

A C*-ALGEBRA WITH THE REPRESENTATION EXTENSION PROPERTY

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Every C*-algebra A sits inside its regular completion $\hat{A}$ very nicely. $\hat{A}$ is a monotone complete C*-algebra, which is, in general, not a von Neumann algebra. $\hat{A}$ reflects closely the structure of A , for examples, any bounded derivation of A extends to a unique inner derivation of $\hat{A}$, any automorphism of A extends to a unique automorphism of $\hat{A}$, and if A is simple, any outer automorphism of A extends to a unique outer automorphism of $\hat{A}$.

The following question naturally arises: Let A be any C*-algebra and let π be any irreducible representation of A .

(*) Can we extend π to a *-representation $\hat{\pi}$ from $\hat{A}$ onto $\pi(A)^\wedge$?

If A is abelian, every character of A can be extended to a character of $\hat{A}$ and so (*) has an affirmative answer.

Unfortunately, the answer to this question (*) is negative in general.

Definition. Let A be a C*-algebra. Then A is said to have the representation extension property (the (RE)-property) if, for every irreducible representation π of A , π can be extended to a *-homomorphism $\hat{\pi}$ from $\hat{A}$ onto $\pi(A)^\wedge$. We say that A has the strong representation extension property (the

(SRE)-property) if, any quotient of A has the (RE)-property.

In [6], a characterization of C^* -algebras with the (SRE)-property is given in the following form:

Theorem. Let A be a separable C^* -algebra. Then A has the (SRE)-property if, and only if, A is a restricted direct sum of a sequence $\{B_n\}$, consisting of infinite dimensional simple C^* -algebras or homogeneous C^* -algebras of finite orders (including $\{0\}$).

In this note, we would like to give you an outline of its proof with negative examples to the question (*), which, we do hope, would be helpful to understand the proof of Theorem. We also would like to give an example to show that there is a C^* -algebra which has the (RE)-property, but has not the (SRE)-property.

Detailed proof will appear in the J. London Math. Soc. (see [6]).

It is a pleasure to thank Professor J.D.M. Wright and the staffs of the Mathematics Department of the University of Reading for their warm hospitality during the author's visit. The author also wishes to thank the SERC, who financed his extended visit to the University of Reading.

Note first that if A is abelian, every character of A can be extended to a character of $\hat{A}$ and so A has the (SRE)-property.

Example 1. Let A be the set of all continuous functions F on $[0,1]$, taking values in the C^* -algebra $M_2(\mathbb{C})$ of all 2 by 2 matrices over $\mathbb{C}$, such that $F(0)_{2j} = F(0)_{12} = 0$ for $i, j = 1, 2$. A becomes a C^* -algebra with respect to the pointwise operations and the norm $\|F\| = \sup\{\|F(t)\| \mid t \in [0,1]\}$. Then A has not got the (RE)-property.

In fact, the one dimensional representation of A defined by $F \in A \rightarrow F(0)_{11}$ cannot be extended to $\hat{A} = M_2(C[0,1])^\wedge$.

Example 2. Let H be an infinite dimensional separable Hilbert space and let $A = K(H) \otimes C[0,1]$, the C^* -tensor product of the algebra of compact operators on H , by the C^* -algebra $C[0,1]$. Then A has never got the (RE)-property.

Note, first, that $\hat{A} = L(H) \otimes C[0,1]^\wedge$ and $\hat{A}$ is a properly infinite and σ -finite AW^* -algebra. Let $\pi = i \otimes \pi_t$ be the irreducible representation of A on H induced by $t \in [0,1]$. Then $\pi(A) = K(H)$. If it were possible to extend π to a $*$ -homomorphism $\hat{\pi}$ from $\hat{A}$ onto $K(H)^\wedge = L(H)$, then, since H is separable, this would imply that π is normal (see [1]). Let ξ be any unit vector in H , and let $\phi_t(a) = (\hat{\pi}(1 \otimes a)\xi, \xi)$ for any $a \in C[0,1]^\wedge$. Then $\phi_t(1) = 1$, and so ϕ_t is a state on $C[0,1]^\wedge$, which is completely additive on projections. This were, however, a contradiction, because $C[0,1]^\wedge$ has no normal states.

Example 3. Let F be the Fermion algebra acting irreducibly on an infinite dimensional separable Hilbert space H .

Let $B = F + K(H)$ (note that $F \cap K(H) = \{0\}$). Let π be the irreducible representation of B which is canonically defined by $B/K(H) (\cong F)$. Then π cannot be extended to a $*$ -homomorphism from $\hat{B}$ ($= L(H)$) onto (into) $\hat{F}$.

In fact, if it were possible, then, since $\hat{B}$ is properly infinite and $\hat{F}$ is σ -finite, π would be a direct summand. This is, however, a contradiction, because $\hat{B}$ is a factor and π is not faithful.

The fact that the regular completion $\hat{A}$ of a separable C^* -algebra A has no type II direct summand is used to prove our Theorem. See for details ([3], Theorem 2).

Also we use the following Theorem proved by Feldman and Fell ([1]).

Lemma 1. Let A, B be two AW^* -algebras. Let A be properly infinite and σ -finite. Suppose that B is σ -finite. Then any $*$ -homomorphism from A into B is completely additive on projections.

Some routine calculations in the regular completions of C^* -algebras (see [5], [7], [10] and [11]) give us the following:

Lemma 2. Let $\{B_n\}$ be any sequence of separable C^* -algebras and let $A = \sum_{n=1}^{\infty} B_n$ be the restricted direct sum of $\{B_n\}$. Then $\hat{A} = \sum_{n=1}^{\infty} \hat{B}_n$.

To prove this, we only have to check the following two properties by the uniqueness of the regular completions.

Let $\tilde{A} = \bigcup_{n=1}^{\infty} \hat{B}_n$. Then $\tilde{A}$ has the following two properties:

- 1) $C^*(A,1)_h$ is order dense in $\tilde{A}_h$,
- and
- 2) $C^*(A,1)_h$ σ -generates $\tilde{A}_h$,

where $C^*(A,1)$ is the C^* -algebra obtained from the adjunction of a unit to A if A is non-unital, otherwise, $C^*(A,1) = A$ and B_h is the self-adjoint part of a C^* -algebra B .

By an homogeneous C^* -algebra of order n ($n < \infty$), we mean a C^* -algebra A such that $A/J \cong M_n(\mathbb{C})$ for every primitive ideal J of A ([2], [8]).

By using the fact that the Borel envelop of a separable homogeneous C^* -algebra of order n is $*$ -isomorphic to the n by n matrix algebra over the C^* -algebra $\mathcal{B}(\text{Prim}A)$ of all bounded complex Borel functions on the 2nd countable Hausdorff locally compact space $\text{Prim}A$, where $\text{Prim}A$ is the primitive ideal space of A , we can show the following lemma (see [2], [4] and [5]).

Lemma 3. Let A be any separable homogeneous C^* -algebra of finite order n , then $\hat{A}$ is an AW^* -algebra of type I_n , whose centre is $*$ -isomorphic to the regular completion $C_b(\text{Prim}A)^\wedge$ of the ideal centre $C_b(\text{Prim}A)$ of A , where $C_b(\text{Prim}A)$ means the C^* -algebra of all bounded complex continuous functions on $\text{Prim}A$.

A careful examination of an irreducible representation of an homogeneous algebra of a finite order tells us that every separable homogeneous C^* -algebra of a finite order has the (SRE)-property. By using this fact, together with Lemma 2 and Lemma 3, we can show the following proposition, which states the "if" part of Theorem is valid.

Proposition 1. Let A be a restricted direct sum of a sequence $\{B_n\}$ of simple C^* -algebras or homogeneous C^* -algebras of finite orders, then A has the (SRE)-property.

Next, we shall sketch a proof of "only if" part of Theorem. First of all, we shall start with the following proposition, which treats a C^* -algebra whose regular completion is properly infinite.

Proposition 2. Let B be a separable C^* -algebra whose regular completion $\hat{B}$ is properly infinite. Suppose that B has the (RE)-property. Then $\text{Prim} B$ is discrete, and so B is a restricted direct sum of an at most countable family of separable infinite dimensional simple C^* -algebras.

Let $J \in \text{Prim} B$. Let $\hat{\pi}_J$ be a $*$ -homomorphism from $\hat{B}$ onto $(B/J)^\wedge$ which is an extension of π_J . Since $\hat{B}$ is σ -finite, properly infinite and $(B/J)^\wedge$ is σ -finite, $\hat{\pi}_J$ is normal by Lemma 1, that is, there is a unique central projection e_J in $\hat{B}$ such that

$$J = \hat{B}e_J \cap B.$$

Moreover, $\hat{B}(1 - e_J) \cong (B(1 - e_J))^\wedge \cong (B/J)^\wedge$, which implies that $\hat{B}(1 - e_J)$ is a non-zero factor and so every $J \in \text{Prim}B$ is maximal in $\text{Prim}B$.

For any $J \in \text{Prim}B$, let $J^c = \hat{B}(1 - e_J) \cap B (\neq \{0\})$, which is a closed two-sided ideal of B .

Let $K \subset \text{Prim}B$. Let $J_0 \in \bar{K}$, that is, $J_0 \supset \bigcap \{J | J \in K\}$. We shall show that there is $J \in K$ such that $J_0 + J \neq B$.

Suppose that $J_0 + J = B$ for all $J \in K$. Then, $J^c \subset J_0$ for all $J \in K$. In fact, for any fixed $J \in K$, if $y \in J^c$ ($y \geq 0$), then one can find $x_0 \in J_0$ ($x_0 \geq 0$) and $x_1 \in J$ ($x_1 \geq 0$) such that $y = x_0 + x_1$. Since $y \in J^c$, $e_J y = 0$ follows and so $e_J x_0 e_J + x_1 = 0$, that is, $x_1 = 0$. This means that $y \in J_0$. Hence $1 - e_{J_0} \leq e_J$ for all $J \in K$.

On the other hand, $J_0 \supset \bigcap \{J | J \in K\}$ tells us that $1 - e_{J_0} \leq \sup\{1 - e_J | J \in K\}$ and hence $1 = e_{J_0}$. This is a contradiction and so, if $J_0 \in \bar{K}$, then there is $J \in K$ such that $J_0 + J \neq B$. Since J and J_0 are maximal, this means that $J_0 = J \in K$, that is, $\bar{K} = K$ for all $K \subset \text{Prim}B$. Hence $\text{Prim}B$ is discrete. This completes the proof.

Let A be a separable C^* -algebra and let

$$X_0 = \{J \in \text{Prim}A \mid (A/J)^\wedge \text{ is properly infinite}\},$$

$$Y = \{J \in \text{Prim}A \mid (A/J)^\wedge \text{ is finite}\},$$

and for each k ($k \geq 1$),

$$X_k = \{J \in \text{Prim}A \mid (A/J)^\wedge \cong M_k(\mathbb{C})\}.$$

Since $(A/J)^\wedge$ has no type II-direct summand for each $J \in \text{Prim}A$,

$$Y = \bigcup_{k=1}^{\infty} X_k \quad (\text{some } X_k \text{ may be } \emptyset).$$

Suppose that A has the (SRE)-property. Then, by a rather lengthy speculation, we have the following:

Lemma 4. Keep the notations and assumptions above in mind.
 X_k ($k \geq 0$) is open and closed in $\text{Prim}A$.

The following F.B. Wright's theorem is used to prove Lemma 4, but we shall omit the details (see [6] and [9]).

Lemma 5([9]). Let B be a finite AW*-algebra of type I and let X be the set of all maximal ideals of B . Then B/M is a finite AW*-factor of type I for every M except possibly for a closed nowhere dense set in X . The exceptional set is empty if, and only if, the number of homogeneous summands of B is finite. If this set is non-empty, then B/M is a von Neumann factor of type II_1 for every such M .

So one can find closed two-sided ideals B and C such that

$$A = B \oplus C$$

and $\text{Prim}B \approx X_0$, $\text{Prim}C \approx \bigcup_{k=1}^{\infty} X_k$.

By our assumption, B has the (RE)-property and so, by Proposition 2, $\text{Prim}B$ is discrete. So either $B = \{0\}$ or else

B is a restricted direct sum of at most countable, infinite dimensional simple C^* -algebras.

Let f_k be the projection in the ideal centre $Z(M(C))$ of the multiplier algebra of C corresponding to the continuous characteristic function of X_k ($k \geq 1$) on $\text{Prim} C$, then, since C is separable, it is easy to check that the mapping

$$x \in C \longrightarrow (xf_k) \in \sum_{k=1}^{\infty} Cf_k$$

is a $*$ -isomorphism between C and $\sum_{k=1}^{\infty} Cf_k$. Since Cf_k is k -homogeneous, this completes the proof.

Next, we shall construct an example of a C^* -algebra which has the (RE)-property but not the (SRE)-property.

Let A be the set of all continuous functions F on $[0,1]$, taking values in $M_2(\mathbb{C})$ such that $F(t)_{12} = F(t)_{2j} = 0$ for $i,j = 1,2$ and for t with $0 \leq t \leq 1/2$. Then A becomes a C^* -algebra with respect to the pointwise operations and the norm $\|F\| = \sup\{\|F(t)\| \mid t \in [0,1]\}$. Let

$$I_1 = \{F \in A \mid F(t) = 0 \text{ for } t \in [1/2,1]\}$$

and let

$$I_2 = \{F \in A \mid F(t) = 0 \text{ for } t \in [0,1/2]\}.$$

Then I_1 and I_2 are closed ideals of A such that

- (1) $I_1 \cap I_2 = \{0\}$,
- (2) $A/I_2 \cong C[0,1/2]$,

$$(3) \quad A/I_1 \cong B$$

where B is the C^* -algebra of all continuous $M_2(\mathbb{C})$ -valued functions G on $[1/2, 1]$ such that $G(1/2)_{12} = G(1/2)_{2j} = 0$ for $j = 1, 2$,

$$(4) \quad I_1 \cong C_0[0, 1/2]$$

where $C_0[0, 1/2]$ is the C^* -algebra of all complex-valued continuous functions on $[0, 1/2]$, vanishing at $1/2$,

$$(5) \quad I_2 \cong B_0$$

where B_0 is the C^* -subalgebra of B , vanishing at $1/2$, and

$$(6) \quad I_1 + I_2 \text{ is essential in } A.$$

Lemma 6. For any $J \in \text{Prim} A$, one can find a unique $t \in [0, 1]$, such that, either $t \in [0, 1/2]$ and $F + J = F(t)_{11}$ for all $F \in A$ or else, $t \in (1/2, 1]$ and $F + J = F(t)$ for all $F \in A$.

Since, by (1)-(6),

$$\hat{A} \cong \begin{pmatrix} C[0, 1/2]^\wedge & 0 \\ 0 & 0 \end{pmatrix} \oplus M_2(C[1/2, 1]^\wedge),$$

by making use of Lemma 6, it is easy to check that A has the (RE)-property. On the other hand, since $A/I_1 \cong B$ and B has not got the (RE)-property (see example 1), A has not got the (SRE)-property.

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