

DEGENERACY OF HOLOMORPHIC MAPS OMITTING HYPERSURFACES

Fumio Sakai

Let W be a projective algebraic manifold of dimension n and D a hypersurface on W . Let $f: \mathbb{C}^n \rightarrow W-D$ be a holomorphic map. We say that f is degenerate if the Jacobian of f vanishes identically. In this note, we shall deal with the influence of the singularity of D on degeneracy theorems of f .

1. Notations

A hypersurface D on W is said to have simple normal crossings if each irreducible component of D is non-singular and D has normal crossings, i.e., D is locally given by $w_1 \cdots w_j = 0$, where $(w_1, \dots, w_n)$ are local coordinates of W .

Let L be a line bundle on W . Let $h^i(L) = \dim H^i(W, \mathcal{O}(L))$. The L -dimension $\kappa(L, W)$ of W is roughly the polynomial order of $h^0(mL)$ as a function of positive integers m . Note that $\kappa(L, W)$ takes one of the values $-\infty, 0, 1, \dots, n$. Here we need the following fact: $\kappa(L, W) = n$ if and only if

$$\limsup_{m \rightarrow +\infty} m^{-n} h^0(mL) > 0.$$

If $c_1(L) > 0$, then $\kappa(L, W) = n$. For a divisor D , we denote by $[D]$ the associated line bundle. We write $\kappa(D, W) = \kappa([D], W)$.

2. Degeneracy theorem

Picard's theorem states that any holomorphic map $f: \mathbb{C} \rightarrow \mathbb{P}_1$

omitting three points is a constant map. We begin with the following generalization of this theorem.

Theorem 1 ([15], see also [2],[10]). Let W be a projective algebraic manifold of dimension n and D a hypersurface on W .

Suppose that

- (i) $\kappa(K_W + D, W) = n$, where K_W is the canonical bundle of W ,
- (ii) D has simple normal crossings.

Then any holomorphic map $f: \mathbb{C}^n \longrightarrow W - D$ is degenerate.

Remark. In case $W = \mathbb{P}_n$ and $D =$ a hypersurface of degree d , the hypothesis (i) is satisfied if and only if $d \geq n+2$.

The following example shows that we cannot remove the hypothesis (ii).

Example 1. Let $W = \mathbb{P}_2$ and $D = \{w_0^{d-1}w_2 - w_1^d = 0\}$, where $[w_0:w_1:w_2]$ are homogeneous coordinates of $\mathbb{P}_2$. By the above remark, if $d \geq 4$, the hypothesis (i) is satisfied. D has only one singularity at $[0:0:1]$. Define a holomorphic map $f: \mathbb{C}^2 = (z_1, z_2) \longrightarrow \mathbb{P}_2$ by $f(z_1, z_2) = [1:z_1:z_1^d + e^{z_2}]$. Then f omits D and the Jacobian of f is

$$J_f = \begin{vmatrix} 1 & 0 \\ dz_1^{d-1} & e^{z_2} \end{vmatrix} = e^{z_2} \neq 0.$$

In what follows, we shall consider the question: what happens when D has worse singularities than simple normal crossings in Theorem 1?

3. Resolution of singularities

Let W be a projective algebraic manifold of dimension n and

D a hypersurface on W . If D does not satisfy the hypothesis (ii) in Theorem 1, by desingularizing D , we can find W^* and D^* satisfying the following conditions:

- (i) $\pi: W^* \longrightarrow W$ is a composite of monoidal transformations,
- (ii) $\pi: W^* - D^* \longrightarrow W - D$ is biholomorphic,
- (iii) $D^* =$ the support of π^*D ,
- (iv) D has simple normal crossings.

From Theorem 1, it follows

Theorem 2. If $\kappa(K_{W^*+D^*}, W^*) = n$, then any holomorphic map $f: \mathbb{C}^n \longrightarrow W - D$ is degenerate.

Proof. It suffices to consider f as a holomorphic map to $W^* - D^*$, q.e.d.

$$\begin{array}{ccc} & & W^* - D^* \\ & \nearrow & \downarrow \pi \\ f: \mathbb{C}^n & \longrightarrow & W - D \end{array}$$

To calculate $\kappa(K_{W^*+D^*}, W^*)$, we study the process of the desingularization, in which we have a sequence of monoidal transformations $\pi_i: W_{i+1} \longrightarrow W_i$ with non-singular centers C_i such that

- (i) $W_0 = W, W_\ell = W^*,$
 - (ii) $D_i =$ the support of $\pi_{i-1}^* D_{i-1},$
 - (iii) $D_\ell = D^*$, which has simple normal crossings.
- $$\begin{array}{ccc} W^* = W_\ell & \supset & D_\ell = D^* \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ W_1 & \supset & D_1 \\ \pi_0 \downarrow & & \downarrow \pi_0 \\ W = W_0 & \supset & D_0 = D \end{array}$$

Define

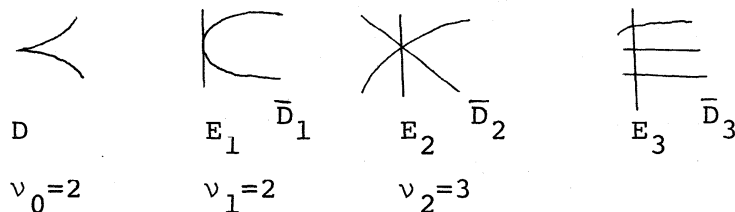
- $\bar{D}_i =$ the strict transform of D_{i-1} by $\pi_{i-1},$
- $E_i =$ the exceptional locus of π_{i-1} , i.e., $\pi_{i-1}^{-1}(C_{i-1}),$
- $\delta_i =$ the codimension of C_i in $W_i,$

v_i = the multiplicity of the singular locus of D_i at C_i .

Then we have

$$(1) \quad \begin{aligned} D_i &= \bar{D}_i + E_i, \quad \pi_{i-1}^* D_{i-1} = \bar{D}_i + v_{i-1} E_i, \quad K_{W_i} = \pi_{i-1}^* K_{W_{i-1}} + [(\delta_{i-1} - 1) E_i], \\ K_{W_i} + [D_i] &= \pi_{i-1}^* (K_{W_{i-1}} + [D_{i-1}]) + [(\delta_{i-1} - v_{i-1}) E_i]. \end{aligned}$$

Example 2. We examine the cusp $D = \{y^2 = x^3\}$.



Proposition 1. $\kappa(K_{W^*} + D^*, W^*) \leq \kappa(K_W + D, W)$.

This follows from properties (1), and further if D has simple normal crossings, the equality holds. Moreover we have the following

Proposition 2. Let $f: V' \rightarrow V$ be a birational morphism, where V', V are projective algebraic manifolds. Let D be a hypersurface on V and put $D' =$ the support of f^*D . Then

$$\kappa(K_{V'} + D', V') \leq \kappa(K_V + D, V).$$

Remark. $\kappa(K_{W^*} + D^*, W^*)$ is independent of the desingularization W^*, D^* . In fact let W_1, D_1 be another desingularization of D . There exists a desingularization W^{**}, D^{**} , which is obtained by a sequence of monoidal transformations of W^* , with centers over D^* , such that there is a birational morphism $\phi: W^{**} \rightarrow W_1$. The assertion follows from the above propositions.

Definition. We say that D has quasi-negligible singularities if $v_i \leq \delta_i$ holds for $i=0, \dots, l-1$.

Proposition 3 ([15]). If D has quasi-negligible singularities, then $\kappa(K_{W^*} + D^*, W^*) = \kappa(K_W + D, W)$.

Proof. By (1), we have

$$\kappa(K_{W_i} + D_i, W_i) \geq \kappa(\pi_{i-1}^*(K_{W_{i-1}} + D_{i-1}), W_i) = \kappa(K_{W_{i-1}} + D_{i-1}, W_{i-1}).$$

Hence we have $\kappa(K_{W^*} + D^*, W^*) \geq \kappa(K_W + D, W)$, which proves Proposition 3.

Thus the hypothesis (ii) in Theorem 1 can be weakened as:

(ii)* D has quasi-negligible singularities.

Examples of quasi-negligible singularities

- (i) normal crossing is quasi-negligible,
- (ii) a curve has quasi-negligible singularities if and only if its singularities are ordinary double points,
- (iii) the isolated singularity $w_1^d + \dots + w_n^d = 0$ is quasi-negligible if $d \leq n$ (this type appeared in Carlson [1]),
- (iv) on surfaces the singularity defined by $w_1^2 + w_2^2 + w_3^{k+1} = 0$ (type A_k) is quasi-negligible.

Proposition 4 ([15]). If the Kodaira dimension $\kappa(W) = \kappa(K_W, W) \geq 0$, then we have $\kappa(K_{W^*} + D^*, W^*) = \kappa(K_W + D, W)$.

Therefore in case $\kappa(W) \geq 0$, the hypothesis (ii) can be removed. This leads us to study the case $\kappa(W) < 0$. Note that $\kappa(W) < 0$ if and only if $h^0(mK_W) = 0$, for every positive integer m .

In case $n=2$, a surface S with $\kappa(S) < 0$ is birationally equivalent to $\mathbb{P}_1 \times C$, where C is a curve (ruled surface). We have

Proposition 5. Let S be an algebraic surface and D a curve on S . If $(K_S + D)^2 - (v_0 - 2)^2 - \dots - (v_{\ell-1} - 2)^2 > 0$, then $\kappa(K_S + D, S) = 2$ implies $\kappa(K_{S^*} + D^*, S^*) = 2$.

Proof. By (1), putting $\delta_i = 2$, we get

$$\begin{aligned} (K_{S_i} + D_i)^2 &= (\pi_{i-1}^* (K_{S_{i-1}} + D_{i-1}) + (2 - v_{i-1}) E_i)^2 \\ &= (K_{S_{i-1}} + D_{i-1})^2 - (v_{i-1} - 2)^2. \end{aligned}$$

Hence we obtain

$$(K_{S^*} + D^*)^2 = (K_S + D)^2 - (v_0 - 2)^2 - \dots - (v_{\ell-1} - 2)^2 > 0.$$

Let $\Gamma = K_S + [D]$, $\Gamma^* = K_{S^*} + [D^*]$. Using this notations we have $\Gamma^*{}^2 > 0$.

We infer from the Riemann-Roch theorem that

$$h^0(m\Gamma^*) + h^2(m\Gamma^*) \geq \frac{1}{2} m\Gamma^* (m\Gamma^* - K_{S^*}) + p_a(S^*).$$

Note that $h^2(m\Gamma^*) = h^0(K_{S^*} - m\Gamma^*) \leq h^0(-(m-1)\Gamma^*)$. Thus the above inequality shows that either $h^0(m\Gamma^*) > 0$, or $h^0(-(m-1)\Gamma^*) > 0$, for large m . By (1), $\Gamma^* = \pi^*(\Gamma) - [\mathcal{E}]$, $\mathcal{E}$ is an exceptional divisor of π . If $h^0(-(m-1)\Gamma^*) > 0$, then $h^0(-(m-1)(\Gamma^* + [\mathcal{E}])) = h^0(-(m-1)\Gamma) > 0$, which is a contradiction. Hence $h^0(m\Gamma^*) \geq \frac{1}{2} (\Gamma^*)^2 m^2 + \dots$, which proves $\kappa(\Gamma^*, S^*) = 2$, q.e.d.

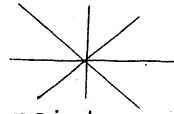
Example 3. Let $S = \mathbb{P}_2$ and $D =$ a hypersurface of degree d . The hypothesis in the above proposition is

$$(d-3)^2 - (v_0 - 2)^2 - \dots - (v_{\ell-1} - 2)^2 > 0.$$

Let $\kappa^* = \kappa(\Gamma^*, S^*)$. We give some examples:

(i) four lines meeting at one point,

$$\Gamma^2 = -1, \quad \kappa^* = -\infty,$$



(ii) a conic and two lines meeting at one point,

$$\Gamma^2 = 0, \quad \kappa^* = 1,$$



(iii) a quintic with two cusps,

$$\Gamma^2 = 2, \quad \kappa^* = 2,$$



(iv) the curve in Example 1,

$$\Gamma^2 < 0, \quad \kappa^* = -\infty.$$



Remark. Let V', V be projective algebraic manifold of dimension n and let $f: V' \rightarrow V$ be a finite morphism. For a hypersurface D on V , let $D' =$ the support of f^*D . Then it is easily seen that $\kappa(K_{V'} + D', V') \geq \kappa(K_V + D, V)$. Let V'^*, D'^* and V^*, D^* be desingularizations of V', D' and V, D , respectively. Is it true that

$$\kappa(K_{V'^*} + D'^*, V'^*) \geq \kappa(K_{V^*} + D^*, V^*) ?$$

4. Concluding Remarks

Let M be a complex manifold of dimension n . We define the following properties of M .

(ED)_k Every holomorphic map $f: \mathbb{C}^k \times D^{n-k} \rightarrow M$ is degenerate, where D is the unit disk $\{z \mid |z| < 1\}$,

(HD)_k Every holomorphic map $f: \mathbb{C}^k \rightarrow M$ is degenerate in the sense that the rank of the Jacobian matrix of f is not maximal anywhere.

$(AD)_k$ Every holomorphic map $f: \mathbb{C}^k \longrightarrow M$ is algebraically degenerate, i.e., the image $f(\mathbb{C}^k)$ is contained in a proper subvariety of M .

Obviously we have the following relations:

$$\begin{array}{ccc} (HD)_k & \longrightarrow & (ED)_k \\ \downarrow & & \downarrow \\ (HD)_{k+1} & \longrightarrow & (ED)_{k+1} \end{array} \quad (HD)_n = (ED)_n$$

The proof of Theorem 1 ([15]) implies the following stronger form of Theorem 1. (The conclusion of Theorem 1 is $(ED)_n$.)

Theorem 4. Under the same assumptions on W and D as in Theorem 1, $M = W - D$ satisfies property $(ED)_1$.

Proof.^{*)} Let $D_r = \{z \mid |z| < r\}$. Replacing $B[r]$ in [15] by D_r^n , we have the following Schottky-Landau theorem.

Theorem ([15]). Assume the same assumptions on W and D as in Theorem 1. Let $f: D_r^n \longrightarrow W - D$ be a holomorphic map with $J_f(0) \neq 0$. Then $r^{2n} < C |J_f(0)|^{-2}$, where C is a constant depending only on $f(0)$.

We proceed to the proof of Theorem 4. Assume that there exists a non-degenerate holomorphic map $f: \mathbb{C} \times D^{n-1} \longrightarrow W - D$. By a translation of coordinates, we may assume that there exists a holomorphic map $f: \mathbb{C} \times D_{r_0}^{n-1} \longrightarrow W - D$, with $r_0 < 1$, $J_f(0) \neq 0$. Define a holomorphic map $\psi: D_r^n \longrightarrow \mathbb{C} \times D_{r_0}^{n-1}$ by

*) This type of argument is due to I. Nakamura.

$$\psi : (z_1, \dots, z_n) \longrightarrow \left(\left(\frac{r}{r} \right)^{n-1} \frac{z_1}{a}, \frac{r}{r} z_2, \dots, \frac{r}{r} z_n \right),$$

where $a = |J_f(0)|$. Let $g = f \circ \psi$. Since $|J_g(0)| = |J_f(\psi(0))| |J_\psi(0)| = 1$, we obtain a holomorphic map $g: \mathbb{D}_r^n \longrightarrow W-D$, with $|J_g(0)| = 1$ for arbitrary r , which contradicts the above theorem, q.e.d.

Several degeneracy theorems are known.

Theorem (Fujimoto [4], Green [6]). Let $H_1, \dots, H_{n+k}$ be hyperplanes in general position in $\mathbb{P}_n$. Let $M = \mathbb{P}_n - H_1 \cup \dots \cup H_{n+k}$. Then any holomorphic map $\mathbb{C}^i \longrightarrow M$ is contained in a linear subspace of dimension $\lfloor \frac{n}{k} \rfloor$.

Corollary. If $k \geq n+1$, then M satisfies property $(HD)_1$.

Corollary (Green [6]). Let $H_1, \dots, H_d$ be hyperplanes in $\mathbb{P}_n$ in arbitrary position. Then $M = \mathbb{P}_n - H_1 \cup \dots \cup H_d$ satisfies properties $(AD)_1$ and $(ED)_n$ if $d \geq n+2$.

Corollary (Green [6], p.39). Let W be a complex manifold of dimension n and let $D_1, \dots, D_k$ be hypersurfaces on W such that each $D_i \in |L|$, for a fixed line bundle L . Let s_i be the section defining D_i . If the algebraic dimension a of $(s_1, \dots, s_k) \leq k-2$, then $M = W - D_1 \cup \dots \cup D_k$ satisfies properties $(AD)_1$ and $(ED)_{n-a+1}$.

Theorem (Green [6], cf. Fujimoto [5]). Let D be a Fermat variety $w_0^d + \dots + w_n^d = 0$, in $\mathbb{P}_n$, where $[w_0 : \dots : w_n]$ are homogeneous coordinates of $\mathbb{P}_n$. If $d > n(n+1)$, then $\mathbb{P}_n - D$ satisfies properties $(AD)_1$ and $(ED)_1$ ($(ED)_1$ is a consequence of Theorem 4).

Theorem(Green[8]). Let D be a non-singular curve of degree d in $\mathbb{P}_2$. Let D^* be the dual curve of D in the dual projective space $\mathbb{P}_2^*$. If $d \geq 3$, then $\mathbb{P}_2^* - D^*$ satisfies property $(HD)_1$.

Theorem([15]). Let A be an abelian variety of dimension n and D an arbitrary hypersurface on A . Then $A - D$ satisfies property $(ED)_n$.

Example 4. Let $X_a = \{z_1^{a_1} + \dots + z_n^{a_n} = 0\}$ in $\mathbb{C}^n$. If $\sum_{i=1}^n \frac{1}{a_i} < 1$, then $\mathbb{C}^n - X_a$ satisfies property $(ED)_2$.

Proof. Let $U_a = \{z_1^{a_1} + \dots + z_n^{a_n} = 1\}$. Then $\mathbb{C} \times U_a$ is an unramified covering of $\mathbb{C}^n - X_a$. By the assumption, U_a satisfies property $(ED)_1$ (see [15]). So $\mathbb{C} \times U_a$ and $\mathbb{C}^n - X_a$ satisfy property $(ED)_2$.

References

- [1] Carlson, J.: Some degeneracy theorems for entire functions in an algebraic variety. Trans. Math. Soc. 168, 278-301 (1972)
- [2] _____, Griffiths, P.: A defect relation for equidimensional holomorphic mappings between algebraic varieties. Ann. of Math. 95, 557-584 (1972)
- [3] Dahlberg, B.: Holomorphic mappings into smooth projective varieties. Arkiv för matematik. 12, 59-71 (1974)
- [4] Fujimoto, H.: Extensions of the big Picard's theorem. Tohoku Math. J. 24, 415-422 (1972)
- [5] _____: On meromorphic maps into the complex projective space. J. Math. Soc. Japan 26, 272-288 (1974)
- [6] Green, M. L.: Some Picard theorems for holomorphic maps to

- algebraic varieties. Thesis, Princeton University 1972
- [7] _____: Holomorphic maps into complex projective space omitting hyperplanes. Trans.Amer.Math.Soc. 169,89-103(1972)
- [8] _____: The complement of the dual of a plane curve and some new hyperbolic manifolds. In:Value-distribution theory. Part A,pp.119-131. New York, Marcel Dekker 1974
- [9] _____: On the functional equation $f^2 = e^{2\phi_1} + e^{2\phi_2} + e^{2\phi_3}$ and a new Picard theorem. (to appear in Trans.Amer.Math.Soc.)
- [10] Griffiths,P.,King,J.: Nevanlinna theory and holomorphic maps to algebraic varieties. Acta Math. 130, 145-220 (1972)
- [11] Iitaka,S.: On D-dimensions of algebraic varieties. J.Math. Soc.Japan 23,356-373 (1971)
- [12] Kiernan,P.: Hyperbolically imbedded spaces and the big Picard theorem. Math.Annalen 204, 203-209 (1973)
- [13] _____,Kobayashi,S.: Holomorphic mappings into projective space with lacunary hyperplanes. Nagoya Math.J.50,199-216 (1973)
- [14] Kodaira,K.: Nevanlinna theory(in Japanese). Seminary Notes 34, University of Tokyo 1974
- [15] Sakai,F.: Degeneracy of holomorphic maps with ramification. (to appear in Inventiones math.)
- [16] _____: Defect relations and ramification. (to appear in Proc. Japan Acad.)
- [17] Shiffman,B.: Applications of geometric measure theory for meromorphic maps. In :Value-distribution theory. Part A, pp. 63-95. New York, Marcel Dekker 1974