

Borel summability of a formal solution for Cauchy problem of some linear partial differential equations

By

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Abstract

In this paper we study the Borel summability of a certain divergent formal power series solution for some linear partial differential equations. We show the Borel summability of the formal solution under the condition that an initial value function $\phi(x)$ is an entire function of exponential order at most 2.

§ 1. Introduction

Let $(t, x) \in \mathbb{C}^2$. In this paper we consider the following Cauchy problem:

$$(E) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = \left(\frac{\partial}{\partial x}\right)^2 u(t, x) + \sum_{2 \leq j+\alpha \leq m, j>0} a_{j,\alpha} t^{j+\alpha-2} \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t, x) \\ u(0, x) = \phi(x) \end{cases}$$

where $a_{j,\alpha} \in \mathbb{C}$ and $a_{m,0} \neq 0$ and the initial value function $\phi(x)$ is a holomorphic function in a neighbourhood of the origin.

Let us introduce the following notations. Let $D_R = \{x \in \mathbb{C}; |x| < R\}$ and $S_{d,\theta} = \{\xi \in \mathbb{C} \setminus \{0\}; |d - \arg \xi| < \theta\}$. $\mathcal{O}(D_R)$ (resp. $\mathcal{O}(S_{d,\theta} \times D_R)$) is the set of all holomorphic functions on D_R (resp. $S_{d,\theta} \times D_R$). $\mathcal{O}(D_R)[[t]] := \{\sum_{i=0}^{\infty} u_i(x) t^i; u_i(x) \in \mathcal{O}(D_R)\}$.

Let us introduce some known results. We consider the following example:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = a \left(\frac{\partial}{\partial x}\right)^2 u(t, x) + bt \left(t \frac{\partial}{\partial t}\right)^3 u(t, x) \\ u(0, x) = \phi(x) \end{cases}$$

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where a and b are any complex numbers.

Let us recall some known results. If $a = 1$ and $b = 0$, then the equation (1.1) is the heat equation. Then we have the following two results i) and ii).

i) Assume that the initial value function $\phi(x)$ is an entire function and satisfies with some positive constants C and K ,

$$|\phi(x)| \leq Ce^{K|x|^2} \quad \text{for } x \in \mathbb{C}.$$

Then the formal power series solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ of (1.1) is holomorphic in a neighborhood of $t = 0$. This is a classical result (see [2]).

ii) The following result is that of Lutz-Miyake-Schäfke in [3]. The following two statements a) and b) are equivalent: Let $d \in \mathbb{R}$ be fixed.

a). The initial value function $\phi(x)$ is analytic on $\Omega_0 = S_{d/2, \theta} \cup S_{d/2+\pi, \theta} \cup D_R$ and satisfies with some positive constants C and K ,

$$|\phi(x)| \leq Ce^{K|x|^2} \quad \text{on } \Omega_0.$$

b). The formal power series solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ of (1.1) is Borel summable in a direction d .

If $a = 0$, then Ōuchi treated this type some linear/nonlinear partial differential equations in [5] and [6] ('02, '06).

Set $\Xi_0 = \{re^{i\theta}; r > 0, \theta \equiv -(1/2)\arg(b) \bmod(\pi)\}$ for the equation (1.1) with $a = 0$. By the result of Ōuchi [5] we have: Assume that the initial value function $\phi(x)$ is in $\mathcal{O}(D_R)$. Then the formal solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ is Borel summable in a direction d with $\overline{S_{d, \theta}} \cap \Xi_0 = \emptyset$.

Author considered the case $ab \neq 0$ as the mixed type equations of [3] and [5], and had a result of the Borel summability of the formal solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ for (1.1) in [8]: Under the condition that the initial value function $\phi(x)$ is an entire function with $|\phi(x)| \leq Ce^{K|x|^2}$ on $\mathbb{C}$, the formal solution $\hat{u}(x) \in \mathcal{O}(D_R)[[t]]$ ($\forall R > 0$) is Borel summable in a direction d with $\overline{S_{d, \theta}} \cap \Xi_0 = \emptyset$.

This result is not obvious. In the case of $b = 0$ the formal solution $\hat{u}(t, x)$ of (1.1) converges in a neighborhood of the origin $t = 0$. But in the case $a = b = 1$ and $\phi(x) = e^{x^2}$, the formal solution $\hat{u}(t, x) = \sum_{i=0}^{\infty} u_i(x)t^i$ of (1.1) satisfies for $x \in \mathbb{R}$

$$(1.2) \quad \begin{aligned} u_i(x) &\geq 2^{i-3} \frac{(\frac{i-2}{2})!^3}{(\frac{i}{2})!} \left(\frac{\partial}{\partial x}\right)^4 \phi(x) \geq AB^i i! \left(\frac{\partial}{\partial x}\right)^4 \phi(x) \quad i \geq 2, i : \text{even} \\ u_i(x) &\geq \frac{1}{2^{i-4}} \frac{(\frac{i-1}{2})!}{i!} \frac{(i-2)!^3}{(\frac{i-3}{2})!^3} \left(\frac{\partial}{\partial x}\right)^2 \phi(x) \geq AB^i i! \left(\frac{\partial}{\partial x}\right)^2 \phi(x) \quad i \geq 3, i : \text{odd}. \end{aligned}$$

By the estimate (1.2), the formal solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ ($\forall R > 0$) is a certain

divergent power series. So it is our purpose to study the summability of the formal solution $\hat{u}(t, x)$ of (1.1).

Let us show the estimate (1.2). For the Cauchy problem (1.1) we have a formal solution $\hat{u}(t, x) = \sum_{i=0}^{\infty} u_i(x)t^i$ with

$$\begin{cases} u_0(x) = \phi(x) \\ iu_i(x) = \left(\frac{\partial}{\partial x}\right)^2 u_{i-1}(x) + (i-2)^3 u_{i-2}(x), \end{cases}$$

where $u_{-i}(x) \equiv 0$ for $i \geq 1$. By the above relations the coefficients $u_i(x)$ have the following form.

$$u_i(x) = \sum_{j=0}^i C_{0,i,j} \left(\frac{d}{dx}\right)^{2j} \phi(x) = \sum_{j=0}^i C_{0,i,j} \sum_{k=0}^j C_{1,j,k} x^{2k} e^{x^2} \quad \text{with } C_{0,i,j} \geq 0, C_{1,j,k} \geq 0.$$

Then we have $u_i(x) \geq 0$ for $x \in \mathbb{R}$ and $i \geq 0$. In the case $i = 2n \geq 2$ we have

$$\begin{aligned} (1.3) \quad u_i(x) &\geq \frac{(i-2)^3}{i} u_{i-2} \\ &\geq \frac{(i-2)^3}{i} \frac{(i-4)^3}{(i-2)} \cdots \frac{2^3}{4} u_2(x) = \frac{(2^{(i-2)/2})^3}{2^{(i-2)/2}} \frac{((i-2)/2)!^3}{(i/2)!} u_2(x), \end{aligned}$$

and $2u_2(x) = (d/dx)^4 u_0(x) = (d/dx)^4 \phi(x)$. Hence we get the estimate (1.2) by the estimate (1.3). In the case $i = 2n + 1$ we can show the estimate (1.2) by a similar way to the case $i = 2n$.

In this paper we will study the Borel summability of formal solutions for the Cauchy problem (E) as a general case of the Cauchy problem (1.1).

§ 2. Definition and Main result

In this section we give a definition of the Borel summability and the main theorem. Let us give a definition of the Borel summability.

Definition 2.1. Let $\hat{u}(t, x) = \sum_{i=0}^{\infty} v_i(x)t^i \in \mathcal{O}(D_R)[[t]]$. Then the formal Borel transform $(\hat{\mathcal{B}}\hat{u})(\xi, x)$ is defined by

$$(\hat{\mathcal{B}}\hat{u})(\xi, x) = v_0(x)\delta(\xi) + \sum_{i=1}^{\infty} \frac{v_i(x)}{\Gamma(i)} \xi^{i-1}$$

where $\delta(\xi)$ means the delta function with support at $\xi = 0$. In the following the notation $\xi^{-1}/\Gamma(0)$ means $\delta(\xi)$.

Definition 2.2. The formal series $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ is Borel summable in a direction d if one can find some $0 < r < R$ so that the following two properties hold:

1. The series $V(\xi, x) = (\hat{\mathcal{B}}\hat{u})(\xi, x) - v_0(x)\delta(\xi)$ converges for $|\xi| < \rho$ and $x \in \overline{D_r}$.
2. There exists a $\theta > 0$ such that for any $x \in \overline{D_r}$ the function $V(\xi, x)$ can be holomorphically continued with respect to ξ into the sector $S_{d,\theta}$. Moreover for any $0 < \theta_1 < \theta$ there exist constants $C, K > 0$ such that

$$\sup_{|x| \leq r} |V(\xi, x)| \leq Ce^{K|\xi|} \quad \text{for } \xi \in S_{d,\theta_1}.$$

Then $v_0(x) + (\mathcal{L}_d V)(t, x)$ is called the Borel summation in a direction d of $\hat{u}(t, x)$, where $\mathcal{L}_d$ is the Laplace transform that is defined by

$$(\mathcal{L}_d \phi)(t, x) := \int_0^{\infty e^{id}} \exp\left(-\left(\frac{\xi}{t}\right)\right) \phi(\xi, x) d\xi.$$

Let us introduce the main result. For the equation (E) set

$$(2.1) \quad A_0(\xi) = 1 - \sum_{j=2}^m a_{j,0} \xi^{j-1}.$$

Definition 2.3. Set $Z = \{\xi; A_0(\xi) = 0\}$. A singular direction is an argument of an element of Z . We denote by Ξ the totality of singular directions.

Remark. For the example (1.1), set $A_0(\xi) = 1 - b\xi^2$. Then $\Xi_0 = \{re^{i\theta}; r > 0, \theta \equiv -(1/2) \arg(b) \bmod(\pi)\}$ by $A_0(\xi) = 0$.

Theorem 2.4 (Main theorem). Assume that the initial value function $\phi(x)$ is an entire function and satisfies

$$|\phi(x)| \leq Ce^{K|x|^2} \quad \text{on } \mathbb{C}.$$

Then the Cauchy problem (E) has a formal power series solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$ ($\forall R > 0$) and the formal solution is Borel summable in a direction d with $\overline{S_{d,\theta}} \cap \Xi = \emptyset$ for a sufficiently small $\theta > 0$.

§ 3. Formal solution

In this section we will get a formal power series solution of the Cauchy problem (E) and give an estimate to all coefficients of the formal power series solution. Moreover we will introduce the Newton polygon as a tool to give the estimate.

If the initial value function $\phi(x)$ is in $\mathcal{O}(D_R)$, then for (E) we have a formal solution $\hat{u}(t, x) = \sum_{i=0}^{\infty} u_i(x)t^i$ with

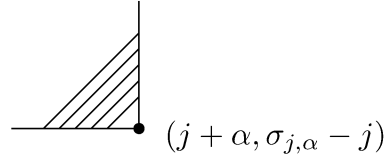
$$(3.1) \quad \begin{cases} u_0(x) = \phi(x) \\ iu_i(x) = \left(\frac{\partial}{\partial x}\right)^2 u_{i-1}(x) + \sum_{2 \leq j+\alpha \leq m, j>0} a_{j,\alpha} (i - (j + \alpha - 1))^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{i-(j+\alpha-1)}(x) \end{cases}$$

where $u_{-i}(x) \equiv 0$ for $i \geq 1$. By the relation (3.1) we have the formal solution $\hat{u}(t, x) \in \mathcal{O}(D_R)[[t]]$.

§ 3.1. Newton polygon

We want to give an estimate for the coefficients $u_i(x)$. Then let us define the Newton polygon for the equation (E). For an operator $t^{\sigma_{j,\alpha}} \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha$, we set

$$\Sigma(j, \alpha) := \{(a, b) \in \mathbb{R}^2; a \leq j + \alpha, b \geq \sigma_{j,\alpha} - j\}.$$



Set an operator

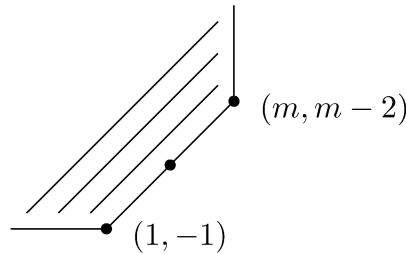
$$L = \sum_{(j,\alpha)}^{finite} a_{j,\alpha} t^{\sigma_{j,\alpha}} \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha.$$

Then for the operator L let us define the Newton polygon $NP(L)$ by

$$NP(L) := CH \bigcup_{(j,\alpha)}^{finite} \{\Sigma(j, \alpha); a_{j,\alpha} \neq 0\},$$

where $CH\{\cdot\}$ denotes the convex hull of a set $\{\cdot\}$.

Then the Newton polygon $NP(E)$ for the equation (E) is the following figure:



The Newton polygon $NP(E)$ has the side with a slope 1.

§ 3.2. Formal Gevrey estimate

We give an estimate for the coefficients $u_i(x)$. The estimate is characterized by the Newton polygon and is called the formal Gevrey estimate. For some linear or non linear partial differential equations, the formal Gevrey estimate of a formal solution is given by many mathematicians (M. Miyake, S. Ōuchi, etc). We omit a proof of the following lemma. We refer the details to Ōuchi [4].

Lemma 3.1. Assume that for the Cauchy problem (E), the initial value function $\phi(x)$ is in $\mathcal{O}(D_R)$. Then for the formal solution $\hat{u}(t, x) = \sum_{i=0}^{\infty} u_i(x)t^i$, there exist positive constants A and B such that

$$(3.2) \quad |u_i(x)| \leq AB^i \Gamma(i+1) \quad \text{on } D_r \quad \text{for } 0 < r < R.$$

§ 4. Preparatory lemmas

In order to prove Theorem 2.4, we give a definition of the convolution and two lemmas (Lemma 1.4 and 3.1) in [5]. For two lemmas we omit the proof.

Definition 4.1. Let $\phi_i(\xi, x) \in \mathcal{O}(S_{d,\theta} \times D_R)$ ($i = 1, 2$) satisfying $|\phi_i(\xi, x)| \leq C|\xi|^{\epsilon-1}$ for $\epsilon > 0$. Then the convolution of $\phi_1(\xi, x)$ and $\phi_2(\xi, x)$ is defined by

$$(\phi_1 * \phi_2)(\xi, x) = \int_0^\xi \phi_1(\xi - \eta, x) \phi_2(\eta, x) d\eta.$$

Let us introduce two lemmas in [5].

Lemma 4.2 (Lemma 1.4 in [5]). Assume that functions $\phi_i(\xi, x) \in \mathcal{O}(S_{d,\theta} \times D_R)$ satisfy

$$|\phi_i(\xi, x)| \leq c_i \frac{|\xi|^{s_i-1}}{\Gamma(s_i)} \quad \text{on } S_{d,\theta} \times D_R$$

with $s_i > 0$ for $i = 1, 2$. Then the convolution $\phi_1 * \phi_2$ satisfies

$$|(\phi_1 * \phi_2)(\xi, x)| \leq c_1 c_2 \frac{|\xi|^{s_1+s_2-1}}{\Gamma(s_1+s_2)} \quad \text{on } S_{d,\theta} \times D_R.$$

Lemma 4.3 (Lemma 3.1 in [5]). For a series $\hat{v}(t, x) = \sum_{n=1}^{\infty} v_n(x)t^n$ set $(\hat{\mathcal{B}}\hat{v})(\xi, x) = V(\xi, x)$. For $1 \leq k \leq \delta$ we have

$$\hat{\mathcal{B}}(t^\delta (t \frac{\partial}{\partial t})^k \hat{v})(\xi, x) = \sum_{s=1}^k C_{k,s} \frac{\xi^{\delta-(s+1)}}{\Gamma(\delta-s)} * (\xi^s V(\xi, x))$$

where numbers $C_{k,s}$ ($1 \leq s \leq k$) are constant numbers in [5] and satisfy

$$(4.1) \quad C_{1,1} = 1, \quad C_{k,s} = -sC_{k-1,s} + C_{k-1,s-1} \quad \text{for } k \geq 2,$$

with $C_{k,0} = C_{k-1,k} = 0$. By the relation (4.1) we have $C_{k,k} = 1$ for $k \geq 1$.

§ 5. Proof of Main theorem

In this section we will give a proof of Theorem 2.4. Firstly we shall construct a convolution equation from the Cauchy problem (E). Secondly we shall show that the

convolution equation has a unique solution $V_{loc}(\xi, x)$ which is holomorphic in a neighborhood of $(\xi, x) = (0, 0)$ (Lemma 5.2). Moreover we shall show that the convolution equation has an analytic solution $V_S(\xi, x)$ on $S_{d,\theta} \times D_r$ ($0 < r < R$) with the exponential growth estimate of order at most 1 there (Corollary 5.5). Finally we shall give that $V_{loc}(\xi, x) = V_S(\xi, x)$ on $S_{d,\theta} \cap \{\xi; |\xi| < \rho\} \times D_r$ (Proposition 5.6).

Let us construct a convolution equation from the Cauchy problem (E). We substitute $u(t, x) = \phi(x) + v(t, x)$ into the equation (E), where the function $v(t, x)$ is a new unknown function. Then we get the following equation:

$$(5.1) \quad \frac{\partial}{\partial t} v(t, x) = \varphi(x) + \left(\frac{\partial}{\partial x}\right)^2 v(t, x) + \sum_{2 \leq j + \alpha \leq m, j > 0} a_{j,\alpha} t^{j+\alpha-2} \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha v(t, x)$$

where $\varphi(x) = \left(\frac{\partial}{\partial x}\right)^2 \phi(x)$.

Set $\hat{v}(t, x) = \hat{u}(t, x) - \phi(x)$ where $\hat{u}(t, x)$ is the formal solution of the Cauchy problem (E). Then the formal series $\hat{v}(t, x)$ is a formal solution of the equation (5.1).

Set $V(\xi, x) = (\hat{\mathcal{B}}\hat{v})(\xi, x)$. By Lemma 3.1 we have the following result.

Lemma 5.1. *The series $V(\xi, x)$ converges on $\{\xi \in \mathbb{C} : |\xi| < \rho\} \times D_R$ ($\forall R > 0$) for a sufficiently small $\rho > 0$.*

Let us find a formal convolution equation that $V(\xi, x)$ satisfies. We multiply each term of the equation (5.1) by t^2 , and apply the formal Borel transform. Then we get the following convolution equation by Lemma 4.3:

$$(5.2) \quad \begin{aligned} \xi V(\xi, x) &= \xi \varphi(x) + \xi * \left(\frac{\partial}{\partial x}\right)^2 V(\xi, x) \\ &+ \sum_{2 \leq j + \alpha \leq m, j > 0} a_{j,\alpha} \sum_{s=1}^j C_{j,s} \frac{\xi^{j+\alpha-(s+1)}}{\Gamma(j+\alpha-s)} * (\xi^s \left(\frac{\partial}{\partial x}\right)^\alpha V(\xi, x)). \end{aligned}$$

Lemma 5.2. *The function $V(\xi, x)$ in Lemma 5.1 is a solution of the convolution equation (5.2) on $\{\xi \in \mathbb{C} : |\xi| < \rho\} \times D_R$ for a sufficiently small $\rho > 0$.*

Let $V_{loc}(\xi, x)$ be the solution in Lemma 5.2. We will show that the solution $V_{loc}(\xi, x)$ is holomorphically extensible to $S_{d,\theta} \times D_r$ for $0 < r < R$. In order to show that, we construct another solution $V_S(t, x)$ of the convolution equation (5.2) on $S_{d,\theta} \times D_r$. Moreover we show $V_{loc}(\xi, x) = V_S(\xi, x)$ on $S_{d,\theta}(\rho) \times D_r$ where $S_{d,\theta}(\rho) = S_{d,\theta} \cap \{\xi \in \mathbb{C} : |\xi| < \rho \ll 1\}$.

Set $A(\xi) := \xi - \sum_{j=2}^m a_{j,0} \xi^j$. Then we have $A(\xi) = \xi A_0(\xi)$ where $A_0(\xi)$ is defined in (2.1). By

$$C_{j,j} \frac{\xi^{-1}}{\Gamma(0)} * (\xi^j V(\xi, x)) = \delta(\xi) * (\xi^j V(\xi, x)) = \xi^j V(\xi, x)$$

we can rewrite the equation (5.2) as follows;

$$(CE) \quad A(\xi)V(\xi, x) = \xi\varphi(x) + \xi * \left(\frac{\partial}{\partial x}\right)^2 V(\xi, x) + \sum_{2 \leq j \leq m} B_j(V) + \sum_{2 \leq j+\alpha \leq m, j, \alpha > 0} B_{j,\alpha}(V)$$

where

$$B_j(V) := a_{j,0} \sum_{s=1}^{j-1} C_{j,s} \frac{\xi^{j-(s+1)}}{\Gamma(j-s)} * (\xi^s V(\xi, x)) \quad \text{for } j = 2, \dots, m, \text{ and}$$

$$B_{j,\alpha}(V) := a_{j,\alpha} \sum_{s=1}^j C_{j,s} \frac{\xi^{j+\alpha-(s+1)}}{\Gamma(j+\alpha-s)} * (\xi^s \left(\frac{\partial}{\partial x}\right)^\alpha V(\xi, x)) \quad \text{for } 2 \leq j+\alpha \leq m \text{ and } j, \alpha > 0.$$

Let us construct a solution of the equation (CE) on $S_{d,\theta}$. Then for the equation (CE) we determine a sequence $\{V_k(\xi, x)\}_{k=0}^\infty$ by the following recurrences:

$$(5.3) \quad \begin{cases} A(\xi)V_0(\xi, x) = \xi\varphi(x) \\ A(\xi)V_k(\xi, x) = \xi * \left(\frac{\partial}{\partial x}\right)^2 V_{k-2}(\xi, x) \\ \quad + \sum_{2 \leq j \leq m} B_j(V_{k-2}(\xi, x)) + \sum_{2 \leq j+\alpha \leq m, j, \alpha > 0} B_{j,\alpha}(V_{k-2}(\xi, x)) \quad \text{for } k \geq 1 \end{cases}$$

where $V_{-k}(\xi, x) \equiv 0$ for $k \geq 1$. Set $\Omega_{d,\theta}(\rho) = \{|\xi| < \rho\} \cup S_{d,\theta}$ with $\overline{S_{d,\theta}} \cap \Xi = \emptyset$. Then we see that $V_k(\xi, x) \in \mathcal{O}(\Omega_{d,\theta}(\rho) \times D_R)$ and $V_S(\xi, x) := \sum_{k=0}^\infty V_k(\xi, x)$ is a formal solution of the equation (CE).

For each $V_k(\xi, x)$ let us give an estimate. To estimate functions $V_k(\xi, x)$ we need the following lemma, which can be found in [2] and [7].

Lemma 5.3. *The following two statements are equivalent:*

(i) *A function $\phi(x)$ is an entire function and satisfies that there exist positive constants C and K such that*

$$|\phi(x)| \leq C e^{K|x|^2} \quad \text{on } \mathbb{C}.$$

(ii) *For any $R > 0$, there exist positive constants D and E depending on R such that*

$$\|(\frac{\partial}{\partial x})^i \phi\|_R \leq D E^i \Gamma(\frac{i}{2} + 1)$$

for all $i \geq 0$ where $\|\cdot\|_R = \sup_{|x| \leq R} |\cdot|$.

The following proposition is important to show the main theorem.

Proposition 5.4. *For the initial value function $\varphi(x) = (\frac{\partial}{\partial x})^2 \phi(x)$, we assume*

$$\|(\frac{\partial}{\partial x})^i \varphi\|_R \leq D E^i \Gamma(\frac{i}{2} + 1)$$

for all $i \geq 0$. Then we have that for $k = 0, 1, \dots$, there exist positive constants D_0 and K such that

$$\|(\frac{\partial}{\partial x})^i V_k\|_R \leq D_0 E^{i+k} K^k \Gamma(\frac{i+k}{2} + 1) \frac{|\xi|^{\frac{k}{2}}}{k!^{\frac{1}{2}} (k+1)!^{\frac{1}{2}}} \quad \text{for } \xi \in \Omega_{d,\theta}(\rho)$$

for all $i \geq 0$.

We give a proof of Proposition 5.4 in Section 6.

By Proposition 5.4, we have the following corollary.

Corollary 5.5. *For the solution $V_S(\xi, x) = \sum_{k \geq 0} V_k(\xi, x)$, we have that there exist positive constants c and D_1 such that*

$$\|V_S\|_R \leq D_1 e^{c|\xi|} \quad \text{for } \xi \in \Omega_{d,\theta}(\rho).$$

Proof. By Proposition 5.4 there exist positive constants D' and E' such that

$$(5.4) \quad \|V_k\|_R \leq D' E'^k \frac{|\xi|^{\frac{k}{2}}}{\Gamma(\frac{k}{2} + 1)} \quad \text{for } \xi \in \Omega_{d,\theta}(\rho).$$

By the estimate (5.4) we get Corollary 5.5. □

Let us show the uniqueness of the solution of (CE) on $\Omega_{d,\theta}(\rho) \times D_R$.

Proposition 5.6. *Let $V_{loc}(\xi, x)$ be the solution in Lemma 5.2 and $V_S(\xi, x)$ be the solution in Corollary 5.5. Then we have $V_{loc}(\xi, x) = V_S(\xi, x)$ on $S_{d,\theta}(\rho) \times D_R$ for $0 < \rho \ll 1$.*

We give a proof of Proposition 5.6 in Section 6.

Let us give a proof of the main theorem.

Proof. By Lemma 5.2, Corollary 5.5 and Proposition 5.6, the formal solution $\hat{u}(t, x)$ of the Cauchy problem (E) is Borel Summable in a direction d . □

§ 6. Proof of Proposition 5.4 and 5.6

In this section we shall give a proof on Proposition 5.4 and 5.6.

Proof of Proposition 5.4.

Proof. We can give a proof by Lemma 4.2. For $A_0(\xi)$ in (2.1) we have

$$(6.1) \quad |\{A_0(\xi)\}^{-1}| \leq C_0 (|\xi|^{m-1} + 1)^{-1} \quad \text{for } \xi \in \Omega_{d,\theta}(\rho).$$

If $k = 0$, by the relation (5.3) and the estimate (6.1) we have

$$\|(\frac{\partial}{\partial x})^i V_0\|_R \leq C_0 D E^i \Gamma(\frac{i}{2} + 1) \quad \text{for } \xi \in \Omega_{d,\theta}(\rho).$$

Then by taking $D_0 = C_0 D$ we get the estimate in Proposition 5.4 on $k = 0$.

For $k \geq 1$ we use an inductive method on k . Let us give an estimate for each term of the right hand side in the relation (5.3). For the first term, by the induction's assumption we have

$$(6.2) \quad \|(\frac{\partial}{\partial x})^i V_{k-2}\|_R \leq D_0 E^{i+k-2} K^{k-2} \Gamma(\frac{i+k-2}{2} + 1) \frac{|\xi|^{\frac{k-2}{2}}}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}}$$

for all $i \geq 0$. Then we have

$$\|(\frac{\partial}{\partial x})^{i+2} V_{k-2}\|_R \leq D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2} + 1) \frac{|\xi|^{\frac{k-2}{2}}}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}}$$

for all $i \geq 0$. Here we use the following inequality

$$\frac{\Gamma(\frac{k-2}{2} + 1)}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}} \frac{1}{\Gamma(\frac{k-2}{2} + 3)} = 2^2 \frac{1}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}} \frac{1}{(k+2)k} \leq 2^2 \frac{1}{k!^{\frac{1}{2}} (k+1)!^{\frac{1}{2}}}.$$

Then by Lemma 4.2 we get

$$(6.3) \quad \begin{aligned} \|\xi * (\frac{\partial}{\partial x})^{i+2} V_{k-2}\|_R &\leq D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2} + 1) \frac{\Gamma(\frac{k-2}{2} + 1)}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}} \frac{|\xi|^{\frac{k-2}{2}+3-1}}{\Gamma(\frac{k-2}{2} + 3)} \\ &\leq D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2} + 1) 2^2 \frac{|\xi|^{\frac{k}{2}+1}}{k!^{\frac{1}{2}} (k+1)!^{\frac{1}{2}}}. \end{aligned}$$

Let us give an estimate for the term $B_j(V_{k-2})$. By the estimate (6.2) and Lemma 4.2 for $s = 1, \dots, j-1$ we get

$$\begin{aligned} &\|\frac{\xi^{j-(s+1)}}{\Gamma(j-s)} * (\xi^s (\frac{\partial}{\partial x})^i V_{k-2})\|_R \\ &\leq D_0 E^{i+k-2} K^{k-2} \Gamma(\frac{i+k-2}{2} + 1) \frac{\Gamma(s + \frac{k-2}{2} + 1)}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}} \frac{|\xi|^{j+\frac{k-2}{2}}}{\Gamma(j + \frac{k-2}{2} + 1)} \\ &\leq D_0 E^{i+k-2} K^{k-2} \Gamma(\frac{i+k-2}{2} + 1) \frac{1}{(k-2)!^{\frac{1}{2}} (k-1)!^{\frac{1}{2}}} \frac{|\xi|^{j+\frac{k-2}{2}}}{(j + \frac{k-2}{2})} \\ &\leq D_0 E^{i+k-2} K^{k-2} \frac{|\xi|^{j+\frac{k-2}{2}}}{k!^{\frac{1}{2}} (k+1)!^{\frac{1}{2}}} \Gamma(\frac{i+k}{2}) \frac{(k-1)^{\frac{1}{2}} k(k+1)^{\frac{1}{2}}}{j + \frac{k-2}{2}}. \end{aligned}$$

For $2 \leq j \leq m$ we have

$$\frac{(k-1)^{\frac{1}{2}} k(k+1)^{\frac{1}{2}}}{j + \frac{k-2}{2}} \leq 2k.$$

Therefore we get

$$(6.4) \quad \|(\frac{\partial}{\partial x})^i B_j(V_{k-2})\|_R \leq |a_{j,0}| \sum_{s=1}^{j-1} |C_{j,s}| \frac{4}{E^2} D_0 E^{i+k} K^{k-2} \Gamma(\frac{i+k}{2} + 1) \frac{|\xi|^{j+\frac{k}{2}-1}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}$$

for all $i \geq 0$.

Let us give an estimate for the term $B_{j,\alpha}(V_{k-\alpha})$. By induction's assumption we have

$$\|(\frac{\partial}{\partial x})^i V_{k-\alpha}\|_R \leq D_0 E^{i+k-\alpha} K^{k-\alpha} \Gamma(\frac{i+k-\alpha}{2} + 1) \frac{|\xi|^{\frac{k-\alpha}{2}}}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}}$$

for all $i \geq 0$. Then we have

$$\|\xi^s (\frac{\partial}{\partial x})^{i+\alpha} V_{k-\alpha}\|_R \leq D_0 E^{i+k} K^{k-\alpha} \Gamma(\frac{i+k}{2} + 1) \frac{|\xi|^{s+\frac{k-\alpha}{2}}}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}}$$

for all $i \geq 0$. For $2 \leq j + \alpha \leq m$ and $j, \alpha > 0$ we get

$$\begin{aligned} & \left\| \frac{\xi^{j+\alpha-(s+1)}}{\Gamma(j+\alpha-s)} * \left\{ \xi^s (\frac{\partial}{\partial x})^{i+\alpha} V_{k-\alpha} \right\} \right\|_R \\ & \leq D_0 E^{i+k} K^{k-\alpha} \Gamma(\frac{i+k}{2} + 1) \frac{\Gamma(s + \frac{k-\alpha}{2} + 1)}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}} \frac{|\xi|^{j+\alpha+\frac{k-\alpha}{2}}}{\Gamma(j+\alpha+\frac{k-\alpha}{2} + 1)} \end{aligned}$$

by Lemma 4.2. For $s = 1, \dots, j$ we have

$$\frac{\Gamma(s + \frac{k-\alpha}{2} + 1)}{(k-\alpha)!^{\frac{1}{2}}(k-\alpha+1)!^{\frac{1}{2}}} \frac{1}{\Gamma(j+\alpha+\frac{k-\alpha}{2} + 1)} \leq 2^\alpha \frac{1}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}.$$

Therefore we get

$$(6.5) \quad \|(\frac{\partial}{\partial x})^i B_{j,\alpha}(V_{k-\alpha})\|_R \leq 2^\alpha |a_{j,\alpha}| \sum_{s=1}^j |C_{j,s}| D_0 E^{i+k} K^{k-\alpha} \Gamma(\frac{i+k}{2} + 1) \frac{|\xi|^{j+\frac{k+\alpha}{2}}}{k!^{\frac{1}{2}}(k+1)!^{\frac{1}{2}}}$$

for all $i \geq 0$. For $|\xi|^{j+(k+\alpha)/2}$ we remark $1 + \frac{k}{2} \leq j + \frac{k+\alpha}{2} \leq j + \alpha + \frac{k}{2} \leq m + \frac{k}{2}$.

We take a sufficiently large $K > 0$ so that

$$1 \geq C_0 \left(\frac{4}{K^2} + 4 \sum_{2 \leq j \leq m} |a_{j,0}| \sum_{s=1}^{j-1} |C_{j,s}| \frac{1}{E^2 K^2} + \sum_{2 \leq j+\alpha \leq m, j, \alpha > 0} 2^\alpha |a_{j,\alpha}| \sum_{s=1}^j |C_{j,s}| \frac{1}{K^\alpha} \right).$$

Hence we complete a proof of Proposition 5.4 by the estimates (6.3), (6.4) and (6.5). $\square$

Proof of Proposition 5.6

Proof. Set $U(\xi, x) = V_{loc}(\xi, x) - V_S(\xi, x)$. The the function $U(\xi, x)$ is a solution of the following equation:

$$(6.6) \quad A(\xi)U(\xi, x) = \xi * \left(\frac{\partial}{\partial x}\right)^2 U(\xi, x) + \sum_{2 \leq j \leq m} B_j(U(\xi, x)) + \sum_{2 \leq j + \alpha \leq m, j, \alpha > 0} B_{j, \alpha}(U(\xi, x)).$$

Further by Lemma 5.2 and Corollary 5.5, we have

$$\|U\|_R \leq D_4 \quad \text{for } \xi \in S_{d, \theta}(\rho)$$

for some $D_4 > 0$. Then by the Cauchy's integral theorem we have

$$(6.7) \quad \left\| \left(\frac{\partial}{\partial x}\right)^i U \right\|_r \leq D_4 (E_r)^i i! \quad \text{for } \xi \in S_{d, \theta}(\rho)$$

where $E_r = 2/(R - r)$ for $0 < r < R$.

We estimate the left hand side in the equation (6.6) by substituting the estimate (6.7) into the right hand side in (6.6). By a similar way to Proposition 5.4, we get for all $k \geq 0$,

$$\|U\|_r \leq D_4 (E_r K_r)^k |\xi|^{k/2} \quad \text{for } \xi \in S_{d, \theta}(\rho)$$

where $K_r > 0$ satisfies

$$1 \geq C_0 \left(\frac{4}{K_r^2} + 2 \sum_{2 \leq j \leq m} |a_{j,0}| \sum_{s=1}^{j-1} |C_{j,s}| \frac{1}{E_r^2 K_r^2} + \sum_{2 \leq j + \alpha \leq m, j, \alpha > 0} 2^\alpha |a_{j,\alpha}| \sum_{s=1}^j |C_{j,s}| \frac{1}{K_r^\alpha} \right).$$

By letting $k \rightarrow \infty$ then we have $U(\xi, x) \equiv 0$ on $S_{d, \theta}(\rho) \times D_r$ for $0 < \rho \ll 1$. $\square$

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