamma, 1) + \gamma(\gamma-1)G(\gamma, 2) \} \quad (6)$$

Proof. According to Lemma (iv), we have

$$(f)_\gamma = \left(((z-b)^2 - c)^{-5} \cdot ((z-b)^2 - c) \right)_\gamma \quad (7)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma+1-k)} \left(((z-b)^2 - c)^{-5} \right)_{\gamma-k} \cdot ((z-b)^2 - c)_k \quad (8)$$

and applying Theorem D.(i) to

$$(((z-b)^2 - c)^{-5})_{\gamma-k}, \quad (9)$$

we obtain

$$\begin{aligned} (f)_\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)} (((z-b)^2 - c)^{-5})_\gamma ((z-b)^2 - c)_0 \\ &\quad + \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)} (((z-b)^2 - c)^{-5})_{\gamma-1} (2(z-b)) \\ &\quad + \frac{\Gamma(\gamma+1)}{2!\Gamma(\gamma-1)} (((z-b)^2 - c)^{-5})_{\gamma-2} \cdot 2 \\ &= (((z-b)^2 - c)^{-5})_\gamma ((z-b)^2 - c) + 2\gamma (((z-b)^2 - c)^{-5})_{\gamma-1} \cdot (z-b) \\ &\quad + \gamma(\gamma-1) (((z-b)^2 - c)^{-5})_{\gamma-2} \\ &= e^{-i\pi\gamma} (z-b)^{-10-\gamma} ((z-b)^2 - c) \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+\gamma)}{k! \Gamma(2k+10)} \left(\frac{c}{(z-b)^2} \right)^k \\ &\quad + 2\gamma (z-b) e^{-i\pi(\gamma-1)} (z-b)^{-10-\gamma+1} \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+\gamma-1)}{k! \Gamma(2k+10)} \left(\frac{c}{(z-b)^2} \right)^k \\ &\quad + \gamma(\gamma-1) e^{-i\pi(\gamma-2)} (z-b)^{-10-\gamma+2} \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+\gamma-2)}{k! \Gamma(2k+10)} \left(\frac{c}{(z-b)^2} \right)^k \end{aligned} \quad (10)$$

Then we have the representation

$$(f(z))_\gamma = e^{-i\pi\gamma} (z-b)^{-8-\gamma} \{(1-S)G(\gamma, 0) - 2\gamma G(\gamma, 1) + \gamma(\gamma-1)G(\gamma, 2)\}. \quad (11)$$

This is the same one as the equation (6).

Next, we choose another process of the fractional calculus which is divided into two stages as like as

$$(f(z))_\gamma = ((f(z))_1)_{\gamma-1}. \quad (12)$$

We have an another result.

Theorem 3. We set $f = f(z)$, and S, R, W as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \quad (13)$$

$$R(k, \gamma, m) = \frac{[5]_k \Gamma(2k + 10 + \gamma - m)}{k! \Gamma(2k + 10)} S^k, \quad (14)$$

$$W(\gamma, m) = \sum_{k=0}^{\infty} R(k, \gamma, m). \quad (15)$$

Then we have

$$(f)_{\gamma} = 8e^{-i\pi\gamma}(z-b)^{-8-\gamma}\{W(\gamma, 1) - (\gamma-1)W(\gamma, 2)\}. \quad (16)$$

Proof. We have

$$\begin{aligned} \left(((z-b)^2 - c)^{-4} \right)_1 &= -4((z-b)^2 - c)^{-5} \cdot 2(z-b) \\ &= -8((z-b)^2 - c)^{-5} \cdot (z-b) \end{aligned} \quad (17)$$

Then

$$\begin{aligned} \left((((z-b)^2 - c)^{-4})_1 \right)_{\gamma-1} &= -8 \left(((z-b)^2 - c)^{-5}(z-b) \right)_{\gamma-1} \\ &= -8 \sum_{k=0}^{\infty} \frac{\Gamma(\gamma)}{k! \Gamma(\gamma-k)} \left(((z-b)^2 - c)^{-5} \right)_{\gamma-1-k} (z-b)_k \\ &= -8 \left\{ \frac{\Gamma(\gamma)}{\Gamma(\gamma)} \left(((z-b)^2 - c)^{-5} \right)_{\gamma-1} (z-b) + \frac{\Gamma(\gamma)}{\Gamma(\gamma-1)} \left(((z-b)^2 - c)^{-5} \right)_{\gamma-1-1} \right\} \\ &= -8 \left\{ e^{-i\pi(\gamma-1)} (z-b)^{-10-\gamma+2} \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+\gamma-1)}{k! \Gamma(2k+10)} \left(\frac{c}{(z-b)^2} \right)^k \right. \\ &\quad \left. + (\gamma-1) e^{-i\pi(\gamma-2)} (z-b)^{-10-\gamma+2} \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+\gamma-2)}{k! \Gamma(2k+10)} \left(\frac{c}{(z-b)^2} \right)^k \right\} \end{aligned} \quad (18)$$

And we put

$$R(k, \gamma, m) = \frac{[5]_k \Gamma(2k+10+\gamma-m)}{k! \Gamma(2k+10)} \left(\frac{c}{(z-b)^2} \right)^k,$$

$$W(\gamma, m) = \sum_{k=0}^{\infty} R(k, \gamma, m).$$

So we have

$$(f(z))_{\gamma} = 8e^{-i\pi\gamma}(z-b)^{-8-\gamma}\{W(\gamma, 1) - (\gamma-1)W(\gamma, 2)\}, \quad (\gamma \notin \mathbb{Z}^-). \quad (19)$$

We have the equation (16) from above equation directly.

3 Identities

We have three kinds of representation on N-fractional calculus of the function $f(z) = ((z-b)^2 - c)^{-4}$ like as Theorem 1, 2 and 3. Accordingly we have the following identities with using S and G and W given in §3.

Theorem 4. We have

(i)

$$\sum_{k=0}^{\infty} \frac{[4]_k \Gamma(2k+8+\gamma)}{k! \Gamma(2k+8)} S^k = (1-S)G(\gamma, 0) - 2\gamma G(\gamma, 1) + \gamma(\gamma-1)G(\gamma, 2), \quad (\gamma \notin Z^-) \quad (1)$$

and

(ii)

$$\sum_{k=0}^{\infty} \frac{[4]_k \Gamma(2k+8+\gamma)}{k! \Gamma(2k+8)} S^k = 8\{W(\gamma, 1) - (\gamma-1)W(\gamma, 2)\}. \quad (\gamma \notin Z^-) \quad (2)$$

Proof. From Theorems 2 and 3 we can obtain above equations directly.

4 A Special Case

In order to make sure of the formulations of Theorem 2 and 3, we consider the case of the integer $\gamma = 1$. When $\gamma = 1$, from Theorem 2, we have

$$\begin{aligned} \left(((z-b)^2 - c)^{-4} \right)_1 &= -(z-b)^{-9} \{ (1-S)G(1, 0) - 2G(1, 1) \} \\ &= -(z-b)^{-9} \left\{ (1-S) \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+1)}{k! \Gamma(2k+10)} S^k \right. \\ &\quad \left. + 2 \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10)}{k! \Gamma(2k+10)} S^k \right\}. \end{aligned} \quad (1)$$

And we notice following relations,

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda} \quad (2)$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{[\lambda]_k k}{k!} T^k &= \sum_{k=0}^{\infty} \frac{[\lambda]_k}{(k-1)!} T^k \\
&= \sum_{k=0}^{\infty} \frac{[\lambda]_{k+1}}{k!} T^{k+1} \\
&= \lambda T \sum_{k=0}^{\infty} \frac{[\lambda+1]_k}{k!} T^k = \lambda T (1-T)^{-1-\lambda} \quad (3)
\end{aligned}$$

$$[\lambda]_{k+1} = \frac{\Gamma(\lambda+1+k)}{\Gamma(\lambda)} = \lambda[\lambda+1]_k \quad (4)$$

Then, we have the following relations with applying to the above euations.

$$\begin{aligned}
(f)_1 &= -(z-b)^{-9} \left\{ (1-S) \sum_{k=0}^{\infty} \frac{[5]_k (2k+10)}{k!} S^k \right. \\
&\quad \left. - 2 \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10)}{k! \Gamma(2k+10)} S^k \right\} \\
&= -(z-b)^{-9} \left\{ 2(1-S) \sum_{k=0}^{\infty} \frac{[5]_k k}{k!} S^k + 10(1-S) \sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^k \right. \\
&\quad \left. - 2 \sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^k \right\} \\
&= -(-b)^{-9} \{ 10S(1-S)^{-5} + 10(1-S)^{-4} - 2(1-S)^{-5} \} \\
&= -8(z-b)^{-9} (1-S)^{-5} \\
&= \frac{-8(z-b)}{((z-b)^2 - c)^5} \quad (5)
\end{aligned}$$

And from Theorem 3, we have

$$\begin{aligned}
(f)_1 &= 8e^{-i\pi} (z-b)^{-9} \{W(1,1)\} \\
&= 8e^{-i\pi} (z-b)^{-9} \sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10)}{k! \Gamma(2k+10)} S^k \\
&= 8e^{-i\pi} (z-b)^{-9} \sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^k \\
&= 8e^{-i\pi} (z-b)^{-9} (1-S)^{-5} \\
&= -8 \frac{(z-b)}{((z-b)^2 - c)^5}. \quad (6)
\end{aligned}$$

Therefore we have the same results from two different forms of N-fractional calculus for the function $((z-b)^2 - c)^{-4}$.

Now these results are consistent with the one of the classical calculus of

$$\frac{d}{dz}((z-b)^2 - c)^{-4}. \quad (7)$$

Here we confirm again the result for Theorem 1.

When $\gamma = 1$, from Theorem 1.(2), we have

$$\begin{aligned} \left(((z-b)^2 - c)^{-4} \right)_1 &= -(z-b)^{-9} \sum_{k=0}^{\infty} \frac{[4]_k [2k+8]_1}{k!} S^k \\ &= -(z-b)^{-9} \sum_{k=0}^{\infty} \frac{[4]_k (2k+8)}{k!} S^k \\ &= -(z-b)^{-9} \left\{ 2 \sum_{k=0}^{\infty} \frac{[4]_k k}{k!} S^k + 8 \sum_{k=0}^{\infty} \frac{[4]_k}{k!} S^k \right\} \\ &= -(z-b)^{-9} \left\{ 8S \sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^k + 8 \sum_{k=0}^{\infty} \frac{[4]_k}{k!} S^k \right\} \\ &= -(z-b)^{-5} \{ 8S(1-S)^{-5} + 8(1-S)^{-4} \} \\ &= -(z-b)^{-9} \frac{8}{(1-S)^5} \\ &= -8(z-b)((z-b)^2 - c)^{-5} \end{aligned} \quad (8)$$

This result also coincides with the one obtained by the classical calculus.

So we conclude that according to the definition of fractional differintegration, we have three forms for γ -th differintegrate of the function $\frac{1}{((z-b)^2 - c)^4}$ by Theorem 1 . 2 and 3.

We made sure that they have the same results as the classical result when the differential order is in the case of $\gamma = 1$.

References

- [1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.
- [2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.
- [3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^ν (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

- [4] K. Nishimoto ; Ring and Field Produced from The Set of N- Fractional Calculus Operator, J. Frac Calc. Vol. 24, Nov. (2003),29 - 36.
- [5] K. Nishimoto ; N-Fractional Calculus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May (2005), 83 - 88.
- [6] K. Nishimoto ; N-Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006), 35 - 44.
- [7] K. Nishimoto ; N-Fractional Calculus of Some Composite Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 11 - 23.
- [8] K. Nishimoto and T. Miyakoda ; N-Fractional Calculus and n -th Derivatives of Some Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 53 - 62.
- [9] T. Miyakoda ; N-Fractional Calculus of Certain Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 63 - 76.
- [10] K. Nishimoto and T. Miyakoda ; N-Fractional Calculus of Some Multiple Power Functions and Some Identities, J. Frac. Calc. Vol. 34, Nov. (2008), 11 - 22.

Tsuyako Miyakoda
Kansai Medical University,
Lab. of Mathematics
Hirakata 573 - 1136, Osaka,