

Endemic threshold results for an age-structured SIR epidemic model with vertical transmission and vaccination

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1 Introduction

In this short note, we consider a mathematical model for the spread of a directly transmitted infectious disease in an age-structured population. We assume that an infection confers permanent immunity, and the infective agent can be transmitted not only by horizontally but also vertically from adult individuals to their newborns. On the other hand, for simplicity, we assume that the demographic process of the host population is not affected by the spread of the disease, since the extra mortality due to the epidemic could be neglected. Then the host population is assumed to be a demographic stable population, that is, its total size is growing exponentially but its age profile is not changing through time. Moreover we take into account existence of a vaccination program.

The age-structured SIR epidemic model with vertical transmission have been analyzed by several authors, especially we can refer to [1], [5], [6] and [7]. For SIS models the reader may refer to [2], [3] and [4]. Under the proportionate mixing assumption (that is, the transmission kernel is given by the type of separation of variable), Cha, *et al.* calculated the basic reproduction ratio R_0 and conclude that if $R_0 < 1$, there is no endemic steady state and the disease-free steady state is locally stable, while if $R_0 > 1$ there exists at least one endemic steady state. They have also provided conditions for unique existence of endemic steady state. Local stability condition for endemic steady state is also given and they give an example of unstable endemic steady state. However, so far there is no result for this model with general transmission rate (non proportionate mixing case). Hence our main purpose of this paper is to establish a most general approach to deal with the age-structured SIR epidemic model with vertical transmission and to extend the above mentioned results to the case of general transmission rate.

Since the space here is limited, we focus on the threshold condition for disease invasion and endemicity. Complete proofs of following propositions (except for some cases), well-posedness of the time evolution problem and stability results for endemic steady states will be published in a separate paper [15].

2 The basic system

First as a host population, we consider a closed one-sex age-structured host population under the demographic stable growth. Let $P(t, a)$ be the age-density at time t of the host population, $\mu(a)$ the age-specific natural death rate and $f(a)$ the age-specific fertility rate. Then we assume that the host population dynamics is described by the McKendrick equation as follows:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) P(t, a) = -\mu(a)P(t, a), \\ P(t, 0) = \int_0^\omega f(a)P(t, a)da, \\ P(0, a) = P_0(a), \end{cases} \quad (2.1)$$

where $P_0(a)$ is a given initial data and $\omega < \infty$ is the upper bound of age. The system (2.1) is well known as the stable population model in demography.

It follows from the stable population theory (see [12], [13]) that the system (2.1) has a unique persistent age profile as

$$\psi(a) := \frac{e^{-r_0 a} \ell(a)}{\int_0^\omega e^{-r_0 a} \ell(a) da},$$

where $\ell(a)$ is the survival rate defined by $\ell(a) := \exp\left(-\int_0^a \mu(\sigma) d\sigma\right)$ and r_0 , called as the *intrinsic rate of natural increase*, is given by the dominant real root of the Euler-Lotka characteristic equation:

$$\int_0^\omega e^{-ra} f(a) \ell(a) da = 1. \quad (2.2)$$

Since ω is the maximum attainable age, that is, $\ell(\omega) = 0$, we assume that $\mu \in L^1_{+,loc}(0, \omega)$ and $\int_0^\omega \mu(\sigma) d\sigma = \infty$.

Moreover, for a given initial data there exists a constant $Q > 0$ and a function $\eta(t, a)$ such that

$$P(t, a) = Qe^{r_0(t-a)} \ell(a) (1 + \eta(t, a)), \quad (2.3)$$

where $\lim_{t \rightarrow \infty} \eta(t, a) = 0$ uniformly for $a \in [0, \omega]$. Then as time evolves, the age distribution converges to the persistent age profile:

$$\lim_{t \rightarrow \infty} \frac{P(t, a)}{\int_0^\omega P(t, a) da} = \psi(a). \quad (2.4)$$

That is, ψ is relatively stable age distribution and if once it is attained, its profile is persistent. In fact, if $P_0(a) = C\psi(a)$ with a positive constant C , then $P(t, a) = Ce^{r_0 t}\psi(a)$ for $t > 0$. In the following we assume that the stable age distribution is already attained, the age density of the host population is given by $P(t, a) = N(t)\psi(a)$ where $N(t) = \int_0^\omega P(t, a)da$ is the total size of the population.

Subsequently let us divide the host population into three subpopulations; the susceptible class, the infective class and the recovered class, the age-density functions of each class are denoted by $S(t, a)$, $I(t, a)$ and $R(t, a)$. Let $\beta(a, \sigma)$ be the transmission rate between the susceptible individual aged a and the infective individual aged σ , $\gamma(a)$ the rate of recovery at age a and $\theta(a)$ the vaccination rate at age a . Then the basic system (age-structured SIR model) with vertical transmission can be formulated as follows:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) S(t, a) = -(\lambda(t, a) + \theta(a) + \mu(a))S(t, a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) I(t, a) = \lambda(t, a)S(t, a) - (\gamma(a) + \mu(a))I(t, a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) R(t, a) = \theta(a)S(t, a) + \gamma(a)I(t, a) - \mu(a)R(t, a), \\ S(t, 0) = \int_0^\omega f(a)[S(t, a) + (1 - q)I(t, a) + R(t, a)]da, \\ I(t, 0) = q \int_0^\omega f(a)I(t, a)da, \\ R(t, 0) = 0. \end{cases} \quad (2.5)$$

where the force of infection $\lambda(t, a)$ is given by

$$\lambda(t, a) = \frac{1}{N(t)} \int_0^\omega \beta(a, \sigma)I(t, \sigma)d\sigma, \quad (2.6)$$

and q is the ratio that newborns produced from infected individuals are vertically infected.

Since we assume that there is no true interaction between demography and epidemics, it is convenient to introduce the fractional age distributions for each epidemiological classes as follows:

$$s(t, a) := \frac{S(t, a)}{P(t, a)}, \quad i(t, a) := \frac{I(t, a)}{P(t, a)}, \quad r(t, a) := \frac{R(t, a)}{P(t, a)}.$$

Then the new system for the fractional age distributions is given as

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s(t, a) = -(\lambda(t, a) + \theta(a))s(t, a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) i(t, a) = \lambda(t, a)s(t, a) - \gamma(a)i(t, a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) r(t, a) = \theta(a)s(t, a) + \gamma(a)i(t, a), \\ s(t, 0) = 1 - q \int_0^\omega \pi(a)i(t, a)da, \\ i(t, 0) = q \int_0^\omega \pi(a)i(t, a)da, \\ r(t, 0) = 0, \\ \lambda(t, a) = \int_0^\omega \beta(a, \sigma)\psi(\sigma)i(t, \sigma)d\sigma, \end{cases} \quad (2.7)$$

where $\pi(a) := e^{-r_0 a}f(a)\ell(a)$, and note that it follows from (2.2) that $\int_0^\omega \pi(a)da = 1$. Moreover of course, it follows from the definition that $s(t, a) + i(t, a) + r(t, a) =$

1. In the following, we mainly consider the basic system (2.7) under the above normalization condition and the following technical assumption:

Assumption 2.1 $\beta \in L_+^\infty((0, \omega) \times (0, \omega))$ and $f, \gamma, \theta \in L_+^\infty(0, \omega)$.

3 The disease invasion process

It is easy to see that the basic system (2.7) has the disease-free steady state

$$(s^*, i^*, r^*) = (\Theta(a), 0, 1 - \Theta(a)),$$

where $\Theta(a) := \exp(-\int_0^a \theta(\sigma) d\sigma)$. If a very small number of infected individuals enter into the disease-free steady state, the initial phase of epidemic could be described by the linearized system at the disease-free steady state. Since the linearized equations for infective population does not include other subpopulation, we can only consider the single equation for infective population as

$$\frac{di(t)}{dt} = A_0 i(t) + F_0 i(t), \quad (3.1)$$

where operators A_0 and F_0 acting on $E_0 := L^1(0, \omega)$ as follows:

$$A_0 \phi := -\frac{d\phi}{da} - \gamma(a)\phi, \quad (F_0 \phi)(a) := \Theta(a)\lambda[a|\phi],$$

where $\phi \in E_0$, $\mathbb{T}$ and the domain of A_0 is given by

$$D(A_0) = \left\{ \phi \in E_0 : v \in AC[0, \omega], \phi(0) = q \int_0^\omega \pi(a)\phi(a) da \right\}.$$

In the following, we adopt the following technical assumption:

Assumption 3.1 *The transmission coefficient β satisfies the following:*

1. $\beta \in L_+^\infty(\mathbf{R} \times \mathbf{R})$ where β is extended as $\beta(a, \sigma) = 0$ for $(a, \sigma) \notin [0, \omega] \times [0, \omega]$,
2. *The following holds uniformly for $\zeta \in \mathbf{R}$:*

$$\lim_{h \rightarrow 0} \int_{-\infty}^{\infty} |\beta(a+h, \zeta) - \beta(a, \zeta)| da = 0$$

3. *There exists a nonnegative function $\eta(\sigma)$ such that $\eta(\sigma) > 0$ for a left neighborhood at $\sigma = \omega$ and $\beta(a, \sigma) \geq \eta(\sigma)$ for almost all $(a, \sigma) \in \mathbf{R} \times \mathbf{R}$.*

From the above assumption and the well known compactness criteria in L^1 , we obtain the following:

Lemma 3.2 For $\phi \in L^1(0, \omega)$, the mapping $\Lambda : \phi \rightarrow \lambda[\cdot|\phi]$ defines a compact operator from $L^1(0, \omega)$ to itself.

Then it is easy to see that $A_0 + F_0$ is a generator of an eventually norm continuous semigroup $T_0(t) = \exp((A_0 + F_0)t)$, since A_0 is a generator of a nilpotent semigroup and F_0 is a compact perturbation (Nagel [19], p. 87). Since the spectral mapping theorem holds for the eventually norm continuous semigroup, we know that

$$\omega_0(A_0 + F_0) = \sup\{\Re \lambda : \lambda \in \sigma(A_0 + F_0)\}, \quad (3.2)$$

where $\omega_0(A)$ denotes the growth bound of the semigroup $\exp(tA)$ and $\sigma(A)$ denotes the spectrum of A . Then if $\gamma > \omega_0(A)$, there exists a number $M(\gamma) \geq 1$ such that $\|\exp(tA)\| \leq M(\gamma) \exp(\gamma t)$ for $t \geq 0$. In particular, if $\omega_0(A) < 0$, the equilibrium $i = 0$ of (3.1) is asymptotically stable. From the principle of linearized stability ([8]), the stability of the equilibrium $i = 0$ in (3.1) implies the local asymptotic stability of the disease-free steady state of (2.7).

For $u \in D(A_0)$ and $v \in E_0$, let us consider the resolvent equation:

$$(z - (A_0 + F_0))^{-1}v = u, \quad z \in \mathbf{C}, \quad (3.3)$$

Then we have

$$\frac{dv}{da} + (z + \gamma(a))v = \Theta(a) \int_0^\omega \beta(a, \sigma) \psi(\sigma) v(\sigma) d\sigma + u(a), \quad (3.4)$$

$$v(0) = q < \pi, v >,$$

where we use the notation as $\langle f, g \rangle := \int_0^\omega f(a)g(a)da$. By the variation of constants formula, we can obtain the expression

$$v(a) = v(0)e^{-za}\Gamma(a) + \int_0^a e^{-z(a-\sigma)} \frac{\Gamma(a)}{\Gamma(\sigma)} [w(\sigma) + u(\sigma)] d\sigma, \quad (3.5)$$

where $w(a) := \Theta(a)\lambda[a|v]$ and $\Gamma(a) := \exp(-\int_0^a \gamma(\sigma) d\sigma)$.

Multiplying $q\pi$ to the both sides of (3.5) and integrating from zero to ω , we have

$$\begin{aligned} v(0) &= q \int_0^\omega e^{-za} \pi(a) \Gamma(a) da v(0) \\ &\quad + q \int_0^\omega \pi(a) \int_0^a e^{-z(a-\sigma)} \frac{\Gamma(a)}{\Gamma(\sigma)} w(\sigma) d\sigma da + \chi_1, \end{aligned} \quad (3.6)$$

where we use $v(0) = q < \pi, v >$ and

$$\chi_1 := q \int_0^\omega \pi(a) \int_0^a e^{-z(a-\sigma)} \frac{\Gamma(a)}{\Gamma(\sigma)} u(\sigma) d\sigma da.$$

Then (3.6) can be written as follows:

$$(1 - a_{11}(z))v(0) - \langle a_{12}(z), w \rangle = \chi_1, \quad (3.7)$$

where

$$a_{11}(z) := q \int_0^\omega e^{-za} \pi(a) \Gamma(a) da,$$

$$\langle a_{12}(z), w \rangle := q \int_0^\omega \pi(a) \int_0^a e^{-z(a-\sigma)} \frac{\Gamma(a)}{\Gamma(\sigma)} w(\sigma) d\sigma da.$$

Again multiplying $\Theta(a)\beta(a, \sigma)\psi(\sigma)$ to the both sides of (3.5) and integrating from zero to ω with respect to σ , we obtain

$$w(a) = v(0)\Theta(a) \int_0^\omega e^{-z\sigma} \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma \quad (3.8)$$

$$+ \Theta(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma e^{-z(\sigma-\eta)} \frac{\Gamma(\sigma)}{\Gamma(\eta)} w(\eta) d\eta d\sigma + \chi_2,$$

where

$$\chi_2 := \Theta(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma e^{-z(\sigma-\eta)} \frac{\Gamma(\sigma)}{\Gamma(\eta)} u(\eta) d\eta d\sigma.$$

Then (3.8) can be written as follows:

$$-a_{21}(z, a)v(0) + [(I - a_{22}(z))w](a) = \chi_2, \quad (3.9)$$

where

$$a_{21}(z, a) := \Theta(a) \int_0^\omega e^{-z\sigma} \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma,$$

and $a_{22}(z)$ is a linear operator from $L^1(0, \omega)$ into itself defined by

$$[a_{22}(z)w](a) := \Theta(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma e^{-z(\sigma-\eta)} \frac{\Gamma(\sigma)}{\Gamma(\eta)} w(\eta) d\eta d\sigma.$$

Let us define a linear operator $T(z)$ from $\mathbf{C} \times L^1(0, \omega)$ into itself as

$$T(z) \begin{bmatrix} x \\ f \end{bmatrix} = \begin{bmatrix} a_{11}(z)x + \langle a_{12}(z), f \rangle \\ a_{21}(z, \cdot)x + a_{22}(z)f \end{bmatrix}, \quad \begin{bmatrix} x \\ f \end{bmatrix} \in \mathbf{C} \times L^1(0, \omega).$$

Then under our condition, $T(z)$, $z \in \mathbf{C}$ is an analytic family of compact operators with respect to z . By using $T(z)$, we can formulate (3.7) and (3.9) as a simultaneous equation as follows:

$$(I - T(z)) \begin{bmatrix} v(0) \\ w \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}. \quad (3.10)$$

Thus the solution $(v(0), w)$ is uniquely determined, that is, the resolvent $(z - (A_0 + F_0))^{-1}$ exists if and only if $I - T(z)$ is invertible. Now we conclude that

Lemma 3.3 *Let Σ be the spectrum set of $A_0 + F_0$. Then it follows that*

$$\Sigma = \{z \in \mathbf{C} : (I - T(z)) \text{ is not invertible}\} \quad (3.11)$$

$$= \{z \in \mathbf{C} : z \text{ is pole of } (I - T(z))^{-1}\} = P_\sigma(A_0 + F_0).$$

Now we can define $T(0)$ as the *next generation operator* for the invasion at the partially immune population $(s^*, i^*, r^*) = (\Theta(a), 0, 1 - \Theta(a))$, since $T(0)$ maps the density of primary cases $(v(0), w)$ to the density of secondary cases. Hence the per-generation growth factor of the infectious population density, called as the *basic reproduction ratio*, denoted by R_0 , is given by the spectral radius, denoted by $r(T(0))$, of the next generation operator $T(0)$ (see [9], [10]).

Here in order to examine the linear operator $T(z)$, we make use of some ideas from positive operator theory. For detail of positive operator theory, the reader may refer to [14], [11], [18] and [20]. Let $B(E)$ be the set of bounded linear operators from a Banach lattice E into itself. From results by Sawashima [20] and Marek [18], we can state the following:

Proposition 3.4 *Let E be a Banach lattice and let $T \in B(E)$ be compact and nonsupporting. Then the following holds:*

- (1) $r(T) \in P_\sigma(T) \setminus \{0\}$ and $r(T)$ is a simple pole of the resolvent, that is, $r(T)$ is an algebraically simple eigenvalue of T .
- (2) The eigenspace corresponding to $r(T)$ is one-dimensional and the corresponding eigenvector $\psi \in E_+$ is a quasi-interior point. The relation $T\phi = \mu\phi$ with $\phi \in E_+$ implies that $\phi = c\psi$ for some constant c .
- (3) The eigenspace of T^* corresponding to $r(T)$ is also one-dimensional subspace of E^* spanned by a strictly positive functional $f \in E_+^*$.
- (4) Let $S, T \in B(E)$ be compact and nonsupporting. Then $S \leq T$, $S \neq T$ and $r(T) \neq 0$ implies $r(S) < r(T)$.

Roughly speaking, we can expect that even for positive operators in the ordered Banach space, the Perron-Frobenius properties hold just like the case of positive irreducible matrices.

Lemma 3.5 *For $z \in \mathbf{R}$, $T(z)$ is compact and nonsupporting.*

By using the above results, we can relate the Malthusian parameter of the infected population to the next generation operator and its spectral radius:

Proposition 3.6 Let $\Sigma := \{z \in \mathbf{C} : 1 \in P_\sigma(T(z))\}$. There exists a unique $z_0 \in \mathbf{R} \cap \Sigma$ such that $r(T(z_0)) = 1$ and $z_0 > 0$ if $r(T(0)) > 1$; $z_0 = 0$ if $r(T(0)) = 1$; $z_0 < 0$ if $r(T(0)) < 1$, and it is the dominant characteristic root as

$$\omega(A_0 + F_0) = z_0 > \sup\{\operatorname{Re} z : z \in \Sigma \setminus \{z_0\}\}. \quad (3.12)$$

From the above result, we can state the threshold criterion as follows:

Proposition 3.7 Let $R_0 = r(T(0))$. If $R_0 < 1$, the disease-free steady state is globally asymptotically stable, while it is unstable if $R_0 > 1$.

As an important special case, we briefly consider the proportionate mixing assumption (in the following, we call it as PMA), that is, the transmission rate β can be written as $\beta(a, \sigma) = \beta_1(a)\beta_2(\sigma)$. In this case we can calculate the threshold condition explicitly:

Proposition 3.8 Suppose that β can be factorized as $\beta(a, \sigma) = \beta_1(a)\beta_2(\sigma)$, where β_1 and β_2 are assumed to be nonnegative essentially bounded functions. Let R be a reproduction number defined by

$$R := q \frac{\int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \Theta(\sigma) \beta_1(\sigma) d\sigma da}{1 - q < \pi, \Gamma >} \int_0^\omega \beta_2(\sigma) \psi(\sigma) \Gamma(\sigma) d\sigma \quad (3.13)$$

$$+ \int_0^\omega \beta_2(\sigma) \psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\eta)} \Theta(\eta) \beta_1(\eta) d\eta d\sigma.$$

Then $R_0 > 1$ if $R > 1$, $R_0 = 1$ if $R = 1$ and $R_0 < 1$ if $R < 1$.

From the above proposition, we know that the reproduction number R can be seen as the basic reproduction ratio for the PMA case.

4 Existence and bifurcation of endemic steady states

We have so far shown that there is no endemic steady state if $R_0 < 1$. In this section, we consider the existence of endemic steady states and their bifurcation from the disease-free steady state at $R_0 = 1$.

Let (s^*, i^*, r^*) be the density vector at the endemic steady state, then it must satisfy the following system:

$$\begin{cases} \frac{d}{da} s^*(a) = -(\theta(a) + \lambda^*(a))s^*(a), \\ \frac{d}{da} i^*(a) = \lambda^*(a)s^*(a) - \gamma(a)i^*(a), \\ \frac{d}{da} r^*(a) = \theta(a)s^*(a) + \gamma(a)i^*(a), \\ s^*(0) = 1 - q \int_0^\omega \pi(a)i^*(a)da = 1 - i^*(0), \\ i^*(0) = q \int_0^\omega \pi(a)i^*(a)da, \\ r^*(0) = 0, \\ \lambda^*(a) = \lambda[a|i^*] := \int_0^\omega \beta(a, \sigma)\psi(\sigma)i^*(\sigma)d\sigma. \end{cases} \quad (4.1)$$

By formal integration, we obtain the following expression:

$$s^*(a) = (1 - i^*(0))e^{-\int_0^a \lambda^*(\sigma) d\sigma} \Theta(a). \quad (4.2)$$

$$i^*(a) = i^*(0)\Gamma(a) + (1 - i^*(0)) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \Theta(\sigma) \lambda^*(\sigma) e^{-\int_0^\sigma \lambda^*(z) dz} d\sigma. \quad (4.3)$$

Applying π to the both sides of (4.2) and integrating from 0 to ω , we obtain

$$\langle \pi, i^* \rangle = i^*(0) \langle \pi, \Gamma \rangle + (1 - i^*(0)) \int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \Theta(\sigma) \lambda^*(\sigma) e^{-\int_0^\sigma \lambda^*(z) dz} d\sigma da,$$

where $\langle \pi, i^* \rangle := \int_0^\omega \pi(a) i^*(a) da$.

Then we know that $i^*(0) = q \langle \pi, \Gamma \rangle$ is given by the functional G as

$$i^*(0) = G(\lambda^*) := \frac{q \int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \Theta(\sigma) \lambda^*(\sigma) e^{-\int_0^\sigma \lambda^*(\zeta) d\zeta} d\sigma da}{1 - q \langle \pi, \Gamma \rangle + q \int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \Theta(\sigma) \lambda^*(\sigma) e^{-\int_0^\sigma \lambda^*(\zeta) d\zeta} d\sigma da}.$$

Inserting (4.3) into the expression of λ in (4.1) and using the functional G , we have

$$\begin{aligned} \lambda^*(a) &= G(\lambda^*) \int_0^\omega \beta(a, \sigma) \psi(\sigma) \Gamma(\sigma) d\sigma \\ &+ (1 - G(\lambda^*)) \int_0^\omega \beta(a, \sigma) \psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\zeta)} \Theta(\zeta) \lambda^*(\zeta) e^{-\int_0^\zeta \lambda^*(\eta) d\eta} d\zeta d\sigma. \end{aligned} \quad (4.4)$$

Let us define a positive operator $H : L_+^1 \rightarrow L_+^1 \cap L^\infty$ by

$$\begin{aligned} H(\lambda)(a) &:= G(\lambda) \int_0^\omega \beta(a, \sigma) \psi(\sigma) \Gamma(\sigma) d\sigma \\ &+ (1 - G(\lambda)) \int_0^\omega \beta(a, \sigma) \psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\zeta)} \Theta(\zeta) \lambda(\zeta) e^{-\int_0^\zeta \lambda(\eta) d\eta} d\zeta d\sigma. \end{aligned} \quad (4.5)$$

for $\lambda \in L^1(0, \omega)$. Then from (4.5) we know that the force of infection at the endemic steady state λ^* is given by positive solutions of fixed point equation:

$$\lambda^*(a) = H(\lambda^*)(a). \quad (4.6)$$

From our basic assumption 3.1, the operator H is a compact operator from $L^1(0, \omega)$ into itself. Then we know that the endemic steady state exists if and only if H has a positive fixed point.

Proposition 4.1 *If $R_0 > 1$, there exists at least one endemic steady state, while there is no endemic steady state if $R_0 \leq 1$.*

Proof. First we can observe that the Fréchet derivative $W_0 := \partial H[0]$ of the operator H at the origin is given by

$$(W_0\lambda)(a) = \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\zeta)} \Theta(\zeta)\lambda(\zeta)d\zeta d\sigma \quad (4.7)$$

$$+ \frac{q \int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \Theta(\sigma)\lambda(\sigma)d\sigma da}{1 - q < \pi, \Gamma >} \int_0^\omega \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma.$$

Since W_0 is also compact and nonsupporting, it has a unique positive eigenvector corresponding to its spectral radius $r(W_0)$. On the other hand, it is easy to see that the strong asymptotic derivative of H is zero; $\partial H[\infty] = 0$. Therefore, we can apply the Krasnoselski's fixed point theorem ([17], p. 135, Theorem 4.11) to conclude that H has at least one non-zero fixed point in the positive cone of L_+^1 if $r(W_0) > 1$. Next we show that $R_0 > 1$ if and only if $r(W_0) > 1$. Observe that for $z \geq 0$, (3.6) and (3.8) can be written as

$$v(0) = \frac{\langle a_{12}(z), w \rangle}{1 - a_{11}(z)} + \frac{\chi_1(u)}{1 - a_{11}(z)}.$$

Inserting the above expression into (3.8) and define an operator $\Psi(z)$ as

$$(\Psi(z)w)(a) := \Theta(a) \int_0^\omega e^{-z\sigma} \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma \frac{\langle a_{12}(z), w \rangle}{1 - a_{11}(z)} \quad (4.8)$$

$$+ \Theta(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma e^{-z(\sigma-\eta)} \frac{\Gamma(\sigma)}{\Gamma(\eta)} w(\eta)d\eta d\sigma,$$

then (3.8) can be written as follows:

$$w_z = \Psi(z)w_z + \chi_2 + \frac{\chi_1 \Theta(a) \int_0^\omega e^{-z\sigma} \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma}{1 - a_{11}(z)}. \quad (4.9)$$

which means that for $z \geq 0$, $z \in \Sigma$ if and only if $1 \in P_\sigma(\Psi(z))$. Define an operator L such that $(L\phi)(a) = \Theta(a)\phi(a)$, then we obtain that $\Psi(0) = LW_0L^{-1}$, hence $r(\Psi(0)) = r(W_0)$. Suppose that $r(W_0) > 1$, then we have $r(\Psi(0)) > 1$. Since $\Psi(z)$, $z \geq 0$ is compact and nonsupporting and it is monotone decreasing with respect to $z \geq 0$, then there exists a unique $z_0 > 0$ such that $r(\Psi(z_0)) = 1$. Thus $z_0 \in \Sigma$ and $R_0 = r(T(0)) > 1$ (see Prop. 4.6). Conversely if $R_0 = r(T(0)) > 1$, there exists a positive $z_0 \in \Sigma$ such that $r(T(z_0)) = 1$ and there exists a positive vector (x, f) satisfying

$$T(z_0) \begin{bmatrix} x \\ f \end{bmatrix} = \begin{bmatrix} x \\ f \end{bmatrix},$$

which implies that $\Psi(z_0)$ has a positive eigenvector f corresponding to the eigenvalue one. Since $\Psi(z_0)$ is compact and nonsupporting, it has unique positive eigenvector corresponding to its spectral radius, hence we conclude that $r(\Psi(z_0)) = 1$. Since $r(\Psi(z))$ is monotone decreasing for $z \geq 0$, we have $r(\Psi(0)) = r(W_0) > 1$. Therefore $R_0 > 1$ if and only if $r(W_0) > 1$ and there exists at least one endemic steady state if $R_0 > 1$. Next suppose that $R_0 \leq 1$, that is, $r(\Psi(0)) = r(W_0) \leq 1$. If there exists a positive fixed point λ^* of H , we have $\lambda^* = H(\lambda^*) \leq W_0 \lambda^*$. Let F_0 be the adjoint eigenvector of W_0 corresponding to $r(W_0)$. Taking the duality pairing, we find that $\langle F_0, W_0 \lambda^* - \lambda^* \rangle = (r(W_0) - 1) \langle F_0, \lambda^* \rangle > 0$, because $W_0 \lambda^* - \lambda^* \in L_+^1 \setminus \{0\}$ and F_0 is a strictly positive eigenfunctional. Then we have $r(W_0) > 1$, which contradicts our assumption. Therefore there is no endemic steady state if $R_0 \leq 1$. $\square$

From the above proof, we know that $r(W_0) > 1$ if $R_0 > 1$, $r(W_0) = 1$ if $R_0 = 1$ and $r(W_0) < 1$ if $R_0 < 1$. Then we know that $r(W_0)$ is acting as a threshold value, so in the following we define $R_* := r(W_0)$ as a basic reproduction ratio.

If we can adopt the proportionate mixing assumption, that is, the transmission rate can be factorized as $\beta(a, \sigma) = \beta_1(a)\beta_2(\sigma)$, the force of infection at the endemic steady state λ^* is given as $\lambda^*(a) = c\beta_1(a)$ with a positive number c . Then the fixed point equation (4.6) is reduced to the following characteristic equation for unknown number c as

$$1 = \mathcal{H}(c) := \frac{G(c\beta_1)}{c} \int_0^\omega \beta_2(\sigma)\psi(\sigma)\Gamma(\sigma)d\sigma \quad (4.10)$$

$$+ (1 - qG(c\beta_1)) \int_0^\omega \beta_2(\sigma)\psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\zeta)} \Theta(\zeta)\beta_1(\zeta)e^{-c \int_0^\zeta \beta_1(\eta)d\eta} d\zeta d\sigma.$$

Since $\mathcal{H}(0) = R_*$ and $\mathcal{H}(\infty) = 0$, we can again confirm that there exists at least one endemic steady state if $R_* > 1$ (equivalently if $R > 1$ in (3.13)).

If $\mathcal{H}$ becomes a monotone function under some additional conditions, we can prove the uniqueness of the endemic steady state. For example, if we assume that there exists an age $A \in (0, \omega)$ such that $\beta_2(a) = 0$ for $a > A$ and $\beta_1(a) = 0$ for $a < A$, then the second term of $\mathcal{H}$ in (4.10) becomes zero and $\mathcal{H}(c)$ is monotone. However, such additional assumptions to guarantee the monotonicity of $\mathcal{H}$ are usually very restrictive, so far we have no biologically reasonable one. Though here we do not examine such additional conditions to guarantee the uniqueness of endemic steady state, the readers who are interested in the uniqueness problem may refer to Cha, *et al.* [5], [6].

More important basic observation is that the endemic steady states are given by forward bifurcation from the disease-free steady state. In fact, this is intuitively clear for the PMA case, since $\mathcal{H}'(0) < 0$. Here we give a proof for the general transmission case by using a bifurcation scenario as follows:

Assumption 4.2 The transmission rate β is given by $\epsilon\beta_0(a, \sigma)$ where ϵ is a bifurcation parameter and β_0 is a given standard schedule such that $R_* = r(\partial H_0[0]) = 1$.

Proposition 4.3 Under the assumption 4.2, the endemic steady states are forwardly bifurcated from the disease-free steady state at $R_* = 1$.

Proof. Under the assumption 4.2, the fixed point equation (4.6) is written as $\lambda = \epsilon H_0(\lambda)$. Define a mapping $F : \mathbf{R} \times L^1 \rightarrow L^1$ as $F(\lambda, \epsilon) := \epsilon H_0(\lambda) - \lambda$ and assume that $F(\lambda, \epsilon)$ is analytic with respect to (λ, ϵ) . Now we are interested in the structure of solution set $F^{-1}(0) := \{(\lambda, \epsilon) \in L^1(0, \omega) \times \mathbf{R}_+ : F(\lambda, \epsilon) = 0\}$. From the Implicit Function Theorem, we can expect a bifurcation from the trivial branch $(0, \epsilon)$ only for those values ϵ such that the linear mapping

$$L(\epsilon) := D_1 F(0, \epsilon) = \epsilon \partial H_0[0] - I,$$

is not boundedly invertible, where D_1 denotes the Fréchet derivative for the first element and I is the identity operator. It follows from our assumption that $\partial H_0[0]$ has a unique positive eigenvalue $r(\partial H_0[0]) = 1$, since $\partial H_0[0]$ is compact and nonsupporting. Then the only possible bifurcation from the trivial branch can occur at $\epsilon = 1$. Let $\sigma(\epsilon)$ be the simple real strictly dominant eigenvalue of $L(\epsilon)$, $\phi(\epsilon)$ the eigenvector of $L(\epsilon)$ and $\phi^*(\epsilon)$ the eigenvector of $L^*(\epsilon)$ (the adjoint operator of $L(\epsilon)$) associated with $\sigma(\epsilon)$ such that $\langle \phi(\epsilon), \phi^*(\epsilon) \rangle = 1$, where $\langle \phi, \phi^* \rangle$ is the value of ϕ^* at ϕ . Since $\phi(1)$ is the Frobenius eigenvector of the nonsupporting operator $\partial H_0[0]$ corresponding to the eigenvalue one, there exist a projection to the one-dimensional eigenspace spanned by $\phi(1)$. Then we can apply the standard argument of Lyapunov-Schmidt method ([21], Chapter VII) to conclude that the bifurcation at $(0, 1)$ is subcritical if $\tau_1 < 0$, and it is supercritical if $\tau_1 > 0$, where the parameter τ_1 is given by

$$\tau_1 = -\frac{1}{2} \langle D_1^2 F(0, 1)(\phi(1), \phi(1)), \phi^*(1) \rangle, \quad (4.11)$$

where D_1^2 denotes the second derivative with respect to the first element. It is easy to see that

$$\tau_1 = - \left. \frac{\partial^2 H_0((h+k)\phi(1))}{\partial h \partial k} \right|_{(h,k)=(0,0)} > 0.$$

Therefore we conclude that the bifurcation at $R_* = 1$ is supercritical. $\square$

Finally note that we can define a next generation operator at the endemic steady state. From the variation of constants formula, it follows from (4.1) that

$$i^*(a) = i^*(0)\Gamma(a) + \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \lambda^*(\sigma) s^*(\sigma) d\sigma. \quad (4.12)$$

Applying $q\pi$ to the both sides of (4.12) and integration from zero to ω , we obtain an expression:

$$i^*(0) = i^*(0)q < \pi, \Gamma > + \int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} \lambda^*(\sigma) s^*(\sigma) d\sigma da. \quad (4.13)$$

Again applying $s^*(a)\beta(a, \sigma)\psi(\sigma)$ to the both sides of (4.12) and integrating from 0 to ω with respect to σ and multiplying s^* , we obtain the following expression:

$$\begin{aligned} s^*(a)\lambda^*(a) &= i^*(0)s^*(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma \\ &+ s^*(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\eta)} s^*(\eta)\lambda^*(\eta)d\eta d\sigma. \end{aligned} \quad (4.14)$$

Now let us define a positive linear operator T^* from $\mathbf{R} \times L^1(0, \omega)$ into itself as

$$T^* \begin{bmatrix} x \\ f \end{bmatrix} = \begin{bmatrix} q < \pi, \Gamma > x + q \int_0^\omega \pi(a) \int_0^a \frac{\Gamma(a)}{\Gamma(\sigma)} f(\sigma) d\sigma da \\ s^*(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma)\Gamma(\sigma)d\sigma + s^*(a) \int_0^\omega \beta(a, \sigma)\psi(\sigma) \int_0^\sigma \frac{\Gamma(\sigma)}{\Gamma(\eta)} f(\eta) d\eta d\sigma \end{bmatrix}.$$

Then from (4.13)-(4.14), the newly infected population $(i^*(0), s^*(a)\lambda^*(a))$ can be formally seen as the eigenvector of the operator T^* corresponding to the eigenvalue one:

$$\begin{bmatrix} i^*(0) \\ s^*\lambda^* \end{bmatrix} = T^* \begin{bmatrix} i^*(0) \\ s^*\lambda^* \end{bmatrix}. \quad (4.15)$$

The equation (4.15) implies that at the endemic steady state the infected population simply reproduce itself. Therefore we can call T^* as the *next generation operator at the endemic steady state*. This fact will be used to show the stability of the endemic steady state (see [15]), and this formulation could provide an intuitive understanding about whether multiple endemic steady states can occur (see [16]).

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