

Gauss-Manin system and the flat coordinate system
(Connection with the expansion of the solutions at ∞)

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In this note, we consider the differential system of Gauss-Manin associated with a versal deformation of a simple singularity. As to such a Gauss-Manin system, it is known by K.Saito [4] that the parameter space, say S , has a canonical linear structure. A flat coordinate system is a "linear" coordinate system with respect to the linear structure of S . The purpose of this note is to give an explicit construction of the flat coordinate system by means of the expansion of the solutions near a point at infinity.

§1. Gauss-Manin system and the flat coordinate system.

We begin with a review of the notion of a flat coordinate system. Let $f(x) = f(x_1, \dots, x_n)$ be one of the following canonical forms of simple isolated singularities:

$$(1.1) \quad \begin{cases} A_\ell : x_1^{\ell+1} \quad (n=1), & D_\ell : x_1^{\ell-1} + x_1 x_2^2 \quad (n=2), \\ E_6 : x_1^4 + x_2^3, \quad E_7 : x_1^3 + x_1 x_2^3, \quad E_8 : x_1^5 + x_2^3 \quad (n=2). \end{cases}$$

Here, the subscript ℓ or $6, 7, 8$ stands for the Milnor number $\mu = \mu(f)$ of f at $x=0$:

$$(1.2) \quad \mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / (\partial_x f), \quad (\partial_x f) = (\partial_{x_1} f, \dots, \partial_{x_n} f).$$

Note that f is weighted homogeneous of degree 1 with respect to a unique weight $\rho = (\rho_1, \dots, \rho_n)$, where ρ_i are positive rational numbers. In our case, the quotient ring $\mathcal{O}_{\mathbb{C}^n, 0} / (\partial_x f)$ has a $\mathbb{C}$ -basis consisting of monomials. To fix the ideas, we take a set $N \subset \mathbb{N}^n$ of multi-indices as follows, so that the residue classes of $x^\nu = x_1^{\nu_1} \dots x_n^{\nu_n}$ ($\nu = (\nu_1, \dots, \nu_n) \in N$) form a basis of the ring $\mathcal{O}_{\mathbb{C}^n, 0} / (\partial_x f)$.

$$(1.3) \quad \left\{ \begin{array}{l} \text{(I) Case of } A_1, E_6, E_8 \text{ (} f = x_1^{p_1} + \dots + x_n^{p_n} \text{):} \\ \quad N = \{ \nu \in \mathbb{N}^n; 0 \leq \nu_i \leq p_i - 2 \text{ (} 1 \leq i \leq n \text{)} \} . \\ \text{(II) Case of } D_1, E_7 \text{ (} f = x_1^{p_1} + x_1 x_2^{p_2} \text{):} \\ \quad N = \{ \nu \in \mathbb{N}^2; 0 \leq \nu_1 \leq p_1 - 1, 0 \leq \nu_2 \leq p_2 - 2 \} \cup \{ (0, p_2 - 1) \} . \end{array} \right.$$

Let $S = \mathbb{C}^\mu$ be the complex affine μ -space with coordinate system $t = (t_\nu)_{\nu \in N}$, which we regard as the space of deformation parameters. We take a versal deformation $F = F(t, x)$ of $f = f(x)$ defined by

$$(1.4) \quad F(t, x) = f(x) + \sum_{\nu \in N} t_\nu x^\nu .$$

The deformation F is versal in the sense that the residue classes of $\partial F / \partial t_\nu |_{t=0}$ ($\nu \in N$) form a basis of $\mathcal{O}_{\mathbb{C}^n, 0} / (\partial_x f)$. Note again that F is weighted homogeneous of degree 1 with respect to a unique weight (r, ρ) of (t, x) , $r = (r_\nu)_{\nu \in N}$, and that all r_ν ($\nu \in N$) are positive rational numbers as an effect of simpleness of the singularity f . We pay a special attention to the parameter t_0 . Let $T = \mathbb{C}^{\mu-1}$ be the affine $(\mu-1)$ -space with coordinate system $t^* = (t_\nu)_{\nu \in N^*}$, where $N^* = N \setminus \{0\}$, and $\pi : S \rightarrow T$ the canonical pro-

jection. Then the projection π is determined by the vector field $D_0 = D_{t_0} = \partial/\partial t_0$ and the parameter t_0 gives the fiber coordinate of π .

The Gauss-Manin system associated with F , denoted by $\underline{H}_F$, is by definition the differential system on S to be satisfied by the integral

$$(1.5) \quad u(t) = \int \mathcal{S}(F(t, x)) dx, \quad dx = dx_1 \wedge \dots \wedge dx_n.$$

For each $\nu \in \mathbb{N}$, define

$$(1.6) \quad u_\nu = \int \frac{\partial F}{\partial t_\nu} \mathcal{S}(F) dx = \int x^\nu \mathcal{S}(F) dx \quad (u_0 = u).$$

Then one can represent the Gauss-Manin system as a differential system on S including the column vector $\vec{u} = (u_\nu)_{\nu \in \mathbb{N}}$ as the unknown.

Proposition 1. The Gauss-Manin system $\underline{H}_F$ has a finite presentation

$$(1.7) \quad \begin{cases} D_{t_0} t_0 \vec{u} = (A_0(t^*) D_{t_0} + A_1(t^*)) \vec{u}, \\ D_{t_\nu} \vec{u} = (B_0^\nu(t^*) D_{t_0} + B_1^\nu(t^*)) \vec{u} \quad (\nu \in \mathbb{N}^*), \end{cases}$$

where $A_i, B_i^\nu \in M(\mu; \mathbb{C}[t^*])$ ($i = 0, 1; \nu \in \mathbb{N}^*$).

Let us recall here the Gauss-Manin system $\underline{H}_{f+t_0}$. For each $\nu \in \mathbb{N}$, define $w_\nu = \int x^\nu \mathcal{S}(f+t_0) dx$ and consider the column vector $\vec{w} = (w_\nu)_{\nu \in \mathbb{N}}$. Then the Gauss-Manin system $\underline{H}_{f+t_0}$ is given by

$$(1.8) \quad D_{t_0} t_0 \vec{w} = -\Lambda \vec{w}, \quad -\Lambda = \text{diag}(\varepsilon_\nu; \nu \in \mathbb{N}).$$

Note that the exponents ε_ν ($\nu \in \mathbb{N}$) of f are determined by the formulas

$$(1.9) \quad L_{\theta_x} (x^\nu dx) = \varepsilon_\nu x^\nu dx \quad (\nu \in N),$$

where L_{θ_x} is the Lie derivative with respect to the Euler vector field $\theta_x = \sum_{i=1}^n \rho_i x_i D_{x_i}$. Comparing the differential system (1.7) with (1.8), one sees that

$$(1.10) \quad A_0|_{t^*=0} = 0 \quad \text{and} \quad A_1|_{t^*=0} = -\Lambda,$$

since $\vec{u}$ is a deformation of $\vec{w} = \vec{u}|_{t^*=0}$.

Now we recall the notion of a flat coordinate system of S .

Let $s = (s_\nu)_{\nu \in N}$ be a coordinate system of S relative to $\pi: S \rightarrow T$. Namely, suppose that π is realized as the projection $s = (s_0, s^*) \mapsto s^* = (s_\nu)_{\nu \in N^*}$ and that $D_{s_0} = D_0$. For such a coordinate system $s = (s_\nu)_{\nu \in N}$, we define

$$(1.11) \quad v_\nu = \int \frac{\partial F}{\partial s_\nu} \delta(F) dx \quad \text{for } \nu \in N.$$

Then, as to the differential system for the column vector $\vec{v} = (v_\nu)_{\nu \in N}$, we have

Theorem 2. There exists a coordinate system $s = (s_\nu)_{\nu \in N}$ of S relative to $\pi: S \rightarrow T$ such that the Gauss-Manin system H_F is represented in the form

$$(1.12) \quad \begin{cases} D_{s_0} s_0 \vec{v} = (A(s^*) D_{s_0} - \Lambda) \vec{v}, \\ D_{s_\nu} \vec{v} = B^\nu(s^*) D_{s_0} \vec{v} \quad (\nu \in N^*), \end{cases}$$

where $A, B^\nu \in M(\mu; \mathbb{C}[s^*])$ ($\nu \in N^*$). Moreover, such a coordinate system $s = (s_\nu)_{\nu \in N}$ is determined uniquely up to linear transformation.

A coordinate system $s = (s_\nu)_{\nu \in N}$ relative to π is called a flat coordinate system if it has the property as stated in Theorem 2. At the same time, Theorem 2 says that the space S of parameters can be endowed with a linear structure. For an intrinsic formulation of the linear structure and the flat coordinate system, we refer the reader to K.Saito [4] or S.Ishiura-M.Noumi [2].

§2. Expansion of the solutions near a point at infinity.

We keep the assumptions and the notations in §1. Here we explain how one can construct the flat coordinate system of S by means of the expansion of the solutions of H_F at infinity.

As to the differential system (1.7) of Proposition 1, we define the discriminant $\Delta = \Delta(t)$ by

$$(2.1) \quad \Delta(t) = \det(t_0 - A_0(t^*)),$$

which is a monic polynomial of degree μ in t_0 with coefficients in $\mathbb{C}[t^*]$. Then the system (1.7) defines an integrable meromorphic connection over S with logarithmic poles along the discriminant set $D = \{\Delta = 0\}$. Hence, one sees that the system (1.7) has a unique fundamental system $\Phi(t)$ of many-valued holomorphic solutions on $S \setminus D$ such that

$$(2.2) \quad \Phi \in GL(\mu; \mathcal{O}(S \setminus D)) \quad \text{and} \quad \Phi|_{t^*=0} = t_0^{-\Lambda-1}.$$

Now we take the compactification $\bar{S} = \mathbb{P} \times T$ of $S = \mathbb{C} \times T$ in the direction of t_0 -axis. Then, it is easy to see that the discriminant set D is closed in $\bar{S}$ and does not intersect with the hyperplane

$\{t_0 = \infty\}$ at infinity. Moreover, the meromorphic connection in question has regular singularities along $\{t_0 = \infty\}$ in the naïve sense. So one sees that $\bar{\Phi}(t)$ can be factorized in the form

$$(2.3) \quad \bar{\Phi}(t) = \bar{\Psi}(t) t_0^{-\Lambda-1} \quad \text{near } (t_0, t^*) = (\infty, 0),$$

where $\bar{\Psi}(t) \in GL(\mu; \mathcal{O}_{\bar{S}, (\infty, 0)})$ and $\bar{\Psi}|_{t^*=0} = 1$. (It is known that this kind of expression gives rise to a hypergeometric representation of the solutions in the case where $f = x_1^{p_1} + \dots + x_n^{p_n}$ and $F = f + t_1 x_1 + \dots + t_n x_n + t_0$.) We formulate this expression (2.3) in connection with the flat coordinate system.

Proposition 3. (a) There is a unique formal differential operator

$$(2.4) \quad P(t^*, D_{t_0}) = \sum_{k=0}^{\infty} P_k(t^*) D_{t_0}^k, \quad P_k \in M(\mu; \mathbb{C}[t^*]),$$

such that

$$(2.5) \quad \bar{\Phi}(t) = P(t^*, D_{t_0}) t_0^{-\Lambda-1} \quad \text{near } (t_0, t^*) = (\infty, 0).$$

(b) As to the operator P in (a), define

$$(2.6) \quad s_\nu = \delta_{0,\nu} t_0 + P_1(t^*)_{0,\nu}, \quad \text{for } \nu \in \mathbb{N}.$$

Then, $s = (s_\nu)_{\nu \in \mathbb{N}}$ gives a flat coordinate system of S.

Remark that the 0-th row of the matrix P of operators represents the expansion of μ independent solutions of $u = \int \delta(F) dx$.

Now we mention how the matrix P can be determined explicitly. The first step is to solve a system of difference equations for a function $c: \mathbb{N}^n \rightarrow \mathbb{C}$. To describe the difference system, we introduce some notations. We denote by T_i ($1 \leq i \leq n$) the transla-

tion by one in the i -th direction. Namely, for a function $c : \mathbb{N}^n \rightarrow \mathbb{C}$, we define a function $T_i c : \mathbb{N}^n \rightarrow \mathbb{C}$ by

$$(2.7) \quad (T_i c)(\alpha) = c(\alpha + l_i) \quad \text{for } \alpha \in \mathbb{N}^n,$$

where l_i stands for the unit vector $(0, \dots, \overset{i}{1}, \dots, 0)$. Then the difference system to be solved is given by

$$(2.8) \quad \alpha_i c(\alpha - l_i) + f_i(T)c(\alpha) = 0 \quad (\alpha \in \mathbb{N}^n; 1 \leq i \leq n),$$

where $T = (T_1, \dots, T_n)$ and $f_i = \partial_{x_i} f$ ($1 \leq i \leq n$). One can see that the difference system (2.8) has μ independent solutions, denoted by c_ν ($\nu \in \mathbb{N}$), as follows.

Case of A_ℓ, E_6, E_8 where $f = x_1^{p_1} + \dots + x_n^{p_n}$. We define a lattice L of $\mathbb{Z}^n$ by $L = \sum_{i=1}^n \mathbb{Z} p_i l_i$. Then, for each $\nu \in \mathbb{N}$, we define a function $c_\nu : \mathbb{N}^n \rightarrow \mathbb{C}$ by

$$(2.9) \quad c_\nu(\alpha) = \prod_{i=1}^n (-)^{k_i} \left(\frac{\nu_i + 1}{p_i}; k_i \right), \quad k_i = \frac{\alpha_i - \nu_i}{p_i},$$

if $\alpha \in (\nu + L) \cap \mathbb{N}^n$ and by $c_\nu(\alpha) = 0$ if $\alpha \in \mathbb{N}^n \setminus (\nu + L)$.

Case of D_ℓ, E_7 where $f = x_1^{p_1} + x_1 x_2^{p_2}$. We define a lattice L of $\mathbb{Z}^n$ by $L = \mathbb{Z} p_1 l_1 + \mathbb{Z} (p_2 l_2 + l_1)$. Then, for each $\nu \in \mathbb{N}$, we define a function $c_\nu : \mathbb{N}^2 \rightarrow \mathbb{C}$ by

$$(2.10) \quad c_\nu(\alpha) = (-)^{k_1} \left(\frac{\nu_1 + 1}{p_1} - \frac{\nu_2 + 1}{p_1 p_2}; k_1 \right) \cdot (-)^{k_2} \left(\frac{\nu_2 + 1}{p_2}; k_2 \right)$$

with $k_1 = \frac{\alpha_1 - \nu_1}{p_1} - \frac{\alpha_2 - \nu_2}{p_1 p_2}$ and $k_2 = \frac{\alpha_2 - \nu_2}{p_2}$ if $\alpha \in (\nu + L) \cap \mathbb{N}^n$ and by $c_\nu(\alpha) = 0$ if $\alpha \in \mathbb{N}^2 \setminus (\nu + L)$.

In (2.9) and (2.10), the expression $(z; k)$ stands for the factorial function $\Gamma(z+k)/\Gamma(z)$. It is directly checked that the functions

c_ν ($\nu \in \mathbb{N}$) defined as above give μ independent solutions of (2.8). As the next step, we introduce a linear mapping $\ell: \mathbb{N}^{\mu-1} \rightarrow \mathbb{N}^n$ by the formula

$$(2.11) \quad \exp\left(\sum_{\nu \in \mathbb{N}^*} t_\nu x^\nu\right) = \sum_{\alpha^* \in \mathbb{N}^{\mu-1}} \frac{t^{*\alpha^*}}{\alpha^*!} x^{\ell(\alpha^*)}.$$

In other words, the multi-index $\ell(\alpha^*) = (\ell_1(\alpha^*), \dots, \ell_n(\alpha^*))$ is given by $\ell_i(\alpha^*) = \sum_{\nu \in \mathbb{N}^*} \nu_i \alpha_\nu$ for $1 \leq i \leq n$.

Theorem 4. By means of the functions $c_\nu: \mathbb{N}^n \rightarrow \mathbb{C}$ and the linear mapping $\ell: \mathbb{N}^{\mu-1} \rightarrow \mathbb{N}^n$ above, the operator $P(t^*, D_{t_0})$ of Proposition 3 is determined as follows:

$$(2.12) \quad P_k(t^*)_{k,\nu} = \sum_{\alpha^* \in \mathbb{N}^{\mu-1}} c_\nu(\ell(\alpha^*) + k) \frac{t^{*\alpha^*}}{\alpha^*!}$$

for $k, \nu \in \mathbb{N}$ and $k \in \mathbb{N}$.

Corollary. The flat coordinate system of Proposition 3 can be represented by the formulas

$$(2.13) \quad s_\nu = \delta_{0,\nu} t_0 + \sum_{\alpha^* \in \mathbb{N}^{\mu-1}} c_\nu(\ell(\alpha^*)) \frac{t^{*\alpha^*}}{\alpha^*!} \quad (\nu \in \mathbb{N}).$$

Note that $P_k(t^*)_{k,\nu}$ and so s_ν are weighted homogeneous polynomials. Details of the above argument are given in M.Noumi [3].

§3. A remark in the case of type A_ℓ .

In the case of type A, it is known by S.Ishiura-M.Noumi [1] that the flat coordinate systems of type A_ℓ ($\ell \geq 1$) can be obtained by a method of reduction from a countable sequence of weighted homogeneous polynomials. Here we remark that this sequence of

polynomials eventually appears in the expansion (2.5) of the solutions of the Gauss-Manin system.

Let $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \dots)$ be a sequence of countably many variables and consider the formal Laurent series

$$(3.1) \quad f(x) = x + \tilde{t}_2 x^{-1} + \tilde{t}_3 x^{-2} + \dots$$

Then one can find a unique Laurent series

$$(3.2) \quad g(y) = y - \tilde{s}_2 y^{-1} - \tilde{s}_3 y^{-2} + \dots$$

such that $g \circ f(x) = x$ and $f \circ g(y) = y$, so that one obtains a new sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \dots)$ of countably many variables. Note that $\tilde{s}_2, \tilde{s}_3, \dots$ are weighted homogeneous polynomials in $\tilde{t}_2, \tilde{t}_3, \dots$ and vice versa. Moreover, the two sequences $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \dots)$ and $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \dots)$ are connected to each other by the formulas

$$(3.3) \quad \begin{cases} \tilde{s}_k = \frac{1}{k-1} \operatorname{Res}(f(x)^k dx) & (k = 2, 3, \dots) \text{ and} \\ \tilde{t}_k = \frac{-1}{k-1} \operatorname{Res}(g(y)^k dy) & (k = 2, 3, \dots) . \end{cases}$$

There is reason to call this sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \dots)$ the "flat coordinate system" associated with $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \dots)$.

Consider the versal deformation

$$(3.4) \quad F(t, x) = x^n + t_2 x^{n-2} + t_3 x^{n-3} + \dots + t_n, \quad t = (t_2, \dots, t_n)$$

of the singularity $x^n = 0$ of type A_{n-1} (in a different way of indexing from that of preceding sections). Then one can take the fractional power $F^{1/n}$ in the form (3.1), so that $\tilde{t}_2, \tilde{t}_3, \dots$ are weighted homogeneous polynomials in $t_2, \dots, t_n$. (Among those,

$\tilde{t}_2, \dots, \tilde{t}_n$ are algebraically independent.) Then the variables $\tilde{s}_2, \tilde{s}_3, \dots$ in (3.2) are determined as polynomials in $t_2, \dots, t_n$. Now, we define a sequence $s_2, s_3, \dots$ of polynomials in $t_2, \dots, t_n$ as follows:

$$(3.5) \quad s_k = n \tilde{s}_k \in \mathbb{C}[t] \quad \text{for } k = 2, 3, \dots$$

Then one can prove that $s_2, \dots, s_n$ are algebraically independent and that the coordinate system $(s_2, \dots, s_n)$ coincides with the flat coordinate system given in Corollary of Theorem 4. It is by this reduction process that the flat coordinate systems of type A_ℓ are obtained from the sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \dots)$ of "flat variables". The sequence of "flat variables" also appears in the expansion of the solutions of the Gauss-Manin system $\underline{H}_F$ at infinity.

Theorem 5. Let φ_k ($k = 2, \dots, n$) be the $n-1$ independent solutions of the Gauss-Manin system $\underline{H}_F$ that appear in the 0-th row of (2.5).

Then we have

$$(3.6) \quad \varphi_k = \delta_{k,n} s^{-\frac{n-1}{n}} + \sum_{r=1}^{\infty} (-)^r \left(\frac{k-1}{n} + r - 1 \right) s'_{(r-1)n+k} s_n^{-\frac{k-1}{n} - r},$$

where $s'_i = s_i|_{s_n=0}$ is the polynomial obtained from s_i by setting $s_n = 0$ for $i = 2, 3, \dots$.

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