

## Metrizability of GO-spaces, and $k$ -spaces

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In this paper, first we shall make a survey of metrizability theorems by means of spaces with certain  $k$ -networks, GO-spaces, or topological groups. Then, we will give some metrizability theorems on GO-spaces or topological groups in terms of weak topology.

**Definition 1:** As is well-known, a *linearly ordered topological space* (abbreviated LOTS) is a triple  $(X, \mathcal{T}, \leq)$ , where  $(X, \leq)$  is a linearly ordered (= totally ordered) set, and  $\mathcal{T}$  is the order topology by the order  $\leq$ ; that is,  $\{(\alpha, +\infty), (-\infty, \alpha) : \alpha \in X\}$  is a subbase for  $\mathcal{T}$ , here  $(\alpha, +\infty) = \{x \in X : x > \alpha\}$ ,  $(-\infty, \alpha) = \{x \in X : x < \alpha\}$ .

A space  $X$  is a *generalized ordered space* (abbreviated GO-space) if  $X$  is a subspace of a LOTS  $Y$ , where the order of  $X$  is the one induced by the order of  $Y$ . For many important properties of GO-spaces, see [8] or [10], for example.

We recall that a space  $(X, \mathcal{T})$  is *orderable* if there exists a linear order  $\leq$  on  $X$  such that the order topology on  $X$  given by  $\leq$  coincides with the topology  $\mathcal{T}$ . Obviously, a space  $X$  is orderable iff it is homeomorphic to a LOTS. A space  $X$  is called *suborderable* if it is homeomorphic to a GO-space.

*Examples:* (1) The Sorgenfrey line, or the Michael line, etc. is a GO-space, but it is not a LOTS with the usual ordering, not even orderable. Also, any Stone-Čech compactification  $\beta(X)$  of a completely regular, non-countably compact space  $X$  is not orderable ([17]).

(2) Let  $S = \{0\} \cup (1, 2)$  be a subspace of the real line  $R$ , and let  $D$  be an infinite countable discrete space. Then,  $S$  is the topological sum  $\{0\} \cup (1, 2)$  of LOTS' in  $R$ , and also,  $S$  is an open and closed subset of the product space  $S \times D$  which is orderable. However,  $S$  is not orderable.

(3) A subspace  $\{0\} \cup (1, 2]$  of  $R$  with the usual ordering is not a LOTS, but it is orderable. Also, a space  $X = [0, \omega_1]$  obtained by isolating every countable limit ordinal is not a LOTS with the usual ordering, but  $X$  is orderable (cf. [8]).

(4) Let  $X$  be the quotient space of the topological sum of three unit intervals  $[0, 1]$  by identifying the point 0 to a point. Then,  $X$  is a union of two closed LOTS', but  $X$  is not suborderable.

In this paper, however, let us say that a space  $X$  is a LOTS (resp. GO-space) if  $X$  is orderable (resp. suborderable), for it will cause no confusion.

**Definition 2:** A space  $X$  is determined by a cover  $\mathcal{C}$  if  $F \subset X$  is closed in  $X$  iff  $F \cap C$  is closed in  $C$  for every  $C \in \mathcal{C}$ . We use “ $X$  is determined by  $\mathcal{C}$ ” instead of the usual “ $X$  has the weak topology with respect to  $\mathcal{C}$ ”.

A space is a  $k$ -space (resp. sequential space) if it is determined by a cover of compact subsets (resp. compact metric subsets). A space is a quasi- $k$ -space (Nagata [9]) if it is determined by a cover of countably compact subsets.

As is well-known, every  $k$ -space (resp. sequential space) is precisely a quotient image of a locally compact space (resp. metric space). Also, every quasi- $k$ -space (resp.  $k$ -space) is characterized as a quotient image of an  $M$ -space (resp. paracompact  $M$ -space); see [9].

A space  $X$  has countable tightness (or,  $t(X) \leq \omega$ ) if, whenever  $x \in clA$ , then  $x \in clB$  for some countable subset  $B$  with  $B \subset A$ . It is well-known that  $t(X) \leq \omega$  iff  $X$  is determined by a cover of countable subsets.

Sequential spaces are  $k$ -spaces of countable tightness, and  $k$ -spaces are quasi- $k$ -spaces.

**Definition 3:** Let  $\mathcal{P}$  be a cover of a space  $X$ . Then,  $\mathcal{P}$  is a  $k$ -network for  $X$ , if whenever  $K \subset U$  with  $K$  compact and  $U$  open in  $X$ ,  $K \subset \bigcup \mathcal{P}' \subset U$  for some finite  $\mathcal{P}' \subset \mathcal{P}$ . Also,  $\mathcal{P}$  is a  $wcs^*$ -network if whenever  $L$  is a sequence converging to a point  $x \in X$  and  $U$  is a nbd of  $x$ , some  $P \in \mathcal{P}$  is contained in  $U$ , but contains the sequence  $L$  frequently. Every  $k$ -network is a  $wcs^*$ -network.

CW-complexes, Lašnev spaces, or quotient  $s$ -images of metric spaces are sequential spaces having a point-countable  $k$ -network.

**Definition 4:** A space  $X$  is a  $w\Delta$ -space if there exists a sequence  $\{\mathcal{U}_n; n \in \mathbb{N}\}$  of open covers of  $X$  such that if  $x \in X$  and  $x_n \in St(x, \mathcal{U}_n)$ , then the sequence  $\{x_n\}$  has an accumulation point in  $X$ . Every developable space, or every  $M$ -space is a  $w\Delta$ -space. Recall that a space  $X$  is a  $\Sigma$ -space, if there exist a  $\sigma$ -locally-finite closed cover  $\mathcal{F}$  in  $X$ , and a cover  $\mathcal{C}$  of countably compact closed subsets in  $X$  such that, for  $C \subset U$  with  $C \in \mathcal{C}$  and  $U$  open in  $X$ ,  $C \subset F \subset U$  for some  $F \in \mathcal{F}$ . Every  $M$ -space,  $\sigma$ -space, or locally compact GO-space [8] is a  $\Sigma$ -space.

A collection  $\mathcal{C}$  in  $X$  is compact-finite if any compact subset of  $X$  meets only finitely many elements of  $\mathcal{C}$ .

We assume that spaces are regular  $T_1$ , and maps are continuous and onto.

## Survey of Metrizable theorems

**Basic Metrizable theorem:** (1) Every  $M$ -space  $X$  is metrizable if  $X$  has a  $\sigma$ -hereditarily closure-preserving network (more generally,  $X$  has a  $G_\delta$ -diagonal), or  $X$  has a point-countable base. This is well-known; see [10], for example.

(2) (Lutzer [8]) Every  $GO$ -space  $X$  is metrizable if  $X$  has a  $\sigma$ -hereditarily closure-preserving network, more generally,  $X$  is a semi-stratifiable space.

(3) (Birkhoff-Kakutani (1936)) Every first countable topological group is metrizable.

(4) (Gruenhage-Michael-Tanaka [3]) Every paracompact  $M$ -space with a point-countable  $k$ -network is metrizable.

(5) (Filippov (1969)) Every quotient  $s$ -image  $X$  of a locally separable metric space is metrizable if  $X$  is first countable.

**Metrizability theorem A:** (1) Every  $M$ -space  $X$  with a point-countable  $k$ -network is metrizable if  $X$  is a  $k$ -space.

(2) Every  $M$ -space  $X$  with a  $\sigma$ -locally countable  $k$ -network is metrizable.

(3) Every  $w\Delta$ -space  $X$  with a  $\sigma$ -closure-preserving  $k$ -network is metrizable (Tanaka and Murota [16]).

(4) Every  $GO$ -space  $X$  with a  $G_\delta$ -diagonal is metrizable if  $X$  is a  $w\Delta$ -space or a  $\Sigma$ -space.

(5) Every  $k$ -space  $X$  with a  $\sigma$ -compact-finite  $k$ -network is metrizable if  $X$  contains no closed copy of the sequential fan  $S_\omega$ , and no the Arens' space  $S_2$ . In particular, every first countable space with a  $\sigma$ -hereditarily closure-preserving  $k$ -network is metrizable.

**Metrizability theorem B:** Let  $G$  be a topological group, and a  $k$ -space with a point-countable  $k$ -network. Then,  $G$  is metrizable if one of the following (a), (b), and (c) holds.

(a)  $G$  contains no closed copy of  $S_\omega$ .

(b)  $G$  contains no closed copy of  $S_2$ .

(c)  $G$  has the sequentially order  $\sigma(G) < \omega_1$  (Shibakov [14]).

**Metrizability theorem C:** (1) Let  $f : X \rightarrow Y$  be a quotient compact map such that  $X$  is metric. If  $Y$  is a  $GO$ -space or topological group, then  $Y$  is metrizable.

(2) Let  $f : X \rightarrow Y$  be a quotient  $s$ -map such that  $X$  is metric.

(i) If  $Y$  is a  $GO$ -space, then  $Y$  has a point-countable base. If  $X$  or  $Y$  is locally separable, then  $Y$  is metrizable.

(ii) If  $Y$  is a topological group satisfying one of (a), (b), and (c) in *Metrizability theorem B*, then  $Y$  is metrizable.

**Remark:** (1) Every topological group  $G$  satisfying the following (a) and (b) need not be metrizable (by a topological group  $G = \varinjlim \{R^n : n \in N\}$ , the inductive limit of  $n$ -dimensional Euclidean spaces  $R^n$ ).

(a)  $G$  is a sequential,  $\aleph_0$ -space with  $\sigma(G) = \omega_1$ .

(b)  $G$  is a quotient, countable-to-one image of a locally compact, separable metric space.

(2) Every GO-space  $M$  with a  $G_\delta$ -diagonal, which is an open  $s$ -image of a metric space need not be metrizable (by the Michael line  $M$ ).

### Main result and Related matters

**Metrizability theorem:** Let  $X$  be a GO-space. If  $X$  has a point-countable  $wcs^*$ -network, then the following (1) and (2) hold.

(1) Suppose that one of the following properties (a), (b), and (c) holds. Then,  $X$  is a paracompact space with a point-countable base.

In particular, if  $X$  has a  $\sigma$ -compact-finite  $wcs^*$ -network, then  $X$  is metrizable.

(a) Each point of  $X$  is a  $G_\delta$ -set.

(b)  $X$  is a quasi- $k$ -space.

(c)  $t(X) \leq \omega$ .

(2) Suppose that one of the following properties (d), (e), (f) holds. Then,  $X$  is metrizable.

(d)  $X$  is locally separable.

(e)  $X$  is a (locally)  $w\Delta$ -space.

(f)  $X$  is a (locally)  $\Sigma$ -space.

**Remark 1:** Related to the previous theorem, the following metrizability theorem holds ([6]).

**Metrizability theorem:** Let  $G$  be a topological group. If  $G$  is a GO-space, then  $G$  is metrizable if one of the above properties (a)  $\sim$  (f) holds.

**Remark 2:** (1) Not every countably compact space with a point-countable  $k$ -network is metrizable ([3]).

(2) Not every countably compact, first countable, LOTS  $X$  with a locally countable network is metrizable (by the order space  $X = [0, \omega_1)$ ).

(3) Not every LOTS  $M^*$  with a  $\sigma$ -point-finite base is metrizable (by the usual LOTS  $M^*$  containing the Michael line  $M$ . Here, for a GO-space  $X$ , the usual LOTS  $X^*$  containing  $X$ , see [8] for example).

**Corollary 1:** Every GO-space with a  $\sigma$ -locally countable  $wcs^*$ -network is metrizable.

**Corollary 2:** Let  $X = \varinjlim \{X_n : n \in N\}$  such that  $X_n$  are metric spaces (resp. metric spaces of covering dimension zero), here  $X_n$  are not necessarily closed in  $X$ . Then,  $X$  is a GO-space  $\Rightarrow X$  is a metrizable space (resp.  $X$  is a GO-space  $\Leftrightarrow X$  is a metrizable space of covering dimension zero).

In particular, for locally separable metric, zero-dimensional spaces  $X_n$ ,  $X$  is a GO-space  $\Leftrightarrow X$  is the topological sum of subspaces of the Cantor set  $2^\omega$ .

**Remark 3:** Let  $X = \varinjlim \{X_n : n \in N\}$  such that  $X_n$  are locally compact, topological groups. Here,  $X_n$  are not necessarily closed in  $X$ . Then,  $X$  is a GO-space  $\Leftrightarrow$  (a), (b), or (c) below holds.

- (a)  $X$  is a discrete space.
- (b)  $X$  is the topological sum of the real lines  $R$ .
- (c)  $X$  is the topological sum of the Cantor sets.

*More details and other properties of GO-spaces and topological groups are investigated in the author's joint papers [6] and [7] with C. Liu and M. Sakai.*

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