

CPN Languages and Codes

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Let $D = (P, X, \delta, \mu_0)$ be a Petri net with a initial marking μ_0 where P is the set of places, X is the set of transitions, δ is the transition function and $\mu_0 \in N_+^P$ is a positive marking, i.e. $\pi_p(\mu_0) > 0$ for any $p \in P$. Notice that $\pi_p(\mu_0)$ is meant the number of tokens at p of the marking μ_0 . A language C is called a *CPN language* over X generated by D and denoted by $C = \mathcal{L}(D)$ if $C = \{u \in X^+ | \exists p \in P, \pi_p(\delta(\mu_0, u)) = 0, \forall q \in P, \pi_q(\delta(\mu_0, u)) \geq 0, \text{ and } \forall q' \in P, \pi_{q'}(\delta(\mu_0, u')) > 0 \text{ for } u' \in P_r(u) \setminus \{u\} \text{ where } P_r(u) \text{ is the set of all prefixes of } u\}$. Then it is obvious that $C = \mathcal{L}(D)$ is a prefix code over X . If C is a maximal prefix code over X , then C is called an *mCPN language* over X .

Theorem 1 *Let $A, B \subseteq X^+$ be finite maximal prefix codes over X . If AB is an mCPN language over X , then A, B are full uniform codes over X .*

Remark 1 In the above theorem, the condition for A and B to be finite is necessary. For instance, let $X = \{a, b\}$ and let $A = B = b^*a$. Then $AB = b^*ab^*a$ is an mCPN language over X but neither A nor B is a full uniform code over X .

Now we consider some constructions of mCPN languages.

Definition 1 Let $A, B \subseteq X^+$. Then by $A \oplus B$ we denote the language $(\cup_{b \in X} \{(P_r(A) \setminus A) \diamond Bb^{-1}\}b) \cup (\cup_{a \in X} \{(P_r(B) \setminus B) \diamond Aa^{-1}\}a)$ where $\diamond$ is meant the shuffle operation.

Proposition 1 *Let $X = Y \cup Z$ where $Y, Z \neq \emptyset, Y \cap Z = \emptyset$. If $A \subseteq Y^+$ is an mCPN language over Y and $B \subseteq Z^+$ is an mCPN language over Z , then $A \oplus B$ is an mCPN language over X .*

Example 1 Let $X = \{a, b\}$. Consider $A = \{a\}$ and $B = \{bb\}$. Then both A and B are *mCPN* languages over $\{a\}$ and $\{b\}$, respectively. Hence $A \oplus B = \{a, ba, bb\}$ is an *mCPN* language over X .

Proposition 2 Let $A, B \subseteq X^+$ be finite *mCPN* languages over X . Then $A \oplus B$ is an *mCPN* language over X if and only if $A = B = X$.

Remark 2 For the class of infinite *mCPN* languages over X , the situation is different. For instance, let $X = \{a, b\}$ and let $A = B = b^*a$. Then $A \oplus B = b^*a$ and A, B and $A \oplus B$ are *mCPN* languages over X .

Proposition 3 Let $A, B \subseteq X^+$ be *mCPN* languages over X . Then there exist an alphabet Y , $D \subseteq Y^+$: an *mPCN* language over Y and a homomorphism h of Y^* onto X^* such that $A \oplus B = h(D)$.

Definition 2 Let $A \subseteq X^+$. By $m(A)$, we denote the language $\{v \in A \mid \forall u, v \in A, \forall x \in X^*, v = ux \Rightarrow x = 1\}$. Obviously, $m(A)$ is a prefix code over X . Let $A, B \subseteq X^+$. By $A \otimes B$, we denote the language $m(A \cup B)$.

Proposition 4 Let A, B be *mCPN* languages over X . Then, $A \otimes B$ is an *mCPN* language over X .

Example 2 It is obvious that a^*b and $(a \cup b)^3$ are *mCPN* languages over $\{a, b\}$. Hence $a^*b \otimes (a \cup b)^3 = \{b, ab, aaa, aab\}$ is an *mCPN* language over $\{a, b\}$.

Remark 3 Proposition 4 does not hold for the classe of *CPN* languages over X . The reason is the following: Suppose that $A \otimes B$ is a *CPN* language over X for any two *CPN* languages A and B over X . Then we can show that, for a given finite *CPN* language A over X , there exists a finite *mCPN* language B over X such that $A \subseteq B$ as follows. Let $A \subseteq X^+$ be a finite *CPN* language over X which is not an *mCPN* language. Let $n = \max\{|u| \mid u \in A\}$. Consider X^n which is an *mCPN* language over X . By assumption, $A \otimes X^n$ becomes a *CPN* language (in fact, an *mCPN* language) over X . By the definition of the operation $\otimes$, it can be also proved that $A \subseteq A \otimes X^n$. Notice that there exists a finite *CPN* language A over X such that there exists no *mCPN* language B over X with $A \subseteq B$. Hence, Proposition 4 does not hold for the class of all *CPN* languages over X .

Remark 4 The set of all $mCPN$ languages over X forms a semigroup under $\otimes$. Moreover, the operation $\otimes$ has the following properties:

$$(1) A \otimes B = B \otimes A, (2) A \otimes A = A, (3) A \otimes X = X.$$

Consequently, the set of all $mCPN$ languages over X forms a commutative band with zero under $\otimes$.

Definition 3 Let $A \subseteq X^+$ be a CPN language over X . By $r(A)$ we denote the value $\min\{|P| \mid D = (P, X, \delta, \mu_0), \mathcal{L}(D) = A\}$.

Remark 5 Let $A \subseteq X^+$ be a finite CPN language over X . Then $r(A) \leq |A|$. Moreover, let $A, B \subseteq X^+$ be $mCPN$ languages over X . Then $r(A \otimes B) \leq r(A) + r(B)$. In the above, if $|A|, |B|$ are finite, then $|A \otimes B| \leq \max(|A|, |B|)$.

We define three language classes as follows: $\mathcal{L}_{CPN} = \{A \subseteq X^+ \mid A \text{ is a } CPN \text{ language over } X\}$, $\mathcal{L}_{mCPN} = \{A \subseteq X^+ \mid A \text{ is an } mCPN \text{ language over } X\}$, $\mathcal{L}_{NmCPN} = \{A \subseteq X^+ \mid A: \text{ an } mCPN \text{ language over } X, \exists D = (P, X, \delta, \mu_0), \forall p \in P, \forall a \in X, \#(p \rightarrow a) \leq 1, \mathcal{L}(D) = A\}$. Then it is obvious that we have the following inclusion relations: $\mathcal{L}_{NmCPN} \subseteq \mathcal{L}_{mCPN} \subseteq \mathcal{L}_{CPN}$. It is also obvious that $\mathcal{L}_{mCPN} \neq \mathcal{L}_{CPN}$.

Problem 1 $\mathcal{L}_{mCPN} \neq \mathcal{L}_{NmCPN}$?

Proposition 5 Let $A \in \mathcal{L}_{CPN}$ and let $r(A) = k$. Then there exist $A_1, A_2, \dots, A_k \in \mathcal{L}_{CPN}$ such that $r(A_i) = 1, i = 1, 2, \dots, k$ and $A = A_1 \otimes A_2 \otimes \dots \otimes A_k$. Moreover, in the above, if $A \in \mathcal{L}_{NmCPN}$, then $A_1, A_2, \dots, A_k$ are in $\mathcal{L}_{NmCPN}$ and context-free.

For $mCPN$ languages with rank 1, we have the following:

Proposition 6 Let $A \subseteq X^+$ be a finite $mCPN$ language with $r(A) = 1$ over X . Then A is a full uniform code over X .

Proposition 7 Let $A \subseteq X^+$ be an $mCPN$ language with $r(A) = 1$ over X and let k be a positive integer. Then A^k is an $mCPN$ language with $r(A^k) = 1$ over X .

Proposition 8 Let $A \in \mathcal{L}_{NmCPN}$ and let $r(A) = k$. Then there exist $A_1, A_2, \dots, A_k \in \mathcal{L}_{NmCPN}$ such that $r(A_i) = 1, i = 1, 2, \dots, k$ and $A =$

$A_1 \otimes A_2 \otimes \dots \otimes A_k$. Let $n_1, n_2, \dots, n_k$ be positive integers. Then $A_1^{n_1} \otimes A_2^{n_2} \otimes \dots, \otimes A_k^{n_k} \in \mathcal{L}_{N\mathbf{m}CPN}$.

Finally, we can prove the following main theorem by two different ways, i.e. the first one is an indirect proof and the second one is a direct proof.

Theorem 2 *Let $C \subseteq X^+$ be a CPN language over X . Then C is a context-sensitive language over X .*