

An Application of the PLK-Method for Second-Order
Nonlinear Ordinary Differential Equations

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§1. Introduction. In 1949, M.J.Lighthill [1] proposed a new method for nonlinear differential equations known as PLK-method. In order to construct a uniform representation of the solution of the initial value problem:

$$(1.1) \left\{ x + \sum_{m=1}^{\infty} \varepsilon^m P_m(x, u) \right\} \frac{du}{dx} + q(x)u = r(x) + \sum_{m=1}^{\infty} \varepsilon^m R_m(x, u), \quad u(1) = b$$

over the interval $0 \leq x \leq 1$ for every ε ($0 < \varepsilon \leq \varepsilon_0$, $\varepsilon_0 > 0$ being sufficiently small), he substitutes

$$(1.2) \quad u = \sum_{m=0}^{\infty} \varepsilon^m u_m(\xi), \quad x = \xi + \sum_{m=1}^{\infty} \varepsilon^m x_m(\xi)$$

into the equation of Problem (1.1) and determines $u_m(\xi)$ and $x_m(\xi)$ in a suitable manner. In 1966, Y. Sibuya and K. Takahasi [4] proved the uniform convergence of the formal solution (1.2) of the initial value problem: $(x + \varepsilon u) du/dx + q(x)u = r(x)$, $u(1) = b$.

In 1955, W.A.Wasow mentioned in his paper [3] that Lighthill's method can be justified if $P_m(x, u)$ and $R_m(x, u)$ are polynomials of degree not greater than m with respect to u . In 1967, the author mentioned in his paper [6] that Lighthill's method can be justified for the problem of the form:

$$\left\{ x + \varepsilon u + \sum_{m=2}^S \varepsilon^m P_m(x, u) \right\} \frac{du}{dx} + q(x)u + r(x) + \sum_{m=1}^{S'} \varepsilon^m R_m(x, u) = 0, \quad u(1) = b$$

under some assumptions, even if $P_m(x, u)$ and $R_m(x, u)$ are polynomials of degree greater than m with respect to u . In this paper,

our purpose is to show that Hightbill's method can be justified for the initial value problem of the second order differential equation:

$$(1.3) \left\{ x + \delta \left(v + p(x) \frac{dv}{dx} \right) \right\} \frac{d^2 v}{dx^2} + q(x) \frac{dv}{dx} + s(x)v = r(x), \quad v(1) = b, \quad v'(1) = b',$$

where $p(x)$, $q(x)$, $s(x)$ and $r(x)$ are real-valued and analytic functions of x for $0 \leq x \leq 1$, and $q_0 = q(0)$ is a positive non-integer.

§2 Theorems. In order to consider this problem for $\delta = 0$, we put $\delta = 0$ in Problem (1.3), then we have

$$(2.1) \quad x \frac{d^2 v}{dx^2} + q(x) \frac{dv}{dx} + s(x)v = r(x), \quad v(1) = b, \quad v'(1) = b'.$$

Putting $\frac{dv}{dx} = u$, we have

$$(2.2) \quad \begin{cases} x \frac{du}{dx} = -q(x)u - s(x)v + r(x), \\ \frac{dv}{dx} = u, \quad u(1) = b', \quad v(1) = b. \end{cases}$$

In order to construct a solution of Problem (2.2), first of all, let us consider a solution of the equation of Problem (2.2) of the form:

$$(2.3) \quad u = \varphi(x) = \sum_{m=0}^{\infty} a_m x^m, \quad v = \psi(x) = \sum_{m=0}^{\infty} b_m x^m$$

in the neighborhood of $x = 0$. Inserting the series (2.3) into the equation of Problem (2.2), we have

$$(2.4) \quad \begin{cases} (m+q_0)a_m = E_m(a_0, \dots, a_{m-1}, b_0, \dots, b_m), \\ (m+1)b_{m+1} = a_m, \quad (m = 0, 1, 2, \dots), \end{cases}$$

where E_m is a polynomial of $a_0, \dots, a_{m-1}, b_0, \dots, b_m$. Since we

assumed that $q_0 > 0$, we can construct a solution of the equation of Problem (2.2) which is analytic and real-valued in the neighborhood of $x = 0$. Since (2.2) is linear, $u = \varphi(x)$, $v = \psi(x)$ is analytic on $0 \leq x \leq 1$. It is evident that $\varphi(x)$, $\psi(x)$ are real-valued on $0 \leq x \leq 1$. Thus the solution of (2.2) is given by

$$(2.5) \quad \begin{cases} u = u_0(x) = C_1 \varphi_1(x) + C_2 x^{-q_0} \varphi_2(x) + \varphi(x), \\ v = v_0(x) = C_1 \psi_1(x) + C_2 x^{-q_0} \psi_2(x) + \psi(x), \end{cases}$$

where $(\varphi_1(x), \psi_1(x))$, $(x^{-q_0} \varphi_2(x), x^{-q_0} \psi_2(x))$ is a fundamental system of solutions of the following system of homogeneous equations:

$$x \frac{du}{dx} = -q(x)u - s(x)v, \quad \frac{dv}{dx} = u$$

and C_1 and C_2 are constants such that $u_0(1) = b'$, $v_0(1) = b$.

Our main theorems are stated as follows:

THEOREM 1. Assume that

(I) $p(x)$, $q(x)$, $s(x)$ and $r(x)$ are real-valued and analytic functions of x for $0 \leq x \leq 1$;

(II) $q_0 = q(0)$ is a positive non-integer.

Then there exist functions $x(\xi, \zeta)$, $u(\xi, \zeta)$ and $v(\xi, \zeta)$ defined by the power series

$$(2.6) \quad \begin{cases} x(\xi, \zeta) = \xi + \sum_{m=1}^{\infty} x_m(\xi) \zeta^m, \\ u(\xi, \zeta) = \sum_{m=0}^{\infty} u_m(\xi) \zeta^m, \\ v(\xi, \zeta) = \sum_{m=0}^{\infty} v_m(\xi) \zeta^m \end{cases}$$

such that (a) the power series $x(\xi, \zeta)$, $u(\xi, \zeta)$ and $v(\xi, \zeta)$ are uniformly convergent for $0 < \xi \leq 1$, $|\zeta| \leq \delta$, ($\delta > 0$ being sufficiently small)

with coefficients $x_m(\xi)$, $u_m(\xi)$ and $v_m(\xi)$ functions real-valued and analytic for $0 < \xi \leq 1$; (b) $x(1, \xi) = 1$, $u(1, \xi) = b'$, $v(1, \xi) = b$; (c) $x = x(\xi, \varepsilon \xi^{-q_0})$, $u = u(\xi, \varepsilon \xi^{-q_0})$, $v = v(\xi, \varepsilon \xi^{-q_0})$ is a parametric representation of the solution $u(x)$, $v(x)$ of the following Problem:

$$(2.7) \quad \begin{cases} \{x + \varepsilon(v + p(x)u)\} \frac{du}{dx} + q(x)u + s(x)v = r(x), \\ \frac{dv}{dx} = u, \quad u(1) = b', \quad v(1) = b. \end{cases}$$

THEOREM 2. Assume the same assumptions (I) and (II) as in Theorem 1. Then there exist positive constants δ and ε_0 such that

(1) if $C_2 p(0) \varphi_2(0) > 0$, (a) the function

$$(2.8) \quad F \equiv x(\xi, \varepsilon \xi^{-q_0}) + \varepsilon \{v(\xi, \varepsilon \xi^{-q_0}) + p(x(\xi, \varepsilon \xi^{-q_0}))u(\xi, \varepsilon \xi^{-q_0})\}$$

never vanishes in the interval

$$(2.9) \quad (\varepsilon / \delta)^{1/q_0} \leq \xi \leq 1$$

for every ε in the interval

$$(2.10) \quad 0 < \varepsilon \leq \varepsilon_0;$$

(b) the equation $x(\xi, \varepsilon \xi^{-q_0}) = 0$ has a unique solution $\xi = \hat{\xi}(\varepsilon)$ in the interval (2.9) for every ε in the interval (2.10) and the function $\hat{\xi}(\varepsilon)$ behaves asymptotically as

$$(2.11) \quad \hat{\xi}(\varepsilon) = \varepsilon^{1/(q_0+1)} \left\{ \left(\frac{C_2 p(0) \varphi_2(0)}{q_0 + 1} \right)^{1/(q_0+1)} + o(1) \right\}, \quad (\varepsilon \rightarrow +0),$$

and, consequently, the value of the solution $u(x)$, $v(x)$ at $x = 0$ is

given by $u(\hat{\xi}(\varepsilon)), v(\hat{\xi}(\varepsilon))$ for $0 < \varepsilon \leq \varepsilon_0$; (c) the solution $u(x), v(x)$ defined by

$$(2.12) \quad x = x(\xi, \varepsilon \xi^{-q_0}), \quad u = u(\xi, \varepsilon \xi^{-q_0}), \quad v = v(\xi, \varepsilon \xi^{-q_0})$$

does not have any singular point on the interval $0 \leq x \leq 1$ for every ε ($0 < \varepsilon \leq \varepsilon_0$);

(ii) if $C_2 p(0) \phi_2(0) < 0$, the function F has a zero $\xi = \tilde{\xi}(\varepsilon)$ in the interval (2.9) for every ε in the interval (2.10), hence the solution $u(x), v(x)$ defined by (2.12) has a singular point with respect to x on the interval $0 \leq x \leq 1$.

§ 3. Construction of a Formal Solution of Equation (2.7). Putting

$$(3.1) \quad \begin{cases} x = \xi + \eta, \\ u = \xi^{-q_0} (\bar{u}_0(\xi) + \bar{u}), \\ v = \xi^{-q_0} (\bar{v}_0(\xi) + \bar{v}) \end{cases}$$

in (2.7), where $\bar{u}_0(\xi) = \xi^{q_0} u_0(\xi)$, $\bar{v}_0(\xi) = \xi^{q_0} v_0(\xi)$, we have

$$(3.2) \quad \begin{cases} \left[\xi + \eta + \xi \left\{ \bar{v}_0 + \bar{v} + p(\xi + \eta) (\bar{u}_0 + \bar{u}) \right\} \right] \left[-q(\xi) \bar{u}_0 - s(\xi) \bar{v}_0 + \xi^{q_0} r(\xi) - q_0 \bar{u} + \xi \frac{d\bar{u}}{d\xi} \right] \\ + \xi \left[q(\xi + \eta) (\bar{u}_0 + \bar{u}) + s(\xi + \eta) (\bar{v}_0 + \bar{v}) - \xi^{q_0} r(\xi + \eta) \right] \left(1 + \frac{d\eta}{d\xi} \right) = 0, \\ \xi \frac{d\bar{v}}{d\xi} = \xi \bar{u} + q_0 \bar{v} + \xi (\bar{u}_0 + \bar{u}) \frac{d\eta}{d\xi}, \quad (\xi = \varepsilon \xi^{-q_0}). \end{cases}$$

In order to satisfy the equation (3.2), it is sufficient to determine $\eta, \bar{u}, \bar{v}$ such that

$$(3.3) \quad \begin{cases} \xi \frac{d\eta}{d\xi} = \eta + \xi \left\{ \bar{v}_0 + \bar{v} + p(\xi + \eta) (\bar{u}_0 + \bar{u}) \right\}, \\ \xi \frac{d\bar{u}}{d\xi} = (q_0 - q(\xi + \eta)) \bar{u} - s(\xi + \eta) \bar{v} - (q(\xi + \eta) - q(\xi)) \bar{u}_0 \end{cases}$$

$$\left\{ \begin{array}{l} -(s(\xi+\eta)-s(\xi))\bar{v}_0 + \xi^q o(r(\xi+\eta)-r(\xi)), \\ \xi \frac{d\bar{v}}{d\xi} = \xi \bar{u} + q_0 \bar{v} + (\bar{u}_0 + \bar{u}) \left[\eta + \zeta \{ \bar{v}_0 + \bar{v} + p(\xi+\eta)(\bar{u}_0 + \bar{u}) \} \right]. \end{array} \right.$$

Putting

$$p(\xi+\eta) = \sum_{m=0}^{\infty} p_m(\xi) \eta^m, \quad q(\xi+\eta) = \sum_{m=0}^{\infty} q_m(\xi) \eta^m,$$

$$s(\xi+\eta) = \sum_{m=0}^{\infty} s_m(\xi) \eta^m, \quad r(\xi+\eta) = \sum_{m=0}^{\infty} r_m(\xi) \eta^m$$

in (3.3), we have

$$(3.4) \quad \left\{ \begin{array}{l} \xi \frac{d\eta}{d\xi} = \eta + g_1(\xi, \eta, \bar{u}, \bar{v}, \zeta), \\ \xi \frac{d\bar{u}}{d\xi} = (q_0 - q_0(\xi))\bar{u} - s_0(\xi)\bar{v} + g_2(\xi, \eta, \bar{u}, \bar{v}, \zeta), \\ \xi \frac{d\bar{v}}{d\xi} = \xi \bar{u} + q_0 \bar{v} + g_3(\xi, \eta, \bar{u}, \bar{v}, \zeta), \end{array} \right.$$

where

$$g_1 = \zeta \{ \bar{v}_0 + \bar{v} + (\bar{u}_0 + \bar{u}) \sum_{m=0}^{\infty} p_m(\xi) \eta^m \},$$

$$g_2 = - \sum_{m=1}^{\infty} \{ (\bar{u}_0 + \bar{u}) q_m(\xi) + (\bar{v}_0 + \bar{v}) s_m(\xi) - \xi^q o r_m(\xi) \} \eta^m,$$

$$g_3 = (\bar{u}_0 + \bar{u}) \left[\eta + \zeta \{ \bar{v}_0 + \bar{v} + (\bar{u}_0 + \bar{u}) \sum_{m=0}^{\infty} p_m(\xi) \eta^m \} \right].$$

We want to construct a formal solution of Equation (3.4) in the form:

$$(3.5) \quad \left\{ \begin{array}{l} \eta = \sum_{m=1}^{\infty} \bar{x}_m(\xi) \zeta^m, \quad (\bar{x}_m(1) = 0, m=1,2,\dots), \\ \bar{u} = \sum_{m=1}^{\infty} \bar{u}_m(\xi) \zeta^m, \quad (\bar{u}_m(1) = 0, m=1,2,\dots), \\ \bar{v} = \sum_{m=1}^{\infty} \bar{v}_m(\xi) \zeta^m, \quad (\bar{v}_m(1) = 0, m=1,2,\dots). \end{array} \right.$$

Denote these series by $S(\xi, \zeta)$, $\bar{S}(\xi, \zeta)$, $\bar{\bar{S}}(\xi, \zeta)$ and let

$$g_1(\xi, S, \bar{S}, \bar{S}, \xi) = \sum_{m=1}^{\infty} H_m(\bar{x}_1, \dots, \bar{x}_{m-1}, \bar{u}_1, \dots, \bar{u}_{m-1}, \bar{v}_{m-1}) \xi^m,$$

$$g_2(\xi, S, \bar{S}, \bar{S}, \xi) = \sum_{m=1}^{\infty} \bar{H}_m(\bar{x}_1, \dots, \bar{x}_m, \bar{u}_1, \dots, \bar{u}_{m-1}, \bar{v}_1, \dots, \bar{v}_{m-1}) \xi^m,$$

$$g_3(\xi, S, \bar{S}, \bar{S}, \xi) = \sum_{m=1}^{\infty} \bar{\bar{H}}_m(\bar{x}_1, \dots, \bar{x}_m, \bar{u}_1, \dots, \bar{u}_{m-1}, \bar{v}_1, \dots, \bar{v}_{m-1}) \xi^m,$$

where the quantities H_m , $\bar{H}_m$, $\bar{\bar{H}}_m$ are polynomials of their arguments with coefficients analytic in a suitable sector S_0 in the complex ξ - plane such that the vertex of S_0 is $\xi = 0$, S_0 contains the line segment $0 < \xi \leq 1$ in its interior. In particular

$$H_1 = \bar{v}_0(\xi) + \bar{u}_0(\xi) p_0(\xi),$$

$$\bar{H}_1 = -\bar{x}_1(\xi) \left\{ \bar{u}_0(\xi) q_1(\xi) + \bar{v}_0(\xi) s_1(\xi) - \xi^{q_0} r_1(\xi) \right\},$$

$$\bar{\bar{H}}_1 = \bar{u}_0(\xi) \left\{ \bar{x}_1 + \bar{v}_0(\xi) + \bar{u}_0(\xi) p_0(\xi) \right\}.$$

Hence we determine $\bar{x}_m$, $\bar{u}_m$ and $\bar{v}_m$ by

$$(3.6) \quad \begin{cases} \xi \frac{d\bar{x}_m}{d\xi} = (mq_0 + 1) \bar{x}_m + H_m(\bar{x}_1, \dots, \bar{v}_{m-1}), & \bar{x}_m(1) = 0, \\ \xi \frac{d\bar{u}_m}{d\xi} = \{(m+1)q_0 - q_0(\xi)\} \bar{u}_m - s_0(\xi) \bar{v}_m + \bar{H}_m(\bar{x}_1, \dots, \bar{v}_{m-1}), & \bar{u}_m(1) = 0, \\ \xi \frac{d\bar{v}_m}{d\xi} = \xi \bar{u}_m + (m+1)q_0 \bar{v}_m + \bar{\bar{H}}_m(\bar{x}_1, \dots, \bar{v}_{m-1}), & \bar{v}_m(1) = 0, (m=1, 2, \dots). \end{cases}$$

The solution of Problem (3.6)₁ ($m=1$) is given by

$$(3.7) \quad \bar{x}_1 = \xi^{q_0+1} \int_1^\xi t^{-q_0-2} H_1(t) dt,$$

where $H_1(t) = \bar{v}_0(t) + \bar{u}_0(t) p_0(t)$, and the integral may be taken along any path from $t = 1$ to $t = \xi$ within the sector S_0 . Noticing that $(\xi^{2q_0} \phi_1(\xi), \xi^{2q_0} \psi_1(\xi)), (\xi^{q_0} \phi_2(\xi), \xi^{q_0} \psi_2(\xi))$ is a fundamental system of solutions of the following system:

$$\begin{cases} \xi \frac{d\bar{u}_1}{d\xi} = (2q_0 - q_0(\xi))\bar{u}_1 - s_0(\xi)\bar{v}_1, \\ \xi \frac{d\bar{v}_1}{d\xi} = \xi\bar{u}_1 + 2q_0\bar{v}_1, \end{cases}$$

we get the following solution of Problem (3.6)_{2,3}(m=1):

$$(3.8) \quad \begin{cases} \bar{u}_1 = \xi^{2q_0} \varphi_1(\xi) \int_1^\xi \frac{t^{-2q_0-1} W_1(t)}{W(t)} dt \\ \quad + \xi^{q_0} \varphi_2(\xi) \int_1^\xi \frac{t^{-q_0-1} \hat{W}_1(t)}{W(t)} dt, \\ \bar{v}_1 = \xi^{2q_0} \psi_1(\xi) \int_1^\xi \frac{t^{-2q_0-1} W_1(t)}{W(t)} dt \\ \quad + \xi^{q_0} \psi_2(\xi) \int_1^\xi \frac{t^{-q_0-1} \hat{W}_1(t)}{W(t)} dt, \end{cases}$$

where

$$W(t) = \begin{vmatrix} \varphi_1(t) & \varphi_2(t) \\ \psi_1(t) & \psi_2(t) \end{vmatrix}, \quad W_1(t) = \begin{vmatrix} \bar{H}_1(t) & \varphi_2(t) \\ \bar{\bar{H}}_1(t) & \psi_2(t) \end{vmatrix}$$

$$\hat{W}_1(t) = \begin{vmatrix} \varphi_1(t) & \bar{H}_1(t) \\ \psi_1(t) & \bar{\bar{H}}_1(t) \end{vmatrix}.$$

$$\bar{H}_1(t) = -\bar{x}_1(t) \{ \bar{u}_0(t)q_1(t) + \bar{v}_0(t)s_1(t) - t^q \text{or}_1(t) \},$$

$$\bar{\bar{H}}_1(t) = \bar{u}_0(t) \{ \bar{x}_1(t) + \bar{v}_0(t) + \bar{u}_0(t)p_0(t) \}.$$

Assuming that we can get the solutions of Problem (3.6)(m=1,...,k-1), the solution of (3.6)₁(m=k) is given by

$$(3.9) \quad \bar{x}_k = \xi^{kq_0+1} \int_1^\xi t^{-kq_0-2} H_k(t) dt,$$

where

$$H_k(t) = H_k(\bar{x}_1(t), \dots, \bar{x}_{k-1}(t), \bar{u}_1(t), \dots, \bar{u}_{k-1}(t), \bar{v}_{k-1}(t)),$$

and then the solutions of Problem (3.6)_{2,3}($m=k$) are given by

$$(3.10) \quad \begin{cases} \bar{u}_k = \xi^{(k+1)q_0} \varphi_1(\xi) \int_1^\xi \frac{t^{-(k+1)q_0-1} W_k(t)}{W(t)} dt \\ \quad + \xi^{kq_0} \varphi_2(\xi) \int_1^\xi \frac{t^{-kq_0-1} \hat{W}_k(t)}{W(t)} dt, \\ \bar{v}_k = \xi^{(k+1)q_0} \psi_1(\xi) \int_1^\xi \frac{t^{-(k+1)q_0-1} W_k(t)}{W(t)} dt \\ \quad + \xi^{kq_0} \psi_2(\xi) \int_1^\xi \frac{t^{-kq_0-1} \hat{W}_k(t)}{W(t)} dt, \end{cases}$$

where

$$W_k(t) = \begin{vmatrix} \bar{H}_k(t) & \varphi_2(t) \\ \bar{\bar{H}}_k(t) & \psi_2(t) \end{vmatrix}, \quad \hat{W}_k(t) = \begin{vmatrix} \varphi_1(t) & \bar{H}_k(t) \\ \psi_1(t) & \bar{\bar{H}}_k(t) \end{vmatrix}.$$

$$\bar{H}_k(t) = \bar{H}_k(\bar{x}_1(t), \dots, \bar{x}_k(t), \bar{u}_1(t), \dots, \bar{u}_{k-1}(t), \bar{v}_1(t), \dots, \bar{v}_{k-1}(t)),$$

$$\bar{\bar{H}}_k(t) = \bar{\bar{H}}_k(\bar{x}_1(t), \dots, \bar{x}_k(t), \bar{u}_1(t), \dots, \bar{u}_{k-1}(t), \bar{v}_1(t), \dots, \bar{v}_{k-1}(t)).$$

Thus, by mathematical induction, we can get the solutions of Problem (3.6).

§ 4. Estimates of the Coefficients $x_m(\xi)$, $u_m(\xi)$ and $v_m(\xi)$. In this section we shall prove the following lemma:

LEMMA 1. There exists a positive constant M such that

$$\sup_{\xi \in S_0} |\bar{x}_m(\xi)| \leq M \sup_{\xi \in S_0} |H_m(\xi)|,$$

$$\sup_{\xi \in S_0} |\bar{u}_m(\xi)|, \sup_{\xi \in S_0} |\bar{v}_m(\xi)| \leq M \left\{ \sup_{\xi \in S_0} |\bar{H}_m(\xi)| + \sup_{\xi \in S_0} |\bar{\bar{H}}_m(\xi)| \right\}.$$

PROOF. We shall prove the first inequality, since the others

can be proved in the same manner. Since the quantities $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\psi_1(\xi)$, $\psi_2(\xi)$ and $|w(\xi)|^{-1}$ are all bounded on the closure of S_0 , there exists a constant C such that

$$|\bar{x}_m(\xi)| \leq C |\xi|^{mq_0+1} \sup_{t \in S_0} |H_m(t)| \left\{ \left| \int_{C_1} t^{-mq_0-2} dt \right| + \left| \int_{C_2} t^{-mq_0-2} dt \right| \right\},$$

where the paths of integration C_1 and C_2 are respectively the line segment $1, |\xi|$ and the circular arc $|\xi|, \xi$. On the other hand, we can prove the following inequality in the same way as in [4]:

$$\left| \int_{C_1} t^{-mq_0-2} dt \right| + \left| \int_{C_2} t^{-mq_0-2} dt \right| \leq C' |\xi|^{-mq_0-1} \quad \text{in } S_0,$$

where C' is a sufficiently large constant. This proves our lemma.

§ 5. Construction of Majorant. Since $p(\xi+\eta)$, $q(\xi+\eta)$, $s(\xi+\eta)$ and $r(\xi+\eta)$ are analytic for $|\eta| \leq \delta_0$ ($\delta_0 > 0$ is sufficiently small) and $\xi \in S_0$, there exists a constant K such that

$$|\bar{u}_0(\xi)|, |\bar{v}_0(\xi)| \leq K,$$

$$|p_m(\xi)|, |q_m(\xi)|, |s_m(\xi)|, |r_m(\xi)| = K/\delta_0^m.$$

Let us consider

$$(5.1) \quad \begin{cases} \eta = MG_1(\eta, \bar{u}, \xi), \\ \bar{u} = \bar{v} = M\{G_2(\eta, \bar{u}, \xi) + G_3(\eta, \bar{u}, \xi)\}, \end{cases}$$

where

$$G_1 = \xi(K+\bar{u}) \left(1 + \frac{K\delta_0}{\delta_0 - \eta}\right), \quad G_2 = \frac{K\eta(2\bar{u}+2K+1)}{\delta_0 - \eta},$$

$$G_3 = (K+\bar{u}) \left\{ \eta + \xi(K+\bar{u}) \left(1 + \frac{K\delta_0}{\delta_0 - \eta}\right) \right\}$$

and M is the constant in Lemma 1. Notice that G_1 , G_2 and G_3 are

respectively majorants of g_1, g_2 and g_3 uniformly for ξ in S_0 .

Equation (5.1) has a solution

$$(5.2) \quad \begin{cases} \eta = S(\xi) = \sum_{m=1}^{\infty} S_m \xi^m, \\ \bar{u} = \bar{v} = \bar{S}(\xi) = \sum_{m=1}^{\infty} \bar{S}_m \xi^m, \end{cases}$$

such that $S_m, \bar{S}_m \geq 0$ and these series converge for $|\xi| \leq \delta$, if $\delta > 0$ is sufficiently small. Let

$$G_1(S(\xi), \bar{S}(\xi), \xi) = \sum_{m=1}^{\infty} K_m \xi^m,$$

$$G_2(S(\xi), \bar{S}(\xi), \xi) + G_3(S(\xi), \bar{S}(\xi), \xi) = \sum_{m=1}^{\infty} \bar{K}_m \xi^m.$$

It is easily seen that $S_m = MK_m$, $\bar{S}_m = M\bar{K}_m$, in particular

$$S_1 = MK_1 = MK(1+K),$$

$$\bar{S}_1 = M\bar{K}_1 = MK^2(1+K)\{M(1+2K) + \delta_0(1+M)\} / \delta_0.$$

Next we shall prove

$$(5.3) \quad \sup_{\xi \in S_0} |\bar{x}_m(\xi)| \leq S_m, \quad \sup_{\xi \in S_0} |\bar{u}_m(\xi)|, \quad \sup_{\xi \in S_0} |\bar{v}_m(\xi)| \leq \bar{S}_m, \quad (m=1, 2, \dots).$$

In order to prove (6.3), first of all let us consider the case $m=1$.

We have

$$\begin{aligned} \sup_{\xi \in S_0} |\bar{x}_1(\xi)| &\leq M \sup_{\xi \in S_0} |\bar{H}_1(\xi)| \leq M \sup_{\xi \in S_0} |\bar{v}_0(\xi) + \bar{u}_0(\xi)p_0(\xi)| \\ &\leq MK(1+K) = S_1. \end{aligned}$$

$$\begin{aligned} \sup_{\xi \in S_0} |\bar{u}_1(\xi)| &= \sup_{\xi \in S_0} |\bar{v}_1(\xi)| \leq M \left\{ \sup_{\xi \in S_0} |\bar{H}_1(\xi)| + \sup_{\xi \in S_0} |\bar{\bar{H}}_1(\xi)| \right\} \\ &= M \left[\sup_{\xi \in S_0} |\bar{x}_1(\xi)| \{ \bar{u}_0(\xi)q_1(\xi) + \bar{v}_0(\xi)s_1(\xi) - \xi^{q_0 r_1}(\xi) \} \right] \end{aligned}$$

$$\begin{aligned}
& + \sup_{\xi \in S_0} |\bar{u}_0(\xi) \{ \bar{x}_1(\xi) + \bar{v}_0(\xi) + \bar{u}_0(\xi) p_0(\xi) \}| \Big] \\
& \leq M \{ MK^2(1+K)(1+2K)/\delta_0 + K^2(1+K)(1+M) \} \\
& = MK^2(1+K) \{ M(1+2K) + \delta_0(1+M) \} / \delta_0.
\end{aligned}$$

Assuming that

$$\sup_{\xi \in S_0} |\bar{x}_n(\xi)| \leq S_n, \quad \sup_{\xi \in S_0} |\bar{u}_n(\xi)| \leq \bar{S}_n, \quad \sup_{\xi \in S_0} |\bar{v}_n(\xi)| = \bar{S}_n, \quad (n=1, \dots, m-1),$$

we have

$$\sup_{\xi \in S_0} |H_m(\xi)| \leq K_m, \quad \sup_{\xi \in S_0} |\bar{H}_m(\xi)| + \sup_{\xi \in S_0} |\bar{\bar{H}}_m(\xi)| \leq \bar{K}_m,$$

because G_1, G_2 and G_3 are respectively majorants of g_1, g_2 and g_3 .

Therefore from Lemma 1 it follows

$$\sup |\bar{x}_m(\xi)| \leq MK_m = S_m, \quad \sup |\bar{u}_m(\xi)|, \quad \sup |\bar{v}_m(\xi)| \leq \bar{MK}_m = \bar{S}_m.$$

Thus, by mathematical induction, we complete the proof of (5.3).

§ 6. Properties of $\bar{x}_1(\xi)$. In Section 3, we derived the differential equation of $\bar{x}_1(\xi)$:

$$(6.1) \quad \xi \frac{d\bar{x}_1}{d\xi} = (q_0+1)\bar{x}_1 + H_1(\xi).$$

whose solution satisfying the initial condition $\bar{x}_1(1) = 0$ is given by

$$(6.2) \quad \bar{x}_1(\xi) = \xi^{q_0+1} \int_1^\xi t^{-q_0-2} H_1(t) dt.$$

Noticing that

$$\int_1^\xi t^{-q_0-2} H_1(t) dt = - \frac{1}{q_0+1} \left[t^{-q_0-1} H_1(t) \right]_1^\xi + \frac{1}{q_0+1} \int_1^\xi t^{-q_0-1} \frac{dH_1}{dt} dt$$

$$= \xi^{-q_0-1} \left\{ -\frac{H_1(0)}{q_0+1} + o(1) \right\}, \quad (\xi \rightarrow +0),$$

we get

$$(6.3) \quad \bar{x}_1(\xi) = -\frac{H_1(0)}{q_0+1} + o(1), \quad (\xi \rightarrow +0).$$

§ 7. Case $C_2 p(0) \varphi_2(0) > 0$. Notice that $C_2 p(0) \varphi_2(0) = H_1(0)$. Substituting (3.1), (3.5) in the function F defined by (2.8), we get $F = \xi + \sum_{m=1}^{\infty} \hat{x}_m(\xi) \xi^m$, where in particular

$$\hat{x}_1(\xi) = \bar{x}_1(\xi) + H_1(\xi) = \frac{q_0 H_1(0)}{q_0+1} + o(1), \quad (\xi \rightarrow +0).$$

Let $M_1 (\geq 1)$, ξ_1 , δ_1 and δ be positive constants such that

$$(7.1) \quad \begin{cases} |\hat{x}_1(\xi)| \leq M_1 & \text{for } 0 \leq \xi \leq 1, \\ \hat{x}_1(\xi) \geq \frac{q_0 H_1(0)}{2(q_0+1)} & \text{for } 0 \leq \xi \leq \xi_1 (\leq 1), \\ \left| \sum_{m=2}^{\infty} \hat{x}_m(\xi) \xi^{m-2} \right| \leq M_1 & \text{for } 0 \leq \xi \leq 1, 0 < \xi \leq \delta_1, \\ \delta = \min \left\{ \delta_1, \frac{\xi_1}{4 M_1}, \frac{q_0 H_1(0)}{4 M_1 (q_0+1)} \right\}. \end{cases}$$

From (7.1) it follows that if $\xi_1 \leq \xi \leq 1$, $0 < \xi \leq \delta$, we get

$$\begin{aligned} F &\geq \xi_1 - \left\{ |\hat{x}_1(\xi)| \xi + \left| \sum_{m=2}^{\infty} \hat{x}_m(\xi) \xi^{m-2} \right| \xi^2 \right\} \\ &\geq \xi_1 - (M_1 \delta + M_1 \delta^2) \geq \xi_1 - 2M_1 \delta > 0, \end{aligned}$$

and that if $0 \leq \xi \leq \xi_1$, $0 < \xi \leq \delta$, we get

$$\begin{aligned} F &\geq \hat{x}_1(\xi) \xi + \sum_{m=2}^{\infty} \hat{x}_m(\xi) \xi^m \geq \frac{q_0 H_1(0)}{2(q_0+1)} \xi - M_1 \xi^2 \\ &\geq \xi^2 \left\{ \frac{q_0 H_1(0)}{2(q_0+1)} \xi^{-1} - M_1 \right\}. \end{aligned}$$

Since we have the inequality $\xi = \varepsilon \xi^{-q_0} \leq \delta$, we have

$$\begin{aligned} F &\geq \xi^2 \left\{ \frac{q_0 H_1(0)}{2\delta(q_0+1)} - M_1 \right\} = \frac{\xi^2}{\delta} \left\{ \frac{q_0 H_1(0)}{2(q_0+1)} - M_1 \delta \right\} \\ &\geq \frac{\xi^2}{\delta} \left\{ \frac{q_0 H_1(0)}{2(q_0+1)} - \frac{q_0 H_1(0)}{4(q_0+1)} \right\} = \frac{\xi^2 q_0 H_1(0)}{4\delta(q_0+1)} > 0, \end{aligned}$$

provided that ξ is in the interval

$$(\varepsilon/\delta)^{1/q_0} \leq \xi \leq \xi_1$$

for a sufficiently small ε , which prove the assertion (a) of Theorem 2. Since $\bar{x}_1 = -H_1(0)/(q_0+1) + o(1)$, $(\xi \rightarrow +0)$, the equation $x(\xi, \varepsilon \xi^{-q_0}) = 0$ has a solution $\xi = \hat{\xi}(\varepsilon)$:

$$\hat{\xi}(\varepsilon) = \left\{ \frac{H_1(0)}{q_0+1} \varepsilon + o(\varepsilon) \right\}^{1/(q_0+1)}, \quad (\varepsilon \rightarrow +0).$$

Therefore if $\varepsilon_0 > 0$ is sufficiently small, we get

$$(\varepsilon/\delta)^{1/q_0} \leq \hat{\xi}(\varepsilon) \leq 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Then it is easily seen that the function F is positive, namely $dx/d\xi > 0$ for (2.9) and (2.10). Hence $x(\xi, \varepsilon \xi^{-q_0})$ has one and only one zero there. This completes the proof of the assertion (i) of Theorem 2.

§ 3. Case $C_2 p(0) \varphi_2(0) < 0$. In the same manner as in § 7, we can complete the proof of the assertion (ii) of Theorem 2.

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