

A class of preorders iterated under a type of RCS

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Abstract

We formulate a class of preorders which we call preproper. This class contains the class of proper preorders and is a modification to the subproper preorders by R. Jensen. We show that this class of preorders is iterable under a type of revised countable support iteration.

Introduction

Jensen formulates classes of notions of forcing in [1]. We take subproper among those. This class is wider than proper, may differ from semiproper of Shelah and is iterable under the revised countable support iteration according to [1]. We formulate a class of notion of forcing whose definition involves less parameters than Jensen's subproper and show that it iterates under a type of revised countable support of [2]. We say a preorder P is preproper, if there exist a set z and a regular cardinal θ with $z, P \in H_\theta$ such that for all (χ, A, N, p, s) such that

- χ is a regular cardinal with $H_\theta \subseteq L_\chi[A]$,
 - $(N, \in, A \cap N)$ is a countable elementary substructure of $(L_\chi[A], \in, A \cap L_\chi[A])$ with $z, P \in N$,
 - there exists a transitive set model M of ZFC^- (i.e. no power set axiom) such that the transitive collapse of N appears as a $(H_\tau)^M$ with a regular τ in M ,
 - $p \in P \cap N$ and $s \in N$,
- there exists $(q, \dot{N})$ such that $q \leq p$ in P , $\dot{N}$ is a P -name and q forces the following three.
- $(\dot{N}, \in, A \cap \dot{N})$ is a countable elementary substructure of $(L_\chi[A], \in, A \cap L_\chi[A])$.
 - There exists an isomorphism between the two structures $(N, \in, A \cap N)$ and $(\dot{N}, \in, A \cap \dot{N})$ which fixes s .
 - For all dense subsets $D \in \dot{N}$ of P , we have $D \cap \dot{N} \cap \dot{G} \neq \emptyset$, where $\dot{G}$ denotes the canonical P -name of the P -generic filters over the ground model V .

Therefore we do not use H_χ 's but $L_\chi[A]$ which are fat enough relative to P . A reason to use the structures $(L_\chi[A], \in, A \cap L_\chi[A])$ is to escape definabilities of H_χ 's calculated in finitely many intermediate stages in iterated forcing. We do not expect genericity over N but over $\dot{N}$ which may exist in the generic extension. The condition on the collapse of N is very technical without which we may have no control over the isomorphic images of N . The role of s is to fix finite-parts in an increasing manner to diagonally build a new isomorphic elementary substructures in the limit stages of iterated forcing.

§ 1. Preliminary

We collect what we think are basics in this subject.

Proposition. Let P be a preorder. Let θ and χ be regular cardinals such that $P \in H_\theta \subseteq L_\chi[A]$. Let $P \in N$ and $(N, \in, A \cap N)$ be an elementary substructure of $(L_\chi[A], \in, A \cap L_\chi[A])$, which we simply denote by

$$(N, \in, A \cap N) \prec (L_\chi[A], \in, A \cap L_\chi[A]).$$

Let G be a P -generic filter over the ground model V . Let $L_\chi[A][G] = \{\sigma_G \mid \sigma \in L_\chi[A], \sigma \text{ is a } P\text{-name}\}$ and $N[G] = \{\sigma_G \mid \sigma \in N, \sigma \text{ is a } P\text{-name}\}$. Let $B = (\{0\} \times (A \cap L_\chi[A])) \cup (\{1\} \times G)$. Then we have

- (1) $(L_\chi[A], \in, A \cap L_\chi[A])$ models ZFC^- in the expanded language.

- (2) $(L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$ models ZFC^- in the expanded language.
- (3) If $\pi \in N$ is a P -name and $\varphi(x, y)$ is a formula in the expanded language, then there exists a P -name $\tau \in N$ such that $\Vdash_P \text{"}(L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A]) \models \text{"}\exists y \varphi(y, \pi) \longrightarrow \varphi(\tau, \pi)\text{"}$. And so we have

$$(N[G], \in, L_\chi[A] \cap N[G], (A \cap L_\chi[A]) \cap N[G]) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A]).$$

- (4) The following five are all equivalent.

- $N[G] \cap L_\chi[A] = N$.
- $N[G] \cap \chi = N \cap \chi$.
- For all dense subsets $D \in N$, we have $D \cap N \cap G \neq \emptyset$.
- $(N[G], \in, N, A \cap N) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$.
- The transitive collapse of $(N, \in)$ gets extended to the transitive collapse of $(N[G], \in)$.

- (5) $L_\chi[A][G] = L_\chi[B]$ holds. If $(N[G], \in, N, A \cap N) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$, then

$$(N[G], \in, B \cap N[G]) \prec (L_\chi[B], \in, B \cap L_\chi[B]).$$

- (6) Let $(N[G], \in, N, A \cap N) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A])$ and two substructures $(N[G], \in, B \cap N[G])$ and $(M, \in, B \cap M) \prec (L_\chi[B], \in, B \cap L_\chi[B])$ be isomorphic fixing P . Let $X = L_\chi[A] \cap M$. Then $P \in X$, $(X, \in, A \cap X) \prec (L_\chi[A], \in, A \cap L_\chi[A])$ and $M = X[G]$ hold. We also have

$$(X[G], \in, X, A \cap X) \prec (L_\chi[A][G], \in, L_\chi[A], A \cap L_\chi[A]).$$

The two substructures $(N[G], \in, N, A \cap N)$ and $(X[G], \in, X, A \cap X)$ are isomorphic by the given isomorphism. So are $(N, \in, A \cap N)$ and $(X, \in, A \cap X)$ by restricting the given isomorphism.

- (7) Let us further assume in (6) that we have a preorder $Q \in H_\theta^{V[G]} \cap M$, H is Q -generic over $V[G]$ and $(M[H], \in, M, B \cap M) \prec (L_\chi[B][H], \in, L_\chi[B], B \cap L_\chi[B])$. Then

$$(X[G][H], \in, X[G], X, A \cap X) \prec (L_\chi[A][G][H], \in, L_\chi[A][G], L_\chi[A], A \cap L_\chi[A]).$$

Proof. Mostly routine interpreting formulas in structures with unary predicates. Notice that we assume every dense subset $D \in V$ of P belongs to H_θ and so to $L_\chi[A]$, though $L_\chi[A]$ may not satisfy the power set axiom. Details are left. □

§ 2. Iteration Lemma

We define the class of preorders under consideration and show that it iterates under the revised countable support iteration of [2].

Definition. A preorder P is *preproper*, if there exist θ and z such that

- θ is a regular cardinal and $z, P \in H_\theta$.
- Given any (A, χ, N, p, s) such that
 - $H_\theta \subseteq L_\chi[A]$, χ is a regular cardinal.
 - $z, P \in N$, N is countable and $(N, \in, A \cap N) \prec (L_\chi[A], \in, A \cap L_\chi[A])$.
 - The transitive collapse $\overline{N}$ of N satisfies $\overline{N} = (H_\tau)^M$ for some M , where M is a transitive set model of ZFC^- (i.e. ZFC minus the power set axiom) and τ is a regular cardinal in M .

- $p \in P \cap N$ and $s \in N$.

There exists $(q, \dot{N}, \dot{f})$ such that

- $q \leq p$ and $\dot{N}, \dot{f}$ are P -names.
- q forces the following.

- (1) $\dot{f}$ is an isomorphism from $(N, \in, A \cap N)$ to $(\dot{N}, \in, A \cap \dot{N}) \prec (L_\chi[A], \in, A \cap L_\chi[A])$ with $\dot{f}(s) = s$.
- (2) For all dense subsets $D \in \dot{N}$ of P , $D \cap \dot{N} \cap \dot{G} \neq \emptyset$.

Remark. (1) The condition (2) in the definition is equivalent to

$$(\dot{N}[\dot{G}], \in, \dot{N}, A \cap \dot{N}) \prec (L_\chi[A][\dot{G}], \in, L_\chi[A], A \cap L_\chi[A]),$$

where $L_\chi[A][\dot{G}]$ denotes the generic extension of $L_\chi[A]$ via P .

- (2) $(L_\chi[A][\dot{G}], \in, L_\chi[A], A \cap L_\chi[A])$ and $(L_\chi[B], \in, B \cap L_\chi[B])$ are suitably interpretable equivalent structures with $L_\chi[A][\dot{G}] = L_\chi[B]$, where $B = (\{0\} \times (A \cap L_\chi[A])) \cup (\{1\} \times \dot{G})$.
- (3) If a preorder P is proper, then P is preproper.
- (4) This formulation of preproper is tentative and is a modification to the subproper of [1]. There remains a chance to further reformulate this notion of forcing. A possible modification may include relativizing things in $L[A]$ so that if $j : V \rightarrow M$ is an elementary embedding and P is preproper in V , then P is preproper in M whenever, say, $H_\theta \in M$.

However, we know little on the notion of preproper except the following.

Lemma. Let $\langle P_\alpha \mid \alpha \leq \nu \rangle$ be a simple iteration such that for all $\alpha < \nu$, $\Vdash_{P_\alpha} "P_{\alpha+1} \text{ is preproper with } \dot{\theta}_\alpha \text{ and } \dot{z}_\alpha"$. Let θ and z be such that

- θ is a regular cardinal such that for all $\alpha < \nu$, $\Vdash_{P_\alpha} "\dot{\theta}_\alpha < \theta"$.
- $\langle P_\alpha \mid \alpha \leq \nu \rangle, \langle (\dot{\theta}_\alpha, \dot{z}_\alpha) \mid \alpha < \nu \rangle \in z \prec H_\theta$ and z is countable.

Then for all (α, β) with $\alpha < \beta \leq \nu$, we have $\Vdash_{P_\alpha} "P_{\alpha\beta} \text{ is preproper with } \theta \text{ and } (\dot{G}_\alpha, z)"$.

Proof. By induction on β (for all $\alpha < \beta$). We assume that for all (i, j) with $i < j < \beta$, $\Vdash_{P_i} "P_{ij} \text{ is preproper with } \theta \text{ and } (\dot{G}_i, z)"$. Fix any $\alpha < \beta$. We want to show $\Vdash_{P_\alpha} "P_{\alpha\beta} \text{ is preproper with } \theta \text{ and } (\dot{G}_\alpha, z)"$.

Case. β is limit. Suppose $w \Vdash_{P_\alpha} "(A, \chi, \dot{N}, p[[\alpha, \beta), \dot{s}])$ as in the hypothesis". Namely, w forces

- $H_\theta^{V[\dot{G}_\alpha]} \subseteq L_\chi[\dot{A}]$, χ is regular.
- $\dot{G}_\alpha, z, P_{\alpha\beta} \in \dot{N}$ is countable and $(\dot{N}, \in, \dot{A} \cap \dot{N}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ and $\dot{N}$ satisfies the condition on the transitive collapse.
- $p[[\alpha, \beta) \in P_{\alpha\beta} \cap \dot{N}$ and $\dot{s} \in \dot{N}$.

Then we have that

- $z, \beta \in \dot{N}$ and so $P_\beta \in \dot{N}$.

We may assume that

- $p \in P_\beta \cap \dot{N}$, $p[[\alpha \in \dot{G}_\alpha$ and p has stages $\langle \dot{\delta}_k^p \mid k < \omega \rangle \in \dot{N}$.

We may also extend w so that

- $w \leq p[[\alpha$.
- $w \frown (1_\beta[[\alpha, \beta)) \Vdash_{P_\beta} "\dot{\delta}_0^p = \alpha"$.

This is because $z \in \dot{N} \prec L_X[\dot{A}]$ and so $\dot{N}$ contains various maps defined in V and so $\dot{N}$ is closed under those maps. For example, we have $\langle p \mapsto \langle \delta_k^p \mid k < \omega \rangle \rangle \in \dot{N}$.

Hence w decides P_α -names $\dot{A}, \dot{N}, \dot{s} \in V$ and the values of $\chi, p, \langle \delta_k^p \mid k < \omega \rangle$ and δ_0^p .

It suffices to find $(q, \dot{X}_\omega, \dot{f}_\omega)$ such that

- $q \in P_\beta, q \restriction \alpha = w, q \leq p$.
- $\dot{X}_\omega$ and $\dot{f}_\omega$ are P_β -names.

q forces the following.

- (1) $\dot{f}_\omega$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{X}_\omega, \in, \dot{A} \cap \dot{X}_\omega) \prec (L_X[\dot{A}], \in, \dot{A} \cap L_X[\dot{A}])$, $\dot{f}_\omega(\dot{s}) = \dot{s}$ and $G_\alpha (= \dot{G}_\beta \restriction \alpha), P_\beta \in \dot{X}_\omega$.
- (2) For all dense subsets $D \in \dot{X}_\omega$ of P_β , $D \cap \dot{X}_\omega \cap \dot{G}_\beta \neq \emptyset$.

This is because, given any dense subset $D' \in \dot{X}_\omega$ of $P_{\alpha\beta}$, let

$$D = \{y \in P_\beta \mid y \restriction \alpha \notin G_\alpha \parallel y \restriction [\alpha, \beta) \in D'\}.$$

Then $D \in \dot{X}_\omega$ is a dense subset of P_β with $D \restriction [\alpha, \beta) = D'$. Since $D \cap \dot{X}_\omega \cap \dot{G}_\beta \neq \emptyset$, we have $D' \cap \dot{X}_\omega \cap G_{\alpha\beta} (= \dot{G}_\beta \restriction [\alpha, \beta)) \neq \emptyset$.

In order to get $(q, \dot{X}_\omega, \dot{f}_\omega)$ as such, we construct a nested antichain T with associated structures. To do so, we present the following general construction.

Claim. Given any $(i, x, \langle \delta_k^x \mid k < \omega \rangle, a, \dot{X}, \dot{f}, \dot{D})$ such that

- $\alpha \leq i, a \in P_i, x \in P_\beta, a \restriction \alpha \leq w, a \leq x \restriction i$ and $x \leq p$.
 - $\langle \delta_k^x \mid k < \omega \rangle$ are stages for x and $a \restriction 1 \Vdash_{P_\beta} \text{"}\delta_0^x = i\text{"}$.
 - $a \in P_i$ forces the following, where $\dot{X}, \dot{f}, \dot{D} \in V$ are P_i -names.
- (1) $\dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{X}, \in, \dot{A} \cap \dot{X}) \prec (L_X[\dot{A}], \in, \dot{A} \cap L_X[\dot{A}])$ with $\dot{f}(G_\alpha, z, \dot{s}, P_\beta) = (G_\alpha, z, \dot{s}, P_\beta)$.
 - (2) $P_i \in \dot{X}$ and $(\dot{X}[G_{\alpha i}], \in, \dot{X}, \dot{A} \cap \dot{X}) \prec (L_X[\dot{A}][G_{\alpha i}], \in, L_X[\dot{A}], \dot{A} \cap L_X[\dot{A}])$.
 - (3) $x \in P_\beta \cap \dot{X}$ and $\langle \delta_k^x \mid k < \omega \rangle \in \dot{X}$.
 - (4) $\dot{D} \in \dot{X}$ is a dense subset of P_β .

Get $(j, y, \langle \delta_k^y \mid k < \omega \rangle, b, \dot{Y}, \dot{g})$ such that

- $i < j, b \in P_j, y \in P_\beta, b \restriction i \leq a, b \leq y \restriction j$ and $y \leq x$.
 - $\langle \delta_k^y \mid k < \omega \rangle$ are stages for y and are a step ahead of $\langle \delta_k^x \mid k < \omega \rangle$, namely, $\Vdash_{P_\beta} \text{"}\delta_{k+1}^x \leq \delta_k^y\text{"}$ for all $k < \omega$.
 - $b \restriction 1 \Vdash_{P_\beta} \text{"}\delta_0^y = j\text{"}$.
 - $b \in P_j$ forces the following, where $\dot{Y}, \dot{g} \in V$ are P_j -names.
- (1*) $\dot{g}$ is an isomorphism from $(\dot{X}[G_{\alpha i}], \in, \dot{X}, \dot{A} \cap \dot{X})$ to

$$(\dot{Y}[G_{\alpha i}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_X[\dot{A}][G_{\alpha i}], \in, L_X[\dot{A}], \dot{A} \cap L_X[\dot{A}])$$

with

$$\dot{g}(G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta_k^y \mid k < \omega \rangle) = (G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta_k^y \mid k < \omega \rangle).$$

(2*) $j \in \dot{X} \cap \dot{Y}$ and so $P_{\alpha j} \in \dot{Y}$, $L_\chi[\dot{A}][G_{\alpha j}] = L_\chi[\dot{A}][G_{\alpha i}][G_{ij}]$ and $\dot{Y}[G_{\alpha j}] = \dot{Y}[G_{\alpha i}][G_{ij}]$. We have

$$(\dot{Y}[G_{\alpha i}][G_{ij}], \in, \dot{Y}[G_{\alpha i}], \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}][G_{ij}], \in, L_\chi[\dot{A}][G_{\alpha i}], L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

And so $\dot{Y}[G_{\alpha j}] \cap L_\chi[\dot{A}][G_{\alpha i}] = \dot{Y}[G_{\alpha i}]$ and $\dot{Y}[G_{\alpha i}] \cap L_\chi[\dot{A}] = \dot{Y}$.

(3*) $y \in \dot{D} \cap \dot{X} \cap \dot{Y}$ and $\langle \delta_k^y \mid k < \omega \rangle \in \dot{X} \cap \dot{Y}$.

Hence, we have

- $b \in P_j$ forces the following.

(1) $\dot{g} \circ \dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ with

$$\dot{g} \circ \dot{f}(G_\alpha, z, \dot{s}, P_\beta) = (G_\alpha, z, \dot{s}, P_\beta).$$

(2) $P_j \in \dot{Y}$ and $(\dot{Y}[G_{\alpha j}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha j}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$.

(3) $y \in P_\beta \cap \dot{Y}$ and $\langle \delta_k^y \mid k < \omega \rangle \in \dot{Y}$.

Proof. Let G_i be P_i -generic over V with $a \in G_i$. In $V[G_i]$, let $G_\alpha = G_i[\alpha]$, $G_{\alpha i} = G_i[\alpha, i]$, $A = \dot{A}_{G_\alpha}$, $X = \dot{X}_{G_i}$, $f = \dot{f}_{G_i}$ and $D = \dot{D}_{G_i}$. Then we have

- $x \in X \cap P_\beta$, $P_{\alpha i} \in X$ and

$$(X[G_{\alpha i}], \in, X, A \cap X) \prec (L_\chi[A][G_{\alpha i}], \in, L_\chi[A], A \cap L_\chi[A]).$$

- $D \in X$ is a dense subset of P_β .

Get $y \in D \cap X[G_{\alpha i}] = D \cap X$, $y[i \in G_i]$, $y \leq x$ and stages $\langle \delta_k^y \mid k < \omega \rangle \in X$ for y which is a step ahead of $\langle \delta_k^x \mid k < \omega \rangle \in X$.

Then decide the value of δ_0^y as $u \restriction 1 \Vdash_{P_\beta} \delta_0^y = j$. We may assume $j \in \beta \cap X[G_{\alpha i}] = \beta \cap X$, $u \in P_j \cap X[G_{\alpha i}] = P_j \cap X$, $u \leq y[i \in G_i]$ and $u[i \in G_i]$.

Since P_{ij} is preproper with θ, G_i, z and in $V[G_i]$, we have

- $H_\theta^{V[G_i]} = H_\theta^{V[G_\alpha]}[G_{\alpha i}] \subseteq L_\chi[A][G_{\alpha i}]$, χ is regular.
- $G_i, z, P_{ij} \in X[G_{\alpha i}]$, $(X[G_{\alpha i}], \in, X, A \cap X) \prec (L_\chi[A][G_{\alpha i}], \in, L_\chi[A], A \cap L_\chi[A])$.
- And so the transitive collapse $\overline{X[G_{\alpha i}]}$ of $X[G_{\alpha i}]$ satisfies

$$\overline{X[G_{\alpha i}]} = \overline{X[G_{\alpha i}]} = (H_\tau)^M[\overline{G_{\alpha i}}] = (H_\tau)^{M[\overline{G_{\alpha i}}]} \text{ with } \overline{P_{\alpha i}} \in (H_\tau)^M.$$

- $u \in P_j \cap X[G_{\alpha i}]$, $u[i \in G_i]$ and $G_\alpha, z, G_{\alpha i} \in X[G_{\alpha i}]$.

Hence we may fix $(b, \dot{Y}, \dot{g})$ in V such that

- $b \leq u$ and $b[i \in a]$.
- $b \in P_j$ forces the following.

(1)* $\dot{g}$ is an isomorphism from $(\dot{X}[G_{\alpha i}], \in, \dot{X}, \dot{A} \cap \dot{X})$ to

$$(\dot{Y}[G_{\alpha i}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$$

with

$$\dot{g}(G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta_k^y \mid k < \omega \rangle) = (G_\alpha, z, \dot{s}, P_\beta, P_{\alpha i}, G_{\alpha i}, j, y, \langle \delta_k^y \mid k < \omega \rangle).$$

(2)*

$$(\dot{Y}[G_{\alpha i}][G_{ij}], \in, \dot{Y}[G_{\alpha i}], \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha i}][G_{ij}], \in, L_\chi[\dot{A}][G_{\alpha i}], L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

(3)* $y \in \dot{D} \cap \dot{X} \cap \dot{Y}$ and $\langle \delta_k^y \mid k < \omega \rangle \in \dot{X} \cap \dot{Y}$.

And so, we have

- $b \in P_j$ forces the following.

- (1) $\dot{g} \circ \dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to $(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_X[\dot{A}], \in, \dot{A} \cap L_X[\dot{A}])$.
- (2) $P_j \in \dot{Y}[G_{\alpha j}] \cap L_X[\dot{A}] = \dot{Y}$ and $(\dot{Y}[G_{\alpha j}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_X[\dot{A}][G_{\alpha j}], \in, L_X[\dot{A}], \dot{A} \cap L_X[\dot{A}])$.
- (3) $y \in \dot{D} \cap \dot{Y}$ and $\langle \delta_k^y \mid k < \omega \rangle \in \dot{Y}$.

□

In V , we construct a nested antichain $\langle (a, n) \mapsto p^{(a, n)} \mid a \in T_n, n < \omega \rangle$ together with an associated structures $(\dot{X}^{(a, n)}, p^{(a, n)}, \langle \delta_k^{(a, n)} \mid k < \omega \rangle, \dot{D}^{(a, n)}, \dot{f}^{(a, n)}, S^{(a, n)}, \dot{f}^{(a, n)(b, n+1)})$ for $a \in T_n, b \in \text{succ}_T^n(a), n < \omega$ such that

- $T_0 = \{w\}, \dot{X}^{(w, 0)} = \dot{N}, p^{(w, 0)} = p, \langle \delta_k^{(w, 0)} \mid k < \omega \rangle = \langle \delta_k^p \mid k < \omega \rangle, \dot{D}^{(w, 0)} = \dot{P}_\beta, \dot{f}^{(w, 0)} = (id \text{ on } \dot{N}), S^{(w, 0)} = \{p\}$.

Then for all $a \in T_n$, we have (with $n = 0$)

- $\alpha \leq l(a), a \in P_{l(a)}, p^{(a, n)} \in P_\beta, a \restriction \alpha \leq w, a \leq p^{(a, n)} \restriction l(a)$ and $p^{(a, n)} \leq p$.
- $\langle \delta_k^{(a, n)} \mid k < \omega \rangle$ are stages for $p^{(a, n)}$ and $a \restriction 1 \Vdash_{P_\beta} \text{"}\delta_0^{(a, n)} = l(a)\text{"}$.
- $S^{(a, n)}$ is a finite subset of P_β with $p^{(a, n)} \in S^{(a, n)}$.
- $a \in P_{l(a)}$ forces the following, where $\dot{X}^{(a, n)}, \dot{D}^{(a, n)}, \dot{f}^{(a, n)}$ are $P_{l(a)}$ -names.

- (1) $\dot{f}^{(a, n)}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to

$$(\dot{X}^{(a, n)}, \in, \dot{A} \cap \dot{X}^{(a, n)}) \prec (L_X[\dot{A}], \in, \dot{A} \cap L_X[\dot{A}])$$

with

$$\dot{f}^{(a, n)}(G_\alpha, z, \dot{s}, P_\beta) = (G_\alpha, z, \dot{s}, P_\beta).$$

- (2) $S^{(a, n)} \cup \{G_\alpha, z, \dot{s}, P_{l(a)}\} \subset \dot{X}^{(a, n)}$ and so $G_{l(a)} \in \dot{X}^{(a, n)}[G_{\alpha l(a)}]$ and

$$(\dot{X}^{(a, n)}[G_{\alpha l(a)}], \in, \dot{X}^{(a, n)}, \dot{A} \cap \dot{X}^{(a, n)}) \prec (L_X[\dot{A}][G_{\alpha l(a)}], \in, L_X[\dot{A}], \dot{A} \cap L_X[\dot{A}]).$$

- (3) $p^{(a, n)} \in P_\beta \cap \dot{X}^{(a, n)}$ and $\langle \delta_k^{(a, n)} \mid k < \omega \rangle \in \dot{X}^{(a, n)}$.

- (4) $\dot{D}^{(a, n)} \in \dot{X}^{(a, n)}$ is a dense subset of P_β . We demand the following for a bookkeeping.

$$\dot{D}^{(a, n)} = \begin{cases} \dot{f}^{(a, n)}(\dot{x}_n), & \text{if it is dense in } P_\beta. \\ P_\beta, & \text{otherwise,} \end{cases}$$

where $\dot{N} = \{\dot{x}_n \mid n < \omega\}$ with $\dot{x}_0 = \dot{P}_\beta$.

For $b \in \text{succ}_T^n(a)$,

- $p^{(b, n+1)} \leq p^{(a, n)}$ and $\langle \delta_k^{(b, n+1)} \mid k < \omega \rangle$ is a step ahead of $\langle \delta_k^{(a, n)} \mid k < \omega \rangle$.
- $S^{(b, n+1)} = S^{(a, n)} \cup \{p^{(b, n+1)}\}$.
- $b \in P_{l(b)}$ forces the following, where $\dot{f}^{(a, n)(b, n+1)}$ is a $P_{l(b)}$ -name.

- (1)* $\dot{f}^{(a, n)(b, n+1)}$ is an isomorphism from $(\dot{X}^{(a, n)}, \in, \dot{A} \cap \dot{X}^{(a, n)})$ to

$$(\dot{X}^{(b, n+1)}, \in, \dot{A} \cap \dot{X}^{(b, n+1)}) \prec (L_X[\dot{A}], \in, \dot{A} \cap L_X[\dot{A}])$$

with

$$f^{(a,n)(b,n+1)}(G_\alpha, z, \dot{s}, P_\beta, P_{al(a)}, G_{al(a)}, l(b), S^{(b,n+1)}) = (G_\alpha, z, \dot{s}, P_\beta, P_{al(a)}, G_{al(a)}, l(b), S^{(b,n+1)})$$

and

$$f^{(a,n)(b,n+1)}(\langle \delta_k^{(b,n+1)} \mid k < \omega \rangle) = \langle \delta_k^{(b,n+1)} \mid k < \omega \rangle$$

and furthermore

$$f^{(a,n)(b,n+1)}(f^{(a,n)}(\dot{x}_0), \dots, f^{(a,n)}(\dot{x}_n)) = (f^{(a,n)}(\dot{x}_0), \dots, f^{(a,n)}(\dot{x}_n)).$$

$$f^{(b,n+1)} = f^{(a,n)(b,n+1)} \circ f^{(a,n)} \text{ and so}$$

$$f^{(b,n+1)} \text{ is an isomorphism from } (\dot{N}, \in, \dot{A} \cap \dot{N}) \text{ to } (\dot{X}^{(b,n+1)}, \in, \dot{A} \cap \dot{X}^{(b,n+1)}).$$

$$(2)^* p^{(b,n+1)} \in S^{(b,n+1)} \subset \dot{X}^{(a,n)} \cap \dot{X}^{(b,n+1)} \text{ and}$$

$$(\dot{X}^{(b,n+1)}[G_{al(b)}], \in, \dot{X}^{(b,n+1)}, \dot{A} \cap \dot{X}^{(b,n+1)}) \prec (L_\chi[\dot{A}][G_{al(b)}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

$$(3)^* p^{(b,n+1)} \in \dot{D}^{(a,n)} \cap \dot{X}^{(a,n)} \cap \dot{X}^{(b,n+1)} \text{ and } \langle \delta_k^{(b,n+1)} \mid k < \omega \rangle \in \dot{X}^{(a,n)} \cap \dot{X}^{(b,n+1)}.$$

This completes the construction. Let $q \in P_\beta$ be a fusion of $\langle T_n \mid n < \omega \rangle$. Then $q \leq p$ and $q \restriction \alpha = w$ holds. Let G_β be P_β -generic over V with $q \in G_\beta$. We want to construct X_ω and f_ω . In $V[G_\beta]$, let $\langle a_n \mid n < \omega \rangle$ be the generic cofinal path through T . Let $\alpha_n = l(a_n)$, $X_n = \dot{X}_{G_{\alpha_n}}^{(a_n, n)}$, $p_n = p^{(a_n, n)}$, $D_n = \dot{D}_{G_{\alpha_n}}^{(a_n, n)}$, $f_n = f_{G_{\alpha_n}}^{(a_n, n)}$, $S_n = S^{(a_n, n)}$ and $f_{nn+1} = f_{G_{\alpha_{n+1}}}^{(a_n, n)(a_{n+1}, n+1)}$. We also let $A = \dot{A}_{G_\omega}$, $s = \dot{s}_{G_\omega}$ and $N = \dot{N}_{G_\omega} = \{x_n \mid n < \omega\}$. Then in $V[G_\beta]$ we have

- $a_n \in G_{\alpha_n} (= G_\beta \restriction \alpha_n)$ and $p_n \in G_\beta$.
- $(X_n, \in, A \cap X_n) \prec (L_\chi[A], \in, A \cap L_\chi[A])$.
- $S_0 = \{p\}$, $S_{n+1} = S_n \cup \{p_{n+1}\}$ and so $S_n = \{p_0, \dots, p_n\}$.
- f_n is an isomorphism from $(N, \in, A \cap N)$ to $(X_n, \in, A \cap X_n)$ with

$$f_n(G_\alpha, z, s, P_\beta) = (G_\alpha, z, s, P_\beta)$$

- $f_{n+1} = f_{nn+1} \circ f_n$ and

$$f_{nn+1}(p_0, \dots, p_n, p_{n+1}) = (p_0, \dots, p_n, p_{n+1})$$

$$f_{nn+1}(f_n(x_0), \dots, f_n(x_n)) = (f_n(x_0), \dots, f_n(x_n)).$$

$$D_n = \begin{cases} f_n(x_n), & \text{if it is dense in } P_\beta. \\ P_\beta, & \text{otherwise.} \end{cases}$$

- $p_0 = p \in N$, $D_n \in X_n$ and $p_{n+1} \in D_n \cap X_n \cap X_{n+1}$.

Let $X_\omega = \{f_n(x_n) \mid n < \omega\}$ and f_ω be a map from N to X_ω defined by $f_\omega(x_n) = f_n(x_n)$.

Claim. (1) f_ω is a well-defined isomorphism from $(N, \in, A \cap N)$ to

$$(X_\omega, \in, A \cap X_\omega) \prec (L_\chi[A], \in, A \cap L_\chi[A])$$

such that $f_\omega(s, G_\alpha, P_\beta) = (s, G_\alpha, P_\beta)$ and so $G_\alpha, P_\beta \in X_\omega$.

(2) For all dense subsets $D \in X_\omega$ of P_β , we have $D \cap X_\omega \cap G_\beta \neq \emptyset$.

Proof. (well-defined): Suppose $x_n = x_m$. Then $(N, \in, A \cap N) \models "x_n = x_m"$. Hence $(X_l, \in, A \cap X_l) \models "f_l(x_n) = f_l(x_m)"$, where $l = \max\{n, m\}$. Hence $f_n(x_n) = f_m(x_m)$.

(One-to-one): Suppose $x_n \neq x_m$. Then $(N, \in, A \cap N) \models "x_n \neq x_m"$. Hence $(X_l, \in, A \cap X_l) \models "f_l(x_n) \neq f_l(x_m)"$, where $l = \max\{n, m\}$. Hence $f_n(x_n) \neq f_m(x_m)$.

($\in$ -homo): Suppose $x_n \in x_m$. Then $(N, \in, A \cap N) \models "x_n \in x_m"$. Hence $(X_l, \in, A \cap X_l) \models "f_l(x_n) \in f_l(x_m)"$, where $l = \max\{n, m\}$. Hence $f_n(x_n) \in f_m(x_m)$ and so $(X_\omega, \in, A \cap X_\omega) \models "f_\omega(x_n) \in f_\omega(x_m)"$.

($\notin$ -homo): Suppose $x_n \notin x_m$. Then $(N, \in, A \cap N) \models "x_n \notin x_m"$. Hence $(X_l, \in, A \cap X_l) \models "f_l(x_n) \notin f_l(x_m)"$, where $l = \max\{n, m\}$. Hence $f_n(x_n) \notin f_m(x_m)$ and so $(X_\omega, \in, A \cap X_\omega) \models "f_\omega(x_n) \notin f_\omega(x_m)"$.

(A -homo): Suppose $x_n \in A$. Then $(N, \in, A \cap N) \models "\dot{A}(x_n)"$. Hence $(X_n, \in, A \cap X_n) \models "\dot{A}(f_n(x_n))"$. Hence $f_n(x_n) \in A$ and so $(X_\omega, \in, A \cap X_\omega) \models "\dot{A}(f_\omega(x_n))"$.

($\neg A$ -homo): Suppose $x_n \notin A$. Then $(N, \in, A \cap N) \models "\neg \dot{A}(x_n)"$. Hence $(X_n, \in, A \cap X_n) \models "\neg \dot{A}(f_n(x_n))"$. Hence $f_n(x_n) \notin A$ and so $(X_\omega, \in, A \cap X_\omega) \models "\neg \dot{A}(f_\omega(x_n))"$.

Since f_ω is onto, it is an isomorphism.

(Elementarity): Suppose

$$(L_\chi[A], \in, A \cap L_\chi[A]) \models "\exists y \varphi(y, f_0(x_0), \dots, f_n(x_n))".$$

Since $(X_n, \in, A \cap X_n) \prec (L_\chi[A], \in, A \cap L_\chi[A])$, we have

$$(X_n, \in, A \cap X_n) \models "\exists y \varphi(y, f_0(x_0), \dots, f_n(x_n))".$$

Take $f_n(x_m) \in X_n$ such that

$$(X_n, \in, A \cap X_n) \models "\varphi(f_n(x_m), f_0(x_0), \dots, f_n(x_n))".$$

Then

$$(X_l, \in, A \cap X_l) \models "\varphi(f_m(x_m), f_0(x_0), \dots, f_n(x_n))",$$

where $l = \max\{n, m\}$. Hence

$$(L_\chi[A], \in, A \cap L_\chi[A]) \models "\varphi(f_m(x_m), f_0(x_0), \dots, f_n(x_n))".$$

(Generic): Let $D \in X_\omega$ be a dense subset of P_β . Let $D = f_n(x_n)$. By definition, $D_n = f_n(x_n) = D$ holds. Hence $p_{n+1} \in D_n \cap G_\beta$. But $p_{n+1} \in X_n$ and so $p_{n+1} = f_n(x_m) = f_m(x_m) \in X_\omega$. Therefore, $D \cap X_\omega \cap G_\beta \neq \emptyset$.

Case. β is successor. We write $\beta + 1$ for β for convenience. Since $\alpha < \beta + 1$, we have two subcases.

Subcase. $\alpha = \beta$. We want $\Vdash_{P_\beta} "P_{\beta\beta+1} \text{ is preproper with } \theta \text{ and } (\dot{G}_\beta, z)"$. Let G_β be any P_β -generic filter over V . Argue in $V[G_\beta]$. Let (A, χ, N, p, s) satisfy

- $G_\beta, z, P_{\beta\beta+1} \in H_\theta^{V[G_\beta]} \subseteq L_\chi[A]$.
- $G_\beta, z, P_{\beta\beta+1} \in N$ is countable and $(N, \in, A \cap N) \prec (L_\chi[A], \in, A \cap L_\chi[A])$ and N satisfies the condition on the transitive collapse.
- $p[[\beta, \beta + 1) \in P_{\beta\beta+1} \cap N$ and $s \in N$.

We may assume

- $p \in P_{\beta+1} \cap N$ and $p[\beta \in G_\beta$.

Then $z_\beta = (z_\beta)_{G_\beta} \in N$ and $z_\beta, P_{\beta\beta+1} \in H_{\theta_\beta}^{V[G_\beta]} \subset H_\theta^{V[G_\beta]} \subseteq L_\chi[A]$. By assumption, $P_{\beta\beta+1}$ is preproper with θ_β and z_β . Hence there exists $(q, \dot{N}, \dot{f})$ such that

- $q \in P_{\beta+1}$, $q \restriction \beta \in G_\beta$ and $q \leq p$.
- $q \restriction [\beta, \beta+1)$ forces the following, where $\dot{N}$ and $\dot{f}$ are $P_{\beta\beta+1}$ -names.

(1) $\dot{f}$ is an isomorphism from $(\dot{N}, \in, A \cap \dot{N})$ to

$$(\dot{N}, \in, A \cap \dot{N}) \prec (L_\chi[A], \in, A \cap L_\chi[A])$$

with $\dot{f}(s) = s$.

(2) $(\dot{N}[\dot{G}_{\beta\beta+1}], \in, \dot{N}, A \cap \dot{N}) \prec (L_\chi[A][\dot{G}_{\beta\beta+1}], \in, L_\chi[A], A \cap L_\chi[A])$.

Subcase. $\alpha < \beta < \beta+1$. We want $\Vdash_{P_\alpha}$ “ $P_{\alpha\beta+1}$ is preproper with θ and $(\dot{G}_\alpha, z)$ ”.

Let $w \Vdash_{P_\alpha}$ “ $(\dot{A}, \chi, \dot{N}, p, \dot{s})$ be as in the hypothesis”. Namely, w forces the following.

- $\dot{G}_\alpha, z, P_{\alpha\beta+1} \in H_\theta^{V[\dot{G}_\alpha]} \subseteq L_\chi[\dot{A}]$.
- $\dot{G}_\alpha, z, P_{\alpha\beta+1} \in \dot{N}$ is countable and $(\dot{N}, \in, \dot{A} \cap \dot{N}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ and $\dot{N}$ satisfies the condition on the transitive collapse.
- $p \restriction [\alpha, \beta+1) \in P_{\alpha\beta+1} \cap \dot{N}$ and $\dot{s} \in \dot{N}$.

We may assume

- $w \leq p \restriction \alpha$.
- $w \Vdash_{P_\alpha}$ “ $p \in \dot{N} \cap P_{\beta+1}$ ”.

We want $(q, \dot{Y}, \dot{h})$ such that

- $q \in P_{\beta+1}$, $q \restriction \alpha \leq w$ and $q \leq p$.
- q forces the following, where $\dot{Y}$ and $\dot{h}$ are $P_{\beta+1}$ -names.

(1) $\dot{h}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to

$$(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$$

with $\dot{h}(\dot{s}) = \dot{s}$.

(2) $(\dot{Y}[\dot{G}_{\alpha\beta+1}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][\dot{G}_{\alpha\beta+1}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}])$.

But w forces

- $\dot{G}_\alpha, z, P_{\alpha\beta} \in H_\theta^{V[\dot{G}_\alpha]} \subseteq L_\chi[\dot{A}]$.
- $\dot{G}_\alpha, z, P_{\alpha\beta} \in \dot{N}$ is countable and $(\dot{N}, \in, \dot{A} \cap \dot{N}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$ and $\dot{N}$ satisfies the condition on the transitive collapse.
- $p \restriction [\alpha, \beta) \in P_{\alpha\beta} \cap \dot{N}$ and $\dot{s} \in \dot{N}$.

Since $\Vdash_{P_\alpha}$ “ $P_{\alpha\beta}$ is preproper with θ and $(\dot{G}_\alpha, z)$ ” by induction, we have $(q \restriction \beta, \dot{X}, \dot{f}) \in V$ such that

- $q \restriction \beta \in P_\beta$, $q \restriction \alpha \leq w$ and $q \restriction \beta \leq p \restriction \beta$.
- $q \restriction \beta$ forces the following, where $\dot{X}$ and $\dot{f}$ are P_β -names.

(1) $\dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to

$$(\dot{X}, \in, \dot{A} \cap \dot{X}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$$

with $\dot{f}(G_\alpha, z, \dot{s}, P_\beta, p) = (G_\alpha, z, \dot{s}, P_\beta, p)$.

$$(2) (\dot{X}[G_{\alpha\beta}], \in, \dot{X}, \dot{A} \cap \dot{X}) \prec (L_\chi[\dot{A}][G_{\alpha\beta}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

Now we want $q[\beta, \beta + 1]$. To do so let G_β be any P_β -generic filter over V with $q[\beta \in G_\beta]$. Then argue in $V[G_\beta]$ to specify the interpretation of $q[\beta, \beta + 1]$. Let $G_\alpha = G_\beta[\alpha]$, $G_{\alpha\beta} = G_\beta[\alpha, \beta]$, $A = \dot{A}_{G_\alpha}$, $N = \dot{N}_{G_\alpha}$, $s = \dot{s}_{G_\alpha}$, $X = \dot{X}_{G_\beta}$. Then we have

- $G_\alpha, P_{\alpha\beta} \in X$ and so $G_\beta \in X[G_{\alpha\beta}]$.
- $(\dot{z}_\beta)_{G_\beta}, P_{\beta\beta+1} \in H_{\theta_\beta}^{V[G_\beta]} \subset H_\theta^{V[G_\beta]} = H_\theta^{V[G_\alpha]}[G_{\alpha\beta}] \subseteq L_\chi[A][G_{\alpha\beta}]$, as $P_{\alpha\beta} \in H_\theta^{V[G_\alpha]}$.
- $(\dot{z}_\beta)_{G_\beta}, P_{\beta\beta+1} \in X[G_{\alpha\beta}]$ is countable and $(X[G_{\alpha\beta}], \in, X, A \cap X) \prec (L_\chi[A][G_{\alpha\beta}], \in, L_\chi[A], A \cap L_\chi[A])$ and $X[G_{\alpha\beta}]$ satisfies the condition on the transitive collapse.
- $p[\beta, \beta + 1] \in P_{\beta\beta+1} \cap X[G_{\alpha\beta}]$ and $s \in X[G_{\alpha\beta}]$.

Since $P_{\beta\beta+1}$ is preproper with θ_β and z_β , we have $(q[\beta, \beta + 1], \dot{Y}, \dot{g})$ such that

- $q[\beta, \beta + 1] \leq p[\beta, \beta + 1]$.
- $q[\beta, \beta + 1]$ forces the following, where $\dot{Y}$ and $\dot{g}$ are $P_{\beta\beta+1}$ -names.

(1) $\dot{g}$ is an isomorphism from $(X[G_{\alpha\beta}], \in, X, A \cap X)$ to

$$(\dot{Y}[G_{\alpha\beta}], \in, \dot{Y}, A \cap \dot{Y}) \prec (L_\chi[A][G_{\alpha\beta}], \in, L_\chi[A], A \cap L_\chi[A])$$

with $\dot{g}(s) = s$.

$$(2) (\dot{Y}[G_{\alpha\beta}][\dot{G}_{\beta\beta+1}], \in, \dot{Y}[G_{\alpha\beta}], \dot{Y}, A \cap \dot{Y}) \prec (L_\chi[A][G_{\alpha\beta}][\dot{G}_{\beta\beta+1}], \in, L_\chi[A][G_{\alpha\beta}], L_\chi[A], A \cap L_\chi[A]).$$

Hence we may assume $q \in P_{\beta+1}$, $q \leq p$ and q forces the following.

(1) $\dot{g} \circ \dot{f}$ is an isomorphism from $(\dot{N}, \in, \dot{A} \cap \dot{N})$ to

$$(\dot{Y}, \in, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}], \in, \dot{A} \cap L_\chi[\dot{A}])$$

with $\dot{g} \circ \dot{f}(\dot{s}) = \dot{s}$.

$$(2) (\dot{Y}[G_{\alpha\beta+1}], \in, \dot{Y}, \dot{A} \cap \dot{Y}) \prec (L_\chi[\dot{A}][G_{\alpha\beta+1}], \in, L_\chi[\dot{A}], \dot{A} \cap L_\chi[\dot{A}]).$$

□

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