

ON THE COHOMOLOGY OF THE MODULI SPACE OF PARABOLIC CONNECTIONS

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Introduction

In this poster, we study the moduli space of logarithmic connections of rank 2 on $\mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ with fixed spectral data. We compute the cohomology of such moduli space, and this computation will be used to extend the results of geometric Langlands correspondence of [1] to the case where the parabolic connections have five simple poles on $\mathbb{P}^1$.

Preliminaries

We introduce $\mathfrak{sl}_2$ -connections.

Fix complex numbers $\nu_1, \dots, \nu_n \in \mathbb{C}$. Suppose that $\nu_1 \cdots \nu_n \neq 0$ and

$$\sum_{i=1}^n \epsilon_i \nu_i \notin \mathbb{Z} \quad (1)$$

for any $(\epsilon_i), \epsilon_i \in \{1, -1\}$.

Definition 2.1 A ν - $\mathfrak{sl}_2$ -parabolic connection on $\mathbb{P}^1$ is a triplet (E, ∇, φ) such that

1. E is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla : E \rightarrow E \otimes \Omega_{\mathbb{P}^1}^1(D)$ is a connection, where $D := t_1 + \cdots + t_n$,
3. $\varphi : \wedge^2 E \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\text{res}_i(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^{\pm}$, $1 \leq i \leq n$.

Here, we put

$$\nu_i^{\pm} := \pm \nu_i \quad (i = 1, \dots, n-1), \quad \nu_n^+ := \nu_n, \quad \nu_n^- := 1 - \nu_n.$$

Denote by $\mathcal{M}$ the moduli stack of ν - $\mathfrak{sl}_2$ -parabolic connections on $\mathbb{P}^1$, and by M its coarse moduli space.

For such ν , the parabolic direction $l_i := \ker(\text{res}_i(\nabla) - \nu_i) \subset E|_{t_i}$ is uniquely determined. So, we can get the forgetful map

$$\text{Bun} : M \rightarrow P := (E, \nabla, \varphi) \mapsto (E, l_i)$$

where P is the coarse moduli space of undecomposable quasi-parabolic bundles (E, l_i) on $\mathbb{P}^1$.

Now, we consider the following stratification of M . By the irreducibility of $(E, \nabla, \varphi) \in M$, we have the following proposition.

Proposition 2.2 For $(E, \nabla, \varphi) \in M$, we have

$$E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k-1) \quad \text{where } 0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Denote by M^k the subvariety of M where $E \simeq \mathcal{O}(k) \oplus \mathcal{O}(-k-1)$. Then

$$M = M^0 \cup \cdots \cup M^{\lfloor (n-3)/2 \rfloor}.$$

Note that the stratum M^0 is a Zariski open dense of M .

Geometric description of M^0

Suppose $n = 5$. For computation of the cohomology of M , we introduce some blowing-up of the Hirzebruch surface $\mathbb{F}_3$. Put $L := \mathcal{O}_{\mathbb{P}^1}(D)$. Let $\tilde{L}$ be the total space of the line bundle L . Note that $\tilde{L} = \mathbb{F}_3 \setminus s_{\infty}$ where s_{∞} is the infinity section ($s_{\infty}^2 = -3$). Let $\pi : \tilde{L} \rightarrow \mathbb{P}^1$ be the projection and let $\tau_i : \tilde{F}_i \simeq \pi^{-1}(t_i) \xrightarrow{\sim} \mathbb{C}$ be the residue map. Put $\nu_i^+ := \nu_i$, $\nu_i^- := -\nu_i$ for $i = 1, \dots, 4$, $\nu_5^+ := \nu_5$, $\nu_5^- := 1 - \nu_5$ and $\tilde{\nu}_i^{\pm} := \tau_i^{-1}(\nu_i^{\pm})$. Set

$$K_{5,q}^{\pm} := (\text{Bl}_{\nu^{\pm}} \tilde{L}) \setminus (\tilde{F}_1 \cup \cdots \cup \tilde{F}_5)$$

where $\text{Bl}_{\nu^{\pm}} \tilde{L}$ is the blowing-up of $\tilde{L}$ at $\tilde{\nu}_i^{\pm}$ for $i = 1, \dots, n$, and $\tilde{F}_i$ are the proper pre-images of the fiber F_i $i = 1, \dots, n$.

Lemma 3.1

$$H^i(K_{5,q}^{\pm}, \mathcal{O}_{K_{5,q}^{\pm}}) = \begin{cases} \mathbb{C}, & i = 0, \\ H_n^{\pm}(A) \neq 0, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

where $(A, \mathfrak{m})$ is a local ring such that $\dim(A_{\mathfrak{m}}) = 2$.

We denote by K_5 the image of $K_{5,q}^{\pm}$ under the projection $K_{5,q}^{\pm} \rightarrow \mathbb{F}_3 \setminus s_{\infty}$. We can define the following map

$$M^0 \rightarrow \text{Sym}^2(K_5) \\ (E, \nabla, \varphi) \mapsto \{(q_1, p_1), (q_2, p_2)\}, \quad (2)$$

We consider the composite of the Hilbert-Chow morphism and the blowing-up

$$\text{Hilb}^2(K_5) \rightarrow \text{Sym}^2(K_5) \rightarrow \text{Sym}^2(K_5).$$

We have the next proposition.

Proposition 3.2 ([3] Theorem 5.2) We can extend the map (2) to

$$M^0 \longrightarrow \text{Hilb}^2(K_5)$$

and this map is injective.

Cohomology of $K_{5,q}^{\pm}$

We denote by $Z \subset \text{Sym}^2(K_5)$ the proper pre-image of $\{(q_1, p_1), (q_1, -p_1)\} \subset \text{Sym}^2(K_5)$ under the blowing-up $\text{Sym}^2(K_5) \rightarrow \text{Sym}^2(K_5)$, and by $\tilde{Z} \subset \text{Hilb}^2(K_5)$ the proper pre-image of Z under the Hilbert-Chow morphism $\text{Hilb}^2(K_5) \rightarrow \text{Sym}^2(K_5)$. Denote by

$$\widetilde{\text{Hilb}}^2(K_5) \rightarrow \text{Hilb}^2(K_5) \quad (3)$$

the blowing-up along $\tilde{Z}$ and by $\tilde{Z}$ the strict transform of $\tilde{Z}$. We also denote by $(K_5^{\pm} \times K_5^{\pm})^{\sim}$ the blowing-up of $K_5^{\pm} \times K_5^{\pm}$ along the ideal $(q_1 - q_2, p_1 - p_2)$, and by $(K_5^{\pm} \times K_5^{\pm})^{\sim}$ the blowing-up of $(K_5^{\pm} \times K_5^{\pm})^{\sim}$ along the ideal $(q_1 - q_2, p_1 + p_2)$. Then $\text{Hilb}^2(K_5^{\pm}) = (K_5^{\pm} \times K_5^{\pm})^{\sim} / \mathfrak{S}_2$ and $\widetilde{\text{Hilb}}^2(K_5^{\pm}) = (K_5^{\pm} \times K_5^{\pm})^{\sim} / \mathfrak{S}_2$. Now, using above description, we define another important blowing-up of Hirzebruch surface $\mathbb{F}_3$. Fix $q_i \in \mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ and define the fiber $\tilde{F}_i$ over q_i . We denote by $(\mathbb{F}_3)^{\sim}$ the blowing-up of $\mathbb{F}_3$ at two points $\{(q_1, p_1), (q_1, -p_1)\}$ (when $p_1 = p_2 = 0$, blow up twice at $(q_1, 0)$). Set

$$K_{5,q_1}^{\pm} := (\mathbb{F}_3)^{\sim} \setminus (s_{\infty} \cup \tilde{F}_1 \cup \cdots \cup \tilde{F}_5)$$

where $\tilde{F}_i$ is the strict transform of F_i . We denote by K_{5,q_1} the image of $K_{5,q_1}^{\pm}$ under the projection $K_{5,q_1}^{\pm} \rightarrow \mathbb{F}_3 \setminus s_{\infty}$. Define

$$D_{q_1} := 2s_{\infty} + \tilde{F}_1 + \cdots + \tilde{F}_5 + \tilde{F}_5$$

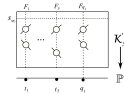
Then, we have

$$(s_{\infty}, \tilde{F}_i) = 1, (\tilde{F}_i, \tilde{F}_i) = 0, (s_{\infty}, s_{\infty}) = -3, (\tilde{F}_i, \tilde{F}_i) = -2$$

This implies

$$(D_{q_1}, D_{q_1}) = (D_{q_1}, s_{\infty}) = (D_{q_1}, \tilde{F}_i) = 0.$$

This is the same condition of Arinkin Lysenko [2], section 5.2.



Proposition 4.1

$$H^i(K_{5,q}^{\pm}, \mathcal{O}_{K_{5,q}^{\pm}}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Cohomology of M

Set $\widetilde{M}_Z := M^0 \cup \tilde{Z}$. By Proposition 3.2, we have injective maps $\iota : M^0 \hookrightarrow \text{Hilb}^2(K_5^{\pm})$ and $\iota : \widetilde{M}_Z \hookrightarrow \widetilde{\text{Hilb}}^2(K_5^{\pm})$. We define the blowing-up parameter $\lambda := p_1 + p_2 = \lambda - (q_1 - q_2)$. Set $T := \widetilde{\text{Hilb}}^2(K_5^{\pm}) \setminus \widetilde{M}_Z$. For a vector bundle $\mathcal{F}$ on $\widetilde{\text{Hilb}}^2(K_5^{\pm})$,

$$H^i(\widetilde{M}_Z, \mathcal{F}|_{\widetilde{M}_Z}) = H^i(\widetilde{\text{Hilb}}^2(K_5^{\pm}), \iota_* \iota^* \mathcal{F}) \\ = \varinjlim H^i(\widetilde{\text{Hilb}}^2(K_5^{\pm}), \mathcal{F}(kT)).$$

To compute $H^i(\widetilde{\text{Hilb}}^2(K_5^{\pm}), \mathcal{F}(kT))$, consider $H^i((K_5^{\pm} \times K_5^{\pm})^{\sim}, \mathcal{F}(kT'))$, where T' is defined by $(\lambda_{\infty} = \infty)$. We can define a map

$$f : (K_5^{\pm} \times K_5^{\pm})^{\sim} \setminus T' \rightarrow K_5^{\pm} \\ (q_1, p_1, q_2, p_2) \mapsto (q_1, p_1),$$

and the fiber $f^{-1}(\{(q_1, p_1)\}) \simeq K_{5,q_1}^{\pm}$. By Leray's spectral sequence, we have

$$H^i((K_5^{\pm} \times K_5^{\pm})^{\sim} \setminus T', \mathcal{F}) \simeq \bigoplus_{p+q=i} H^p(K_{5,q_1}^{\pm}, R^q f_* \mathcal{F}).$$

Using the Base change theorem, we have $(R^q f_* \mathcal{F})_{(q_1, p_1)} \simeq H^q(K_{5,q_1}^{\pm}, \mathcal{F}|_{(q_1, p_1)})$. By using Lemma 3.1 and Proposition 4.1, we have

$$H^i((K_5^{\pm} \times K_5^{\pm})^{\sim} \setminus T', \mathcal{O}) = \begin{cases} \mathbb{C}, & i = 0, \\ H_n^{\pm}(A) \neq 0, & i = 1, \\ 0, & i > 1. \end{cases}$$

Moreover, the action of $\mathfrak{S}_2$ on $H^1((K_5^{\pm} \times K_5^{\pm})^{\sim} \setminus T', \mathcal{O})$ is nontrivial. Therefore,

$$H^i(\widetilde{M}_Z, \mathcal{O}_{\widetilde{M}_Z}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Since $\text{codim}_{\text{Hilb}^2(K_5^{\pm})}(\tilde{Z}) = 2$, and $M^1 = M \setminus M^0 \simeq A^2$, we have

Theorem 5.1

$$H^i(M, \mathcal{O}_M) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

References

- [1] D. Arinkin, *Orthogonality of natural sheaves on moduli stacks of $SL(2)$ -bundles with connections on P^1 minus 4 points*, Selecta Math., New Series 7 (2001), 213-239.
- [2] D. Arinkin, S. Lysenko, *On the moduli of $SL(2)$ -bundles with connections on $\mathbb{P}^1 \setminus \{x_1, \dots, x_4\}$* , Internat. Math. Res. Notices (1997), no. 19, 983-999.
- [3] A. Kuroyo, M.-H. Saito, *Explicit description of jumping phenomena on moduli spaces of parabolic connections and Hilbert schemes of points on surfaces*, accepted in Kyoto Journal of Mathematics, (arXiv:math/1611.00971)