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TENTATIVE STUDY ON EQUIVARIANT SURGERY OBSTRUCTIONS: FIXED POINT SETS OF SMOOTH A_5-ACTIONS

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Abstract. Let G be the alternating group on 5 letters and let F be a closed smooth manifold diffeomorphic to the fixed point set of a smooth G-action on a disk. Marek Kaluba proved that if F is even dimensional then there exists a smooth G-action on a closed manifold X being homotopy equivalent to a complex projective space such that the fixed point set of the G-action is diffeomorphic to F. In this paper we discuss whether series of manifolds diffeomorphic or homotopy equivalent to complex projective spaces, real projective spaces, or lens spaces, admit smooth G-actions with fixed point set diffeomorphic to F.

1. Introduction

Let G be a finite group throughout this paper. For a smooth manifold M, let $\mathfrak{F}_G(M)$ denote the family of all manifolds F such that $F = M^G$ for some smooth G-action on M. For a family $\mathfrak{M}$ of smooth manifolds, let $\mathcal{F}_G(\mathfrak{M})$ denote the union of $\mathfrak{F}_G(M)$ with $M \in \mathfrak{M}$. Let $\mathfrak{D}$, $\mathfrak{S}$, and $\mathfrak{P}_{\mathbb{C}}$ denote the families of disks, spheres, and complex projective spaces, respectively. B. Oliver [19] completely determined the family $\mathfrak{F}_G(\mathfrak{D})$ for G not of prime power order. K. Pawałowski and the author [18, 14] studied $\mathfrak{F}_G(\mathfrak{S})$ for various Oliver groups G.

In order to quote a part of Oliver's result on $\mathfrak{F}_G(\mathfrak{D})$, we adopt the notation $\mathcal{G}_{\mathbb{R}}$, $\mathcal{G}_{\mathbb{C}}^{g}$, $\mathcal{G}_{\mathbb{C}}$ and $\mathcal{E}$ for the families of all finite groups satisfying the following properties, respectively.

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- $G \in \mathcal{G}_{\mathbb{R}}$: G possesses a subquotient K/H isomorphic to a dihedral group of order $2pq$ for some distinct primes p and q, where $H \triangleleft K \leq G$.
- $G \in \mathcal{G}_{\mathbb{C}}^{g}$: G contains an element g being conjugate to its inverse of order pq for some distinct primes p and q.
- $G \in \mathcal{G}_{\mathbb{C}}$: G contains an element g of order pq for some distinct primes p and q.
- $G \in \mathcal{E}$: A Sylow 2-subgroup of G is not normal in G, and any element of G is of prime power order.

Note that $\mathcal{G}_{\mathbb{R}} \subset \mathcal{G}_{\mathbb{C}}^{g} \subset \mathcal{G}_{\mathbb{C}}$. Let A_5 denote the alternating group on 5 letters. Then A_5 belongs to $\mathcal{E}$. B. Oliver [19] says that for $G \in \mathcal{F}_{\mathbb{C}} \cup \mathcal{E}$, a closed manifold F belongs to $\mathfrak{F}_G(\mathcal{D})$ if and only if $\chi(F) \equiv 1 \mod n_G$ and

$$
(1.1)\qquad
\begin{aligned}
&\bullet\ G \in \mathcal{G}_{\mathbb{R}} \Rightarrow \text{no restrictions on } T(F),\\
&\bullet\ G \in \mathcal{G}_{\mathbb{C}}^{g} \smallsetminus \mathcal{G}_{\mathbb{R}} \Rightarrow c_{\mathbb{R}}([T(F)]) \in c_{\mathbb{H}}\left(\widetilde{KSp}(F)\right) + \mathrm{Tor}\left(\widetilde{KU}(F)\right),\\
&\bullet\ G \in \mathcal{G}_{\mathbb{C}} \smallsetminus \mathcal{G}_{\mathbb{C}}^{g} \Rightarrow [T(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \mathrm{Tor}\left(\widetilde{KO}(F)\right),\\
&\bullet\ G \in \mathcal{E} \Rightarrow [T(F)] \in \mathrm{Tor}\left(\widetilde{KO}(F)\right).
\end{aligned}
$$

If $G \in \mathcal{E}$ and $F \in \mathfrak{F}_G(\mathfrak{D})$ then each connected component of F has same dimension. The Oliver number n_G above is equal to 1 whenever G is nonsolvable.

Marek Kaluba [5] obtained the next two theorems concerned with $\mathfrak{F}_G(\mathfrak{P}_{\mathbb{C}})$.

Theorem. [5, Theorem 2.6] *Let G be a nontrivial perfect group in the class $\mathcal{G}_{\mathbb{C}}$ and let F be a closed manifold in $\mathfrak{F}_G(\mathcal{D})$. In the case $G \in \mathcal{G}_{\mathbb{C}} \smallsetminus \mathcal{G}_{\mathbb{R}}$, suppose that some connected component of F is even dimensional. Then F belongs to $\mathfrak{F}_G(\mathfrak{P}_{\mathbb{C}})$.*

Theorem. [5, Theorem 4.11] *Let G be A_5 and F a closed manifold in $\mathfrak{F}_G(\mathfrak{D})$. Suppose that F is even dimensional. Then F is diffeomorphic to the fixed point set of a smooth G-action on a closed manifold X which is homotopy equivalent to some complex projective space.*

Let $P_{\mathbb{C}}^k$ (resp. $P_{\mathbb{R}}^k$) denote the complex (resp. real) projective space of complex (resp. real) dimension k, and let Γ be a cyclic subgroup of $\mathbb{C}^\times$ of order ≥ 3. The orbit space $L^{2k+1} = S(\mathbb{C}^{k+1})/\Gamma$ is a lens space of dimension $2k+1$. Let $\mathfrak{L}$ be the family of lens spaces L^{2k+1}, $k = 2, 3, 4, \ldots$. By examining and improving the proof of [5, Theorem 4.11] by M. Kaluba, we obtain the next result.

Theorem 1.1. *Let G be A_5 and F a closed manifold in $\mathfrak{F}_G(\mathfrak{D})$. Then there exists an integer $N > 0$ possessing the property that for any $k \geq N$,*

(1) $F \in \mathfrak{F}_G(D^k)$,
(2) $F \in \mathfrak{F}_G(S^k)$,
(3) *if* $\dim F \equiv 0 \mod 2$ *then* $F \in \mathfrak{F}_G(P^k_{\mathbb{C}})$,
(4) $F \in \mathfrak{F}_G(X_k)$ *such that* X_k *is a smooth closed manifold homotopy equivalent to* $P^k_{\mathbb{R}}$,
(5) *if* $\dim F \equiv 1 \mod 2$ *then* $F \in \mathfrak{F}_G(Y_k)$ *such that* Y_k *is a smooth closed manifold homotopy equivalent to* L^{2k+1}.

This result follows from Theorem 3.4. In Theorem 1.1, one may conjecture that $P^k_{\mathbb{R}}$ and L^{2k+1} can be chosen as X_k and Y_k respectively, but the author cannot prove the conjecture so far.

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2. Dimension conditions of fixed point sets

Let G be a finite group. Let U be a G-manifold and (H, K) a pair of subgroups $H < K \leq G$. We say that U satisfies the *gap condition*, *cobordism gap condition*, or *strong gap condition* for (H, K) if the inequality

$$2\dim({U^H}_i)^K < \dim {U^H}_i, \tag{2.1}$$

$$2\left(\dim\{({U^H}_i)^K \smallsetminus ({U^H}_i)^{N_G(H)}\} + 1\right) \leq \dim {U^H}_i, \tag{2.2}$$

or

$$2\{\dim({U^H}_i)^K + 1\} < \dim {U^H}_i, \tag{2.3}$$

holds, respectively, for any connected component ${U^H}_i$ of U^H.

Proposition 2.1. *Let G be a perfect group having a cyclic subgroup C_2 of order 2, Y the complex projective space associated with the complex G-module*

$$V = \mathbb{C}^{\oplus m+1} \oplus (\mathbb{C}[G] - \mathbb{C})^{\oplus n},$$

where $m \geq 0$ and $n \geq 1$, and U the G-tubular neighborhood of Y^G.

(1) *Y satisfies the gap condition for $(\{e\}, C_2)$ if and only if $m+1 = n$.*
(2) *U satisfies the gap condition for $(\{e\}, C_2)$ if and only if $m+1 \leq n$.*
(3) *If $m+1 \leq n$ then U satisfies the strong gap condition for (H, K) such that $\{e\} \neq H < K \leq G$ and $|K : H| \geq 3$.*

Proof. We readily see that $Y^G = P_{\mathbb{C}}(\mathbb{C}^{m+1}) = P_{\mathbb{C}}^m$ and Y^{C_2} has two connected components

$$Y_a^{C_2} = P_{\mathbb{C}}(\mathbb{C}^{m+1} \oplus ((\mathbb{C}[G] - \mathbb{C})^{C_2})^{\oplus n}) = P_{\mathbb{C}}^{m+n(|G|/2-1)} \quad \text{and}$$
$$Y_b^{C_2} = P_{\mathbb{C}}(((\mathbb{C}[G] - \mathbb{C})_{C_2})^{\oplus n}) = P_{\mathbb{C}}^{n|G|/2-1}.$$

Thus we have $\dim Y = 2m - 2n + 2n|G|$, $\dim Y_a^{C_2} = 2m - 2n + n|G|$ and $\dim Y_b^{C_2} = n|G| - 2$. Note the equivalences

- $2(2m - 2n + n|G|) < 2m - 2n + 2n|G| \iff m < n$
- $2(n|G| - 2) < 2m - 2n + 2n|G| \iff n - 2 < m.$

Thus Y satisfies the gap condition for $(\{e\}, C_2)$ if and only if $n-2 < m < n$, namely $m+1 = n$. Since $\dim U^{C_2} = \dim Y_a^{C_2}$, U satisfies the gap condition for $(\{e\}, C_2)$ if and only if $m < n$, namely $m+1 \leq n$.

For any $H \leq G$, U^H is connected. Let y denote the point $[1, 0, \ldots, 0]$ in $Y = P_{\mathbb{C}}(\mathbb{C} \oplus \mathbb{C}^{\oplus m} \oplus (\mathbb{C}[G] - \mathbb{C})^{\oplus n})$. The tangential representation $T_y(Y)$ is isomorphic to $\mathbb{C}^{\oplus m} \oplus (\mathbb{C}[G] - \mathbb{C})^{\oplus n}$ as complex G-modules. Since $\dim U^H = \dim T_y(Y)^H$, we get

$$\dim U^H = 2\{m + n(|G|/|H| - 1)\} = 2m - 2n + 2n|G|/|H|$$

and

$$\dim U^H - 2(\dim U^K + 1) = 2\{n - (m+1)\} + 2n|G|\{|K| - 2|H|\}/(|H||K|).$$

In the case $n \geq m+1$ and $|K| \geq 3|H|$, we conclude $2(\dim U^K + 1) < \dim U^H$. □

Theorem 2.2 (Disk Theorem). *Let G be a nontrivial perfect group, $F \in \mathfrak{F}(\mathfrak{D})$, F_0 a connected component of F with $x_0 \in F_0$, $m = \dim F_0$, and W a real G-module* *with* $\dim W^G = m$. *Then there exists a smooth* G*-action on the disk* D *of dimension* $\dim W + N(|G|-1)$ *for some integer* $N \geq 0$ *satisfying the following conditions.*

(1) $D^G = F$.

(2) $T_{x_0}(D)$ *is isomorphic to* $W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus N}$ *as real* G*-modules.*

(3) D *satisfies the strong gap condition for arbitrary pair* (H, K) *such that* $H < K \leq G$.

Corollary 2.3. *Let* G, $F \in \mathfrak{F}(\mathfrak{D})$, F_0, $x_0 \in F_0$, $m = \dim F_0$, W, D *and* N *be as in Theorem 2.2. Let* n *be an arbitrary integer* $\geq N$. *Then there exists a smooth* G*-action on the disk* S *of dimension* $\dim W + n(|G|-1)$ *satisfying the following conditions.*

(1) $S^G = F \amalg F'$ *and* F' *is diffeomorphic to* F.

(2) $T_{x_0}(S)$ *is isomorphic to* $W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus n}$ *as real* G*-modules.*

(3) S *satisfies the strong gap condition for arbitrary pair* (H, K) *such that* $H < K \leq G$.

Proof. We set $X = D \times D((\mathbb{R}[G] - \mathbb{R})^{\oplus(n-N)})$. Let S be the double of X. Then S satisfies the desired conditions. $\square$

3. Deleting theorem and realization theorem

In this section we will give a deleting theorem and a realization theorem. The latter is obtained from the formar and Corollary 2.3. Our main result Theorem 1.1 follows from the realization theorem.

Let A_5 denote the alternating group on 5 letters 1,2, 3, 4, 5, and let A_4 denote the alternating group on 4 letters 1,2, 3, 4. Unless otherwise stated, we use the notation:

- $C_2 = \langle (1,2)(3,4) \rangle$
- $D_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$
- $C_3 = \langle (1,2,5) \rangle$
- $D_6 = \langle (1,2,5), (1,2)(3,4) \rangle$
- $C_5 = \langle (1,3,4,2,5) \rangle$
- $D_{10} = \langle (1,3,4,2,5), (1,2)(3,4) \rangle$.

These groups and A_4 are regarded as subgroups of A_5.

Throughout this section, let G be A_5. Then $N_G(C_2) = D_4$, $N_G(D_4) = A_4$, $N_G(C_3) = D_6$, $N_G(C_5) = D_{10}$, and any maximal proper subgroup of G is conjugate to one of A_4, D_{10}, D_6. We can readily show

Proposition 3.1. *Let H be a maximal proper subgroup of G. Then any two subgroups of H are conjugate in G if and only if they are conjugate in H.*

Proposition 3.2. *Let α be the element of the Burnside ring $\Omega(G)$ given by*

$$\alpha = [G/A_4] + [G/D_{10}] + [G/D_6] - [G/C_3] - 2[G/C_2] + [G/\{e\}].$$

Then for any proper subgroup $H < G$, $\mathrm{res}^G_H\alpha$ coincides with $[H/H]$ in $\Omega(H)$.

Theorem 3.3 (Deleting Theorem). *(Let $G = A_5$.) Let Y be a compact connected smooth G-manifold of dimension ≥ 5, with $|\pi_1(Y)| < \infty$, and with a decomposition $Y^G = Y_0^G \amalg Y_1^G$ such that $\partial Y_0^G = \emptyset$. Let U be the G-tubular neighborhood of Y_0^G. Suppose U satisfies the gap condition for $(\{e\}, C_2)$, $(\{e\}, C_3)$ and $(\{e\}, C_5)$, and the cobordism gap condition for (C_2, D_6), (C_2, D_{10}) and (C_3, A_4). Then there exists a smooth G-manifold X possessing the following properties.*

(1) $X^G = Y_1^G$.
(2) *X is homotopy equivalent to Y.*
(3) *X^H is diffeomorphic to Y^H for any H such that $\{e\} \neq H < G$.*
(4) *In the case that* $\dim Y \equiv 0 \mod 2$ *and* $\pi_1(Y) = 1$, *X is diffeomorphic to Y.*

Guideline for Proof. There exists a compact connected smooth G-submanifold U_1 of $Y \smallsetminus \partial Y$ with $U \subset U_1$ such that G freely acts on $U_1 \smallsetminus U$ and the inclusion induced homomorphism $\pi_1(U_1) \to \pi_1(Y)$ is an isomorphism. First, we construct a G-framed map $f_1 : X_1 \to Y$ rel. $Y \smallsetminus \mathring{U}_1$ (i.e., $X_1 \supset Y \smallsetminus \mathring{U}_1$,

$$f_1|_{Y \smallsetminus \mathring{U}_1} : Y \smallsetminus \mathring{U}_1 \to Y \smallsetminus \mathring{U}_1$$

is the identity map, and $f_1(X_2) \subset U_1$, where

$$X_2 = X_1 \smallsetminus (Y \smallsetminus \mathring{U}_1)^\circ \smallsetminus \partial Y).$$

Next we convert $f_2 = f_1|_{X_2} : X_2 \to U_1$, to a G-framed map $f_3 : X_3 \to U_1$ such that f_3 is a homotopy equivalence by G-surgeries rel. ∂U_1 of isotropy types (H)

for $H < G$. The construction of a G-framed map is discussed in Section 4. We perform the G-surgeries of isotropy types (H) with $\{e\} < H < G$ by means of the reflection method in [8]. We do the G-surgeries of isotropy type $(\{e\})$ by showing triviality of the algebraic G-surgery obstruction in the relevant Bak group described in [9] with the G-cobordism invariance property given in [10] and the induction-restriction property presented in [13, 4]. □

Theorem 3.4 (Realization Theorem). *(Let $G = A_5$.) Let W a real G-module with* $\dim W^G = m$, *N an integer possessing the property described in Theorem 2.2, and Z a compact connected smooth G-manifold of dimension ≥ 5 such that $\partial Z^G = \emptyset$, V the G-tubular neighborhood of Z^G, z_0 a point in Z^G, and n an integer $\geq N$. Suppose $|\pi_1(Z)| < \infty$ and $T_{z_0}(Z)$ is isomorphic to $W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus n}$. Further suppose V satisfies the gap condition for $(\{e\}, C_2)$, $(\{e\}, C_3)$ and $(\{e\}, C_5)$, and the cobordism gap condition for (C_2, D_6), (C_2, D_{10}), and (C_3, A_4). Let F, F_0, $x_0 \in F_0$ be as in Theorem 2.2. Then there exists a compact G-manifold X satisfying the following conditions.*

(1) *X^G is diffeomorphic to F.*
(2) *X is homotopy equivalent to Z.*
(3) *In the case that* $\dim Z \equiv 0 \mod 2$ *and $\pi_1(Z) = 0$, X is diffeomorphic to Z.*

Proof. Take the smooth G-action on the sphere S described in Corollary 2.3 with $x_0 \in F_0 \subset F$ and $S^G = F \amalg F'$ such that $T_{x_0}(S) \cong W \oplus (\mathbb{R}[G] - \mathbb{R})^{\oplus n}$ and $F \cong F'$. Let Y be the connected sum of S and Z at points x_0 and z_0. By setting $Y_0^G = F \# Z^G$ and $Y_1^G = F'$ we have $Y^G = Y_0^G \amalg Y_1^G$. Note Y_0^G is without boundary. The tubular neighborhood U of Y_0^G satisfies the gap condition for $(\{e\}, C_2)$, $(\{e\}, C_3)$ and $(\{e\}, C_5)$, and the cobordism gap condition for (C_2, D_6), (C_2, D_{10}), and (C_3, A_4). Deleting the fixed point submanifold F_0^G from Y by means of Theorem 3.3, there exists a G-manifold X satisfying the desired conditions in the theorem, where $X^G = F' \cong F$. □

4. Equivariant cohomology theory $\omega(\bullet)_G^*$ and G-framed maps

Equivariant surgeries are operated on smooth G-manifolds, but more precisely on G-framed maps. T. Petrie gave an idea to construct G-framed maps by using the equivariant cohomology theory $\omega_G^*(\bullet)$ and the Burnside ring $\Omega(G)$. In order to construct our G-framed maps, we employ a modification given in [11] of Petrie's construction.

Let $\Omega(G)$ denote the Burnside ring, i.e.

$$\Omega(G) = \{[X] \mid X \text{ is a finite } G\text{-CW complex}\},$$

where $[X] = [Y]$ if and only if $\chi(X^H) = \chi(Y^H)$ for all $H \leq G$. The next proposition is well known, see [2, 20, 1].

Proposition 4.1. *Let G be a nontrivial perfect group. Then there exists an idempotnent β in the Burnside ring $\Omega(G)$ such that $\chi_G(\beta) = 1$ and $\chi_H(\beta) = 0$ for all $H \neq G$.*

Let $\mathcal{S}(G)$ and $\mathcal{S}(G)_{\max}$ denote the set of all subgroups and all maximal proper subgroups of G, respectively. For a subgroup H of G, (H) stands for the G-conjugacy class containing H, i.e.

$$(H) = \{gHg^{-1} \mid g \in G\}.$$

The β in Proposition 4.1 has the form

$$\beta = [G/G] - \sum_{(K)\subset \mathcal{S}(G)_{\max}} [G/K] - \sum_{(H)\subset \mathcal{F}} a_H[G/H], \tag{4.1}$$

for some G-invariant lower closed $\mathcal{F} \subset \mathcal{S}(G) \setminus (\mathcal{S}(G)_{\max} \cup (G))$ and $a_H \in \mathbb{Z}$.

Proposition 4.2. *Let $G = A_5$ and let α be the element of the Burnside ring $\Omega(G)$ given by*

$$\alpha = [G/A_4] + [G/D_{10}] + [G/D_6] - [G/C_3] - 2[G/C_2] + [G/\{e\}].$$

Then for any proper subgroup $H < G$, $\mathrm{res}_H^G \alpha$ coincides with $[H/H]$ in $\Omega(H)$.

Proposition 4.3. *Let G be A_5. Then there exists a finite G-CW complex Z fulfilling the following conditions.*

(1) $Z^G = \{z_0, z_1\}$.

(2) $\displaystyle\bigcup_{H\in \mathcal{S}(G)_{\max}} Z^H = \{z_0, z_1\} * \mathcal{S}(G)_{\max}$, *where each subgroup in $\mathcal{S}(G)_{\max}$ is regarded as a point. Thus Z^H is homeomorphic to the 1-dimensional disk $[-1, 1]$ for each $H \in \mathcal{S}(G)_{\max}$.*

(3) $Z^{D_4} = Z^{A_4}$ *and* $Z^{C_5} = Z^{D_{10}}$.

(4) Z^H *is contractible for any subgroup* $H < G$.

The G-CW complex Z in this lemma is constructed using Oliver-Petrie's G-CW-surgery theory [21] and the wedge sum technique [16].

Let $M_n = \mathbb{C}[G]^{\oplus n}$ and let $M_n^\bullet$ be the one-point compactification of M_n, hence we write $M_n^\bullet = M_n \cup \{\infty\}$. For a finite G-CW complex X with base point in X^G,

$$\varpi_G^0(X) = \lim_{n\to\infty} [X \wedge M^\bullet, M^\bullet]_0^G,$$

where $[-,-]_0^G$ stands for the set of all homotopy classes of maps in the category of pointed G-spaces. For a finite G-CW complex Z, $\omega_G^0(Z)$ is defined to be $\varpi_G^0(Z^+)$, where

$$Z^+ = Z \amalg (G/G)$$

and G/G is regarded as the base point of Z^+.

For the set S of all powers β^k, $k \in \mathbb{N}$, the restriction $j^* : S^{-1}\omega_G^0(Z) \to S^{-1}\omega_G^0(Z^G)$ induced by the inclusion map $j : Z^G \to Z$ is an isomorphism. It is obvious that for any element $\gamma \in \omega_H^0(Z)$ and any proper subgroup H of G, $\mathrm{res}_H^G\beta \cdot \gamma = 0_Z$ in $\omega_H^0(Z)$.

Lemma 4.4. *Let G be a nontrivial perfect group, β the element in Proposition 4.1, and Z a finite G-CW complex with*

$$Z^G = \{z_0, z_1\}. \tag{4.2}$$

Then there exists an element $\gamma \in \omega_G^0(Z)$ such that $\gamma|_{z_0} = \beta$ and $\gamma|_{z_1} = 0_{z_1}$ in $\Omega(G)$ and $\beta\gamma = \gamma$. In addition, for any proper subgroup $H < G$, there exists a 'homotopy' $\Gamma_H \in \omega_H^0(Z\times I)$ from $\mathrm{res}_H^G\gamma$ to 0_Z, rel. $z_1 \times I$, i.e. $\Gamma_H|_{Z\times\{0\}} = \mathrm{res}_H^G\gamma$, $\Gamma_H|_{Z\times\{1\}} = 0_Z$, and $\Gamma_H|_{z_1\times I} = 0_{z_1\times I}$, such that $\mathrm{res}_H^G\beta \cdot \Gamma_H = \Gamma_H$. Moreover, for any pair of distinct proper subgroups H and K of G, there exists a 'homotopy' $\overline{\Gamma}_{H,K} \in \omega_{H\cap K}^0(Z \times I \times I)$ from $\mathrm{res}_{H\cap K}^H\Gamma_H$ to $\mathrm{res}_{H\cap K}^K\Gamma_K$, rel. $z_1 \times I \times I$ and $Z \times \partial I \times I$.

As a next step, consider the elements $1_Z - \gamma \in \omega_G^0(Z)$, $1_{Z\times I} - \Gamma_H \in \omega_H^0(Z\times I)$, and $1_{Z\times I\times I} - \overline{\Gamma}_{H,K} \in \omega_{H\cap K}^0(Z \times I \times I)$. Recall that an element $\alpha \in \omega_G^0(Z)$ is represented by a G-map

$$Z^+ \wedge M^\bullet \to M^\bullet$$

preserving the base point ∞, where $M = \mathbb{C}[G]^{\oplus n}$ for some n.

Lemma 4.5. *Let G be a nontrivial perfect group, β the element in Proposition 4.1, and Z a finite G-CW complex with $Z^G = \{z_0, z_1\}$. Then there exist maps α, A_H, and $\overline{A}_{H,K}$ satisfying the following conditions (1)–(3), where H and K range all proper subgroups of G such that $H \neq K$.*

(1) *α is a map of pointed G-spaces $Z^+ \times M^\bullet \to M^\bullet$ such that $[\alpha] = 1 - \gamma$ for the γ above, and $\alpha|_{\{z_1\}^+ \wedge M^\bullet} = id_{\{z_1\}^+ \wedge M^\bullet}$.*

(2) *A_H is a homotopy of pointed H-spaces $(Z \times I)^+ \wedge M^\bullet \to M^\bullet$ from α to 1_Z, where $1_Z : Z^+ \wedge M^\bullet \to M^\bullet$ and $1_Z(z, v) = v$ for all $z \in Z$ and $v \in M$, rel. $\{z_1\}^+ \wedge M^\bullet$.*

(3) *$\overline{A}_{H,K}$ is a homotopy of pointed $H \cap K$-spaces $((Z \times I) \times I)^+ \wedge M^\bullet \to M^\bullet$ from A_H to A_j rel. $(z_1 \times I)^+ \wedge M^\bullet$ and $(Z \times \partial)^+ \wedge M^\bullet$.*

Proposition 4.6. *Let $G = A_5$ and $\mathcal{K} = (A_4) \cup (D_{10}) \cup (D_6) \cup (D_4) \cup (C_5)$. Let Z be the finite G-CW complex in Proposition 4.3. Then there exist maps α, A_H, and $\overline{A}_{H,K}$ of Lemma 4.5 satisfying the additional conditions:*

(1) *$\alpha|_{z_0}^{-1}(0)^G = \emptyset$ and $|(\alpha|_{z_0}{}^{-1}(0))^H| = 1$ for $H \in \mathcal{K}$.*

(2) *For each $H \in \mathcal{K}$, there exists a connected component $X(H)$ of $\alpha^{-1}(0)$ containing both $(\alpha|_{z_0}{}^{-1}(0))^H$ and $(z_1, 0)$ such that α is transversal on $X(H)$ to $0 \subset M$, the normal derivative of α on $X(H)$ is the identity, and the projection $Z \times M \to Z$ diffeomorphically maps $X(H)$ to Z^H.*

(3) *For each pair of $H \in \mathcal{S}(G)_{\max}$ and $L \in \mathcal{K}$ with $L \leq H$, there exists a connected component $W(H, L)$ of $A_H{}^{-1}(0)$ containing $X(H)^L \times \{0\}$ ($\subset Z \times M \times I$) and $Z^L \times 0 \times \{1\}$ ($\subset Z \times M \times I$) such that A_H is transversal on $W(H, L)$ to $0 \subset M$, the normal derivative of A_H on $W(H, L)$ is the identity, and the projection $Z \times I \times M \to Z \times I$ diffeomorphically maps $W(H, L)$ to $Z^L \times I$.*

A *G-framed map* $\boldsymbol{f} = (f, b)$ consists of a G-map $f : X \to Y$ such that X and Y are compact smooth G-manifold and $f(\partial X) \subset \partial Y$, and an isomorphism $b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \to f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^m)$ of real G-vector bundles for some integer $m \geq 0$. In the following we suppose Y is connected and $f : (X, \partial X) \to (Y, \partial Y)$ is of degree 1.

Lemma 4.7. *Let Y be a compact smooth G-manifold with a decomposition $Y^G = Y_0^G \amalg Y_1^G$ such that $\partial Y_0^G = \emptyset$. Let U be the G-tubular neighborhood of Y_0^G. Then there exist a G-framed map $\boldsymbol{f} = (f, b)$, H-framed cobordisms $\boldsymbol{F}_H = (F_H, B_H) : \boldsymbol{f} \sim \boldsymbol{id}_Y$, rel. $Y \smallsetminus \mathring{U}$ for $H < G$, and $H \cap K$-framed cobordisms $\overline{\boldsymbol{F}}_{H,K} = (\overline{F}_{H,K}, \overline{B}_{H,K})$: $\boldsymbol{F}_H \sim \boldsymbol{F}_K$ rel. $((Y \smallsetminus \mathring{U}) \times I) \cup (Y \times \partial I)$, for H, $K < G$ such that $H \neq K$, where*

$$f : X \to Y,$$

$$b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \to f^*(Y) \oplus \varepsilon_X(\mathbb{R}^m),$$

$$F_H : W_H \to Y \times I,$$

$$B_H : T(W_H) \oplus \varepsilon_{W_H}(\mathbb{R}^m) \to F_H^* T(Y \times I) \oplus \varepsilon_{W_H}(\mathbb{R}^m),$$

$$\overline{F}_{H,K} : \overline{W}_{H,K} \to Y \times I \times I,$$

$$\overline{B}_{H,K} : T(\overline{W}_{H,K}) \oplus \varepsilon_{\overline{W}_{H,K}}(\mathbb{R}^m) \to \overline{F}^*_{H,K} T(Y \times I \times I) \oplus \varepsilon_{\overline{W}_{H,K}}(\mathbb{R}^m),$$

for some integer $m > 0$.

This lemma is obtained by the arguments in [11].

Lemma 4.8. *Let $G = A_5$ and $\mathcal{K} = (A_4) \cup (D_{10}) \cup (D_6) \cup (D_4) \cup (C_5)$. Then the framed maps $\boldsymbol{f}$, $\boldsymbol{F}_H$ and $\overline{\boldsymbol{F}}_{H,K}$ in Lemma 4.7 can be chosen so that X^L and ${W_H}^L$ are $N_H(L)$-diffeomorphic to Y^L and $Y^L \times I$, respectively, for all H, $K \in \mathcal{S}(G)_{\max}$ and all $L \in \mathcal{K}$ with $L \leq H$.*

This modification is achieved by using Proposition 4.6 and the reflection method in [8].

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