

# Modular Vector Fields in Non-commutative Geometry

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## Abstract

This is a summary of the paper [Tan24] and the presentation in the conference “Intelligence in Low-dimensional Topology 2025”. We construct a non-commutative geometry analogue of the modular vector field in Poisson geometry based on a (flat) connection and show that, in the case of an oriented surface, it coincides with Turaev’s loop operation  $\mu$  with a suitable choice of the connection.

## 1. INTRODUCTION

In [Gol86], Goldman defined a symplectic structure on the  $GL(n)$ -character variety  $X$  of the fundamental group of a connected, closed and oriented surface  $\Sigma$  based on previous works by Atiyah and Bott. In the same paper, he found the topological counterpart of the Lie algebra structure on the ring  $\mathcal{O}(X)$  of regular functions on  $X$  induced from the symplectic structure, which is now known as the *Goldman Lie algebra*, denoted by  $\mathfrak{g}(\Sigma)$  in this summary. More specifically,  $\mathfrak{g}(\Sigma)$  is the  $\mathbb{R}$ -vector space spanned by homotopy classes of free loops on the surface  $\Sigma$ , and we have the surjective map

$$\mathfrak{g}(\Sigma) \rightarrow \mathcal{O}(X)$$

of Lie algebras. The above construction is extended by Fock and Rosly [FR99] for the case of non-empty boundary, in which case  $X$  only has a Poisson structure. In contrast, the definition of the Lie bracket, the Goldman bracket, works as is since it is written in terms of the intersection of loops; hence, it is called a *loop operation*.

The Goldman Lie algebra  $\mathfrak{g}(\Sigma)$  can be thought of as the non-commutative geometry counterpart of the symplectic structure on  $X$ . For simplicity, we only deal with the case of non-empty boundary and an approach involving double brackets. First of all, let  $\mathbb{R}\pi$  be the group algebra of the fundamental group  $\pi = \pi_1(\Sigma)$ . Then, the vector space  $\mathfrak{g}(\Sigma)$  is identified as the *trace space*

$$|\mathbb{R}\pi| := \mathrm{HH}_0(\mathbb{R}\pi) \cong \mathbb{R}\pi / [\mathbb{R}\pi, \mathbb{R}\pi]$$

of  $\mathbb{R}\pi$ , where the most-right-hand side is the quotient by the *vector space* spanned by commutators and not the abelianisation. In [vdB08], van den Bergh introduced a non-commutative geometry analogue of a (Poisson) bivector: a *double bracket*  $\Pi$  on an associative but not-necessarily-commutative algebra  $A$  is a linear map

$$\Pi: A \otimes A \rightarrow A \otimes A$$

satisfying an appropriate version of the Leibniz rule. More details are covered in the next section. In the case of  $\mathbb{R}\pi$ , we can define the double bracket  $\kappa$  in terms of the intersection of based loops, which induces the Goldman bracket mentioned above.

As in the title, the purpose of my paper [Tan24] is to formulate modular vector fields in non-commutative geometry. With a Poisson manifold  $P$  together with a volume form  $\alpha$ , we can associate the vector field  $\mathbf{m}_\alpha$  as the composition

$$\mathbf{m}_\alpha: C^\infty(P) \xrightarrow{\text{Ham}} \text{Der}(C^\infty(P)) \xrightarrow{\text{div}_\alpha} C^\infty(P).$$

Here,  $\text{Ham}$  is the Hamiltonian flow and  $\text{div}_\alpha$  is the divergence with respect to  $\alpha$ . In this summary of my paper [Tan24], we start with formulating these two maps in the setting of non-commutative geometry and show, in the case of the surface  $\mathbb{R}\pi$ , that it coincides with another kind of loop operation  $\mu$  introduced by Turaev.

## 2. DOUBLE BRACKETS, CONNECTIONS AND THE TRIPLE DIVERGENCE

Let  $\mathbb{K}$  be a field and  $A$  a unital associative  $\mathbb{K}$ -algebra. The two-fold tensor product  $A \otimes A$  has two commuting  $A$ -bimodule structures: for  $a, b, x, y \in A$ , the *outer structure* is given by

$$a \cdot (x \otimes y) \cdot b = ax \otimes yb,$$

while the *inner structure* is given by

$$a * (x \otimes y) * b = xb \otimes ay.$$

Following [AKKN23], we make the following:

**Definition 2.1.** A *double bracket* on  $A$  is a  $\mathbb{K}$ -linear map  $\Pi: A \otimes A \rightarrow A \otimes A$  satisfying

$$\begin{aligned} \Pi(a, bc) &= \Pi(a, b) \cdot c + b \cdot \Pi(a, c) \text{ and} \\ \Pi(ab, c) &= \Pi(a, c) * b + a * \Pi(b, c) \end{aligned}$$

for  $a, b, c \in A$ .

**Remark 2.2.** Van den Bergh defined double brackets in [vdB08] with the additional condition of skew-symmetry. Our definition is based on [AKKN23], and the skew-symmetry is not imposed to deal with the important example below.

**Example 2.3.** Let  $\Sigma$  be a connected, oriented and compact surface with non-empty boundary. Take a base point  $*$  on  $\partial\Sigma$ , and let  $\pi = \pi_1(\Sigma, *)$  be the fundamental group of  $\Sigma$ . Then, there is a double bracket on the group algebra  $\mathbb{K}\pi$ , denoted by  $\kappa$  below, which is independently introduced by Massuyeau and Turaev in [MT14] and by Kawazumi and Kuno in [KK15], respectively. It is given as follows: first, fix an embedded short positive arc  $\nu: [0, 1] \rightarrow \partial\Sigma$  with  $\nu(1) = *$ , and put  $\nu(0) = \bullet$ , by which  $\pi_1(\Sigma, \bullet)$  is identified with  $\pi_1(\Sigma, *)$ . For generic loops  $\alpha$  based at  $\bullet$ , and  $\beta$  at  $*$ , respectively, we define

$$\kappa(\alpha, \beta) = \sum_{p \in \alpha \cap \beta} \text{sign}(\alpha, \beta; p) \beta_{*p} \alpha_{p\bullet} \nu \otimes \nu^{-1} \alpha_{\bullet p} \beta_{p*},$$

where  $\text{sign}(\alpha, \beta; p) \in \{\pm 1\}$  is the local intersection number with respect to the orientation of  $\Sigma$ , and  $\beta_{*p}$  is the path along  $\beta$  from  $*$  to  $p$  and so on.

**Definition 2.4.** A *double derivation* on  $A$  is a  $\mathbb{K}$ -linear map  $\theta: A \rightarrow A \otimes A$  satisfying

$$\theta(ab) = \theta(a) \cdot b + a \cdot \theta(b)$$

for  $a, b \in A$ . We denote the set of all double derivations on  $A$  by  $\text{DDer}(A)$ .

Given a double bracket  $\Pi$ , we have  $\Pi(a, \cdot) \in \text{DDer}(A)$  by the first condition in Definition 2.1. We abusively put

$$\Pi: A \rightarrow \text{DDer}(A) : a \mapsto \Pi(a, \cdot),$$

which is an analogue of a Hamiltonian flow.

Next, we move on to the divergence map. As mentioned above, the divergence map is defined upon the choice of a volume form. More specifically, for a Poisson manifold  $P$  together with a volume form  $\alpha$ , the divergence  $\text{div}_\alpha(\xi)$  of a vector field  $\xi$  on  $P$  is the smooth function uniquely specified by

$$\text{div}_\alpha(\xi)\alpha = L_\xi(\alpha) \text{ in } \Omega^{\text{top}}(P),$$

where  $L_\xi$  is the Lie derivative by  $\xi$ . Recall that  $\Omega^{\text{top}}(P)$  is the exterior product of the space of 1-forms  $\Omega^1(P)$  over the  $\mathbb{R}$ -algebra  $C^\infty(P)$  of smooth functions.

For an arbitrary associative algebra  $A$ , the exterior product of modules over  $A$  is not well-defined. However, notice the following: taking a local frame  $(e_1, \dots, e_r)$  of  $T^*P$  such that

$$e_1 \wedge \dots \wedge e_r = \mu$$

and putting  $L_\xi(e_i) = f_i^j e_j$ , we have

$$L_\xi(\mu) = \sum_i e_1 \wedge \dots \wedge f_i^i e_i \wedge \dots \wedge e_r = \sum_i f_i^i \mu.$$

Then, define a (local) flat connection  $\nabla$  on  $T^*X$  by  $\nabla(e_i) = 0$  and the divergence associated with the connection by

$$\text{Div}^\nabla(\xi) = \text{Tr}(L_\xi - \nabla_\xi). \quad (1)$$

This formulation only uses a connection, the Lie derivative and the trace map, all of which can be defined within non-commutative geometry. In this virtue, we regard a flat connection  $\nabla$  as a substitute for a volume form.

With this seen, we recall some preliminary tools from non-commutative geometry. Let  $A^e = A \otimes A^{\text{op}}$  be the universal enveloping algebra of  $A$ .

**Definition 2.5.**

- The space of 1-forms is defined by

$$\Omega^1 A = \text{Ker}(A \otimes A \xrightarrow{\text{mult}} A)$$

This is naturally an  $A$ -bimodule. We put  $da = 1 \otimes a - a \otimes 1$ . With this notation, we have the Leibniz rule for  $d$ :

$$d(ab) = (da)b + a(db) \text{ for all } a, b \in A.$$

A double derivation  $\theta \in \text{DDer}(A)$  is equivalent to an  $A$ -bimodule map  $i_\theta: \Omega^1 A \rightarrow A \otimes A$ , and they are related by the formula  $i_\theta(da) = \theta(a)$ .

- The *Lie derivative*  $L_\theta$  on  $\Omega^1 A$  by a double derivation  $\theta \in \text{DDer}(A)$  is defined by

$$L_\theta(adb) = \theta(a)' \otimes \theta(a)'' db + ad\theta(b)' \otimes \theta(b)'' + a\theta(b)' \otimes d\theta(b)'.$$

This is an element of  $(\Omega^1 A \otimes A) \oplus (A \otimes \Omega^1 A)$ .

- A *connection* on an  $A$ -bimodule  $E$  is a  $\mathbb{K}$ -linear map

$$\nabla: E \rightarrow \Omega^1 A \otimes_A E \oplus E \otimes_A \Omega^1 A$$

satisfying the Leibniz rule

$$\nabla(aeb) = da \otimes eb + a\nabla(e)b + ae \otimes db$$

for  $a, b \in A$  and  $e \in E$ .

- For a connection  $\nabla$  on  $E$  and a double derivation  $\theta$ , we put

$$\begin{aligned} \nabla_\theta: E &\xrightarrow{\nabla} \Omega^1 A \otimes_A E \oplus E \otimes_A \Omega^1 A \\ &\xrightarrow{i_\theta \otimes \text{id} + \text{id} \otimes i_\theta} (A \otimes A) \otimes_A E \oplus E \otimes_A (A \otimes A) \cong (A \otimes E) \oplus (E \otimes A). \end{aligned}$$

Note that, in the case of  $E = \Omega^1 A$ , the domains and codomains of the maps  $\nabla_\theta$  and  $L_\theta$  are the same.

We want to imitate the formula (1), so we need the trace map.

**Definition 2.6.** Let  $B$  be an associative  $\mathbb{K}$ -algebra,  $E$  a left  $B$ -module and  $W$  a  $B$ -bimodule. Suppose that  $E$  is finitely generated and projective. Then, the trace map  $\text{Tr}$  is defined by the composition

$$\begin{aligned} \text{Tr}: \text{Hom}_B(E, W \otimes_B E) &\cong E^* \otimes_B W \otimes E \rightarrow |W| := \text{HH}_0(B, W) \cong W/[B, W], \\ \varepsilon \otimes w \otimes e &\mapsto \varepsilon(e)w \end{aligned}$$

where  $E^* = \text{Hom}_B(E, B)$  is the dual module, which is a *right*  $B$ -module.

Now set  $B = A^e$ ,  $E = \Omega^1 A$  and  $W = (A \otimes A \otimes A^{\text{op}}) \oplus (A \otimes A^{\text{op}} \oplus A^{\text{op}})$ . Then, we have the following canonical isomorphism:

$$(A \otimes E) \oplus (E \otimes A) \cong W \otimes_B E.$$

**Definition 2.7.** The *triple divergence* associated with a connection  $\nabla$  on  $\Omega^1 A$  is defined by

$$\begin{aligned} \text{TDiv}^\nabla: \text{DDer}(A) &\rightarrow |W| \cong A \otimes |A| \oplus |A| \otimes A, \\ \theta &\mapsto \text{Tr}(L_\theta - \nabla_\theta). \end{aligned}$$

Combining a double bracket  $\Pi$  with the triple divergence above, we have the map

$$\phi_{\Pi, \nabla}: A \xrightarrow{\Pi} \text{DDer}(A) \xrightarrow{\text{TDiv}^\nabla} A \otimes |A| \oplus |A| \otimes A,$$

which is an analogue of a modular vector field.

### 3. TURAEV'S LOOP OPERATION $\mu$

We begin this section by recalling another loop operation  $\mu$  introduced by Turaev [Tur79]. Fix a framing  $\text{fr}$  (i.e., a non-vanishing vector field) on  $\Sigma$ . The base points  $\bullet$ ,  $*$  and an arc  $\nu$  are taken as in Example 2.3.

**Definition 3.1.** The  $\mathbb{K}$ -linear map  $\mu_r: \mathbb{K}\pi \rightarrow |\mathbb{K}\pi| \otimes \mathbb{K}\pi$  is defined as follows: for  $\alpha \in \pi$  represented by a generically immersed path based at  $*$ , first deform  $\alpha$  into a path from  $\bullet$  to  $*$  by sliding the endpoint along the arc  $\nu$ , and insert positive or negative monogons so that  $\text{rot}^{\text{fr}}(\alpha) = -1/2$ . Then,

$$\mu_r(\alpha) = \sum_{p \in \text{Self}(\alpha)} \text{sign}(p; \alpha_{\text{first}}, \alpha_{\text{second}}) |\alpha_{pp}| \otimes \alpha_{\bullet p*}.$$

Here  $\text{Self}(\alpha)$  is the set of self-intersections of  $\alpha$ , and  $\alpha_{\text{first}}$  is the velocity vector of  $\alpha$  passing  $p$  for the first time.  $\alpha_{\text{second}}$  is analogously defined.

Similarly, the  $\mathbb{K}$ -linear map  $\mu_l: \mathbb{K}\pi \rightarrow \mathbb{K}\pi \otimes |\mathbb{K}\pi|$  is defined as follows: for  $\alpha \in \pi$  represented by a generically immersed path based at  $*$ , first deform  $\alpha$  into a path from  $*$  to  $\bullet$  by sliding the endpoint along  $\nu$ , and insert positive or negative monogons so that  $\text{rot}^{\text{fr}}(\alpha) = 1/2$ . Then,

$$\mu_l(\alpha) = - \sum_{p \in \text{Self}(\alpha)} \text{sign}(p; \alpha_{\text{first}}, \alpha_{\text{second}}) \alpha_{*p\bullet} \otimes |\alpha_{pp}|.$$

Set  $\mu = \mu_r + \mu_l: \mathbb{K}\pi \rightarrow (|\mathbb{K}\pi| \otimes \mathbb{K}\pi) \oplus (\mathbb{K}\pi \otimes |\mathbb{K}\pi|)$ .

Now we apply the construction in the last section in the case of the surface  $A = \mathbb{K}\pi$  and see what the map  $\phi_{\Pi, \nabla}$  describes. First, note that the fundamental group  $\pi$  of the surface  $\Sigma$  is a free group since we assumed that the boundary is non-empty. We have to check that  $\Omega^1 A$  satisfies the assumption in Definition 2.6 to apply the construction of the triple divergence. In fact, we have the following:

**Lemma 3.2.** *For a surface  $\Sigma$  above, the  $A$ -bimodule  $\Omega^1 A$  is finitely generated and free. In particular, for any free generating system  $\mathcal{C} = \{\gamma_i\}_{1 \leq i \leq r}$  of  $\pi$ , the set  $\{d\gamma_i\}_{1 \leq i \leq r}$  is an  $A^e$ -free basis of  $\Omega^1 A$ .*

By the lemma above, we can define the connection  $\nabla_{\mathcal{C}}$  associated with  $\mathcal{C}$  by

$$\nabla_{\mathcal{C}}(d\gamma_i \gamma_i^{-1}) = 0$$

for all  $i$ ; this uniquely defines the connection due to the Leibniz rule.

The main result is the following:

**Theorem 3.3** ([Tan24]). *We have  $\phi_{\kappa, \nabla_{\mathcal{C}}} = \mu$  for a suitable free-generating system  $\mathcal{C}$  and a framing  $\text{fr}$  such that  $\text{rot}^{\text{fr}}(c) = 0$  for any  $c \in \mathcal{C}$  represented by a simple curve.*

For the concrete description of the generating system  $\mathcal{C}$ , see Figure 2 of [Tan24].

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