

CONTINUED FRACTIONS AND ERGODIC THEORY

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Let x be an irrational number between 0 and 1,

$$(1) \quad x = [0; a_1, a_2, a_3, \dots]$$

its expansion as a regular continued fraction and $\left(\frac{p_n}{q_n}\right)_{n=1}^{\infty}$ the corresponding sequence of convergents. As is well-known

$$(2) \quad \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Define $\theta_n(x) := q_n |q_n x - p_n|$. Hence, by (2), the sequence $\theta_n(x)$, $n = 0, 1, 2, \dots$ is for every fixed x a sequence in the unit interval. The well-known theorem of Vahlen states that for every x and every $n \in \mathbb{N}$

$$\min \{ \theta_{n-1}(x), \theta_n(x) \} < 0.5.$$

It was conjectured by H. W. Lenstra Jr. and proved by Bosma, Jager and Wiedijk [1] that for almost all x , in the sense of Lebesgue, the sequence $\theta_n(x)$, $n = 0, 1, 2, \dots$ is distributed in the unit interval according to the density function f , where f is given by

$$f(a) = \begin{cases} (\log 2)^{-1} & , 0 \leq a \leq 0.5 \\ (\log 2)^{-1} (a^{-1} - 1) & , 0.5 \leq a \leq 1 \end{cases}.$$

In [2] a theorem was proved which contains the two above mentioned results as special cases. It reads

THEOREM 1

For all x the two-dimensional sequence $(\theta_{n-1}(x), \theta_n(x))$, $n=1,2,\dots$ is a sequence in the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. For almost all x this sequence is distributed over this triangle according to the density function f , where

$$f(a,b) = (\log 2)^{-1} (1 - 4ab)^{-1/2}.$$

The main purpose of this paper is to give a shorter and simpler proof of this theorem than the original one in [2]. At the basis is again the following fundamental result of Sh. Ito, H. Nakada and S. Tanaka, [4] and [5].

THEOREM 2 (Ito, Nakada, Tanaka)

Denote the set of irrational numbers between 0 and 1 by Ω and put $\Omega := \Omega \cap [0,1]$. Let B be the set of all Borel subsets of Ω , and μ the measure induced on B by the density function $(\log 2)^{-1} (1 + xy)^{-2}$. Finally, let the operator $T: \Omega \rightarrow \Omega$ be defined by

$$T(x,y) = (Tx, (a_1 + y)^{-1})$$

where, if x is given by (1), Tx is defined as

$$(3) \quad Tx := [0; a_2, a_3, \dots] = \frac{1}{x} - \left[\frac{1}{x} \right].$$

Then (Ω, B, μ, T) forms an ergodic system.

From this theorem we derive the following

THEOREM 3

For almost all irrational numbers x the two-dimensional sequence

$(T^n x, \frac{q_{n-1}}{q_n})_{n=1}^{\infty}$ is distributed over the unit square according to the density

function g , with $g(x,y) = (\log 2)^{-1} (1 + xy)^{-2}$.

Here $T^n x$ is defined inductively by $T(T^{n-1}x)$ with the T from (3).

Proof.

Denote by A that set of numbers $x \in \Omega$ for which the sequence $(T^n x, \frac{q_{n-1}}{q_n})_{n=1}^{\infty}$ is not distributed according to the density function $(\log 2)^{-1} (1 + xy)^{-2}$.

In view of the well-known relation

$$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, a_{n-2}, \dots, a_1] \quad \text{we see that } T^n(x, 0) = (T^n x, \frac{q_{n-1}}{q_n}).$$

Further it follows from the definition of T that for all $x \in \Omega$ and all pairs y and $y' \in [0, 1]$, the sequence $(T^n(x, y) - T^n(x, y'))$, $n = 1, 2, 3, \dots$ is a null-sequence. Hence, if $A := A \times [0, 1]$, then for every pair $(x, y) \in A$, the sequence $T^n(x, y)$ is not distributed according to the density function $(\log 2)^{-1} (1 + xy)^{-2}$. Now if A had, as a one-dimensional set, a positive Lebesgue measure, so had A as a two-dimensional set. But this would be in conflict with theorem 2. ♦

With this simple consequence of the theorem of Ito, Nakada and Tanaka it is now easy to prove theorem 1 as follows. We have, see [2], (2.1) and (2.2):

$$(4) \quad \theta_{n-1}(x) = \frac{q_{n-1}}{q_n} (1 + \frac{q_{n-1}}{q_n} T^n x)^{-1}, \quad \theta_n(x) = T^n x (1 + \frac{q_{n-1}}{q_n} T^n x)^{-1}.$$

In view of this we consider the function

$$F: (x, y) \rightarrow (\frac{y}{1 + xy}, \frac{x}{1 + xy}), \quad xy \neq -1.$$

It is easily verified that F maps the interior of the unit square bijectively onto the interior of the triangle from theorem 1. Put $a := y(1 + xy)^{-1}$, $b := x(1 + xy)^{-1}$. The determinant of Jacobi, J , of F equals $(xy - 1)(1 + xy)^{-3}$.

For almost all x the sequence $(T^n x, \frac{q_{n-1}}{q_n})_{n=1}^{\infty}$ is distributed according to the density

function g from theorem 3. Hence, for almost all x the sequence $(F(T^n x, \frac{q_{n-1}}{q_n}))_{n=1}^{\infty}$

which is in view of (4) the sequence $(\theta_{n-1}(x), \theta_n(x))$, $n = 1, 2, 3, \dots$ is distributed over the interior of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ according to the density function $g |J|^{-1}$.

Now

$$\begin{aligned} g(x,y) |J|^{-1} &= (\log 2)^{-1} \frac{1+xy}{1-xy} = (\log 2)^{-1} \left(\frac{1-xy}{1+xy} \right)^2^{-1/2} = \\ &= (\log 2)^{-1} \frac{((1+xy)^2 - 4xy)^{-1/2}}{(1+xy)^2} = (\log 2)^{-1} (1-4ab)^{-1/2} = f(a,b). \end{aligned} \quad \blacklozenge$$

In [2], several properties of the sequence $(\theta_n(x))$, $n = 1, 2, 3, \dots$ were given as corollaries of theorem 1. We mention one more.

COROLLARY.

Let $\lambda \geq 0$. Then for almost all x one has

$$\lim_{n \rightarrow \infty} n^{-1} \# \{ j; 1 \leq j \leq n, \theta_{j-1}(x) < \lambda \theta_j(x) \} = \begin{cases} (2 \log 2)^{-1} \log(1+\lambda) & , 0 \leq \lambda \leq 1 \\ 1 - (2 \log 2)^{-1} \log(1+\lambda^{-1}) & , 1 \leq \lambda \end{cases}.$$

Proof.

It follows from (4) that the condition $\theta_{j-1}(x) < \lambda \theta_j(x)$ is equivalent with

$\frac{q_{n-1}}{q_n} < \lambda T^n x$. Hence, by theorem 3, we have, when $0 \leq \lambda \leq 1$, for almost all x

$$\lim_{n \rightarrow \infty} n^{-1} \# \{ j; 1 \leq j \leq n, \theta_{j-1}(x) < \lambda \theta_j(x) \} =$$

$$= \frac{1}{\log 2} \int_0^1 \left(\int_0^{\lambda x} \frac{dy}{(1+xy)^2} \right) dx = \frac{1}{2 \log 2} \int_0^1 \frac{2\lambda x}{1+\lambda x^2} dx = \frac{1}{2 \log 2} \log(1+\lambda).$$

The case $1 \leq \lambda$ follows immediately from the case $0 \leq \lambda \leq 1$. $\blacklozenge$

Final remark.

In [3], C. Kraaikamp extended the method of [2] to the nearest integer continued fraction and Hurwitz's singular continued fraction and obtained several interesting results. This author has now also obtained the results from [3] by the method of the present paper (oral communication). He has also applied the method to general α -expansions. His results will be published in due course.

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