

On Starlikeness and Convexity of Certain Analytic Functions

HIDEHIRO KITAMURA and SHIGEYOSHI OWA

Abstract. Two subclasses $S(\alpha, b)$ and $C(\alpha, b)$ of the class A consisting of all analytic functions with $f(0) = 0$ and $f'(0) = 1$ in the open unit disk U are introduced. The classes $S(\alpha, b)$ and $C(\alpha, b)$ are the generalization classes of classes defined by H.Silverman (cf.[2]) and by T.Sekine and S.Owa (cf.[1]). The object of the present paper is to derive some coefficient inequalities for functions belonging to $S(\alpha, b)$ and $C(\alpha, b)$. Also we consider some necessary conditions for $f(z)$ belonging to the classes $S(\alpha, b)$ and $C(\alpha, b)$. Some interesting examples for our results are also given.

I . Introduction.

Let A denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha$$

for some α ($0 \leq \alpha < 1$) and for all z in U . We denote by $S^*(\alpha)$ the subclass of A consisting of all starlike functions $f(z)$ of order α in U .

Further, a function $f(z) \in A$ is said to be convex of order α if it satisfies

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha$$

for some α ($0 \leq \alpha < 1$) and for all $z \in U$. We denote by $K(\alpha)$ the subclass of A consisting of such convex functions $f(z)$ of order α in U .

Let $S(\alpha, b)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$\left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha \quad (0 \leq \alpha < \operatorname{Re}(b) - |b - 1|)$$

for $b \in \mathbb{C}$.

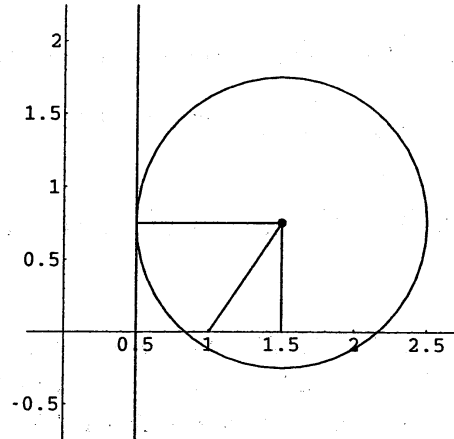


Figure 1: Image of $\left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha$.

Let $C(\alpha, b)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$\left| 1 + \frac{zf''(z)}{f'(z)} - b \right| < \operatorname{Re}(b) - \alpha \quad (0 \leq \alpha < \operatorname{Re}(b) - |b - 1|)$$

for $b \in \mathbb{C}$.

Remark 1. Letting $b = b_1 + ib_2$, the condition

$$0 \leq \alpha < \operatorname{Re}(b) - |b - 1|$$

shows that

$$0 \leq b_2^2 < (1 - \alpha)(2b_1 - (1 + \alpha)).$$

Therefore, b should be in the right half plane of the parabola.

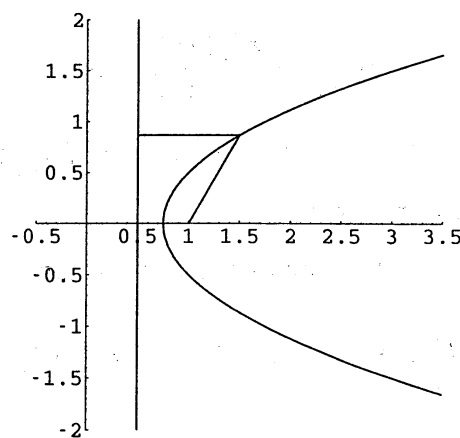


Figure 2: Range of b , that is, $|b - 1| < \operatorname{Re}(b) - \alpha$.

Remark 2. (1) In 1975, Silverman [2] has showed that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in U),$$

so $f(z) \in S(\alpha, 1)$.

(2) Sekine and Owa [1] have showed that

(i) if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha$$

for $1 \leq b \leq 2$, then

$$\left| \frac{zf'(z)}{f(z)} - b \right| < b - \alpha \quad (z \in U),$$

so $f(z) \in S(\alpha, b)$,

(ii) if $f(z) \in A$ satisfies

$$\sum_{n=2}^j (2b - n - \alpha) |a_n| + \sum_{n=j+1}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha$$

for $b > 2$, then

$$\left| \frac{zf'(z)}{f(z)} - b \right| < b - \alpha \quad (z \in U),$$

so $f(z) \in S(\alpha, b)$,

(iii) if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 2b - 1 - \alpha$$

for $\frac{1 + \alpha}{2} < b < 1$, then

$$\left| \frac{zf'(z)}{f(z)} - b \right| < b - \alpha \quad (z \in U),$$

so $f(z) \in S(\alpha, b)$.

II. Coefficient Inequalities.

We shall now prove the following theorems in a same way of Theorem 1 of Silverman [2], or Sekine and Owa [1].

Theorem 1. Let $f(z) \in A$, $b \in \mathbb{C}$ and $0 \leq \alpha < \operatorname{Re}(b) - |b - 1|$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} \{|n - b| + \operatorname{Re}(b) - \alpha\} |a_n| \leq \operatorname{Re}(b) - \alpha - |b - 1|,$$

then

$$\left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha,$$

that is, $f(z) \in S(\alpha, b)$.

Proof. For $f(z) \in A$, it follows that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - b \right| &= \left| \frac{1 - b + \sum_{n=2}^{\infty} (n - b) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\ &= \left| \frac{b - 1 - \sum_{n=2}^{\infty} (n - b) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\ &\leq \frac{|b - 1| + \sum_{n=2}^{\infty} |n - b| |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n| |z|^{n-1}} \\ &< \frac{|b - 1| + \sum_{n=2}^{\infty} |n - b| |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}. \end{aligned}$$

The last expression is bounded by $\operatorname{Re}(b) - \alpha$ if

$$|b - 1| + \sum_{n=2}^{\infty} |n - b| |a_n| \leq (\operatorname{Re}(b) - \alpha) \left(1 - \sum_{n=2}^{\infty} |a_n| \right),$$

which is equivalent to

$$\sum_{n=2}^{\infty} \{|n - b| + \operatorname{Re}(b) - \alpha\} |a_n| \leq \operatorname{Re}(b) - \alpha - |b - 1|.$$

Hence we have $\left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha$, and the Theorem 1 is proved. $\square$

Remark 3. If we take $b = 1$ in Theorem 1, then we have Theorem 1 by Silverman [2]. Further, if we take some real b such that $b > \frac{1 + \alpha}{2}$, then we have Theorem 1 by Sekine and Owa [1].

Corollary 1. Let $f(z) \in A$, $b \in \mathbb{C}$ and $0 \leq \alpha < \operatorname{Re}(b) - |b - 1|$. If $f(z)$ satisfies

$$\sum_{n=2}^{\infty} n\{|n - b| + \operatorname{Re}(b) - \alpha\} |a_n| \leq \operatorname{Re}(b) - \alpha - |b - 1|,$$

then

$$\left| 1 + \frac{zf''(z)}{f'(z)} - b \right| < \operatorname{Re}(b) - \alpha,$$

that is, $f(z) \in C(\alpha, b)$.

Proof. It is well known that $f(z) \in C(\alpha, b)$ if and only if $zf'(z) \in S(\alpha, b)$. Because, suppose that $F(z) = zf'(z)$. Then

$$\begin{aligned} \left| \frac{zF'(z)}{F(z)} - b \right| &= \left| \frac{z(f'(z) + zf''(z))}{zf'(z)} - b \right| \\ &= \left| \frac{f'(z) + zf''(z)}{f'(z)} - b \right| \\ &= \left| 1 + \frac{zf''(z)}{f'(z)} - b \right| < \operatorname{Re}(b) - \alpha. \end{aligned}$$

Since $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$, we may replace a_n with na_n in Theorem 1.

This completes the proof of Corollary 1. $\square$

Remark 4. If we take $b = 1$ in Corollary 1, then we have the result by Silverman [2]. And if we take some real b such that $b > \frac{1 + \alpha}{2}$, then we have the result by Sekine and Owa [1].

III. Distortion Inequalities.

If we consider the function $f(z) \in S\left(\frac{1}{4}, \frac{3}{4}\right)$, then

$$\left| \frac{zf'(z)}{f(z)} - \frac{3}{4} \right| < \frac{1}{2} \quad (z \in U).$$

Recently, Silverman [3] has given the function $f(z) = z - \frac{1}{3}z^2$ which is in the class $S\left(\frac{1}{4}, \frac{3}{4}\right)$.

But this function $f(z)$ does not satisfy the coefficient inequality of Theorem 1. As we mention the above, the inverse of Theorem 1 is not true in general.

Now, let $S_0(\alpha, b)$ be the subclass of $S(\alpha, b)$ consisting of $f(z)$ satisfying the coefficient inequality of Theorem 1. Further, let $C_0(\alpha, b)$ be the subclass of $C(\alpha, b)$ consisting of $f(z)$ which satisfy the coefficient inequality of Corollary 1.

Theorem 2. If $f(z) \in S_0(\alpha, b)$, then

$$|z| - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|^2 \leq |f(z)| \leq |z| + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|^2 \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} z^2 \quad (z = \pm|z|).$$

Proof. By the assumption $f(z) \in S_0(\alpha, b)$, we note that

$$\left\{ \min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha \right\} \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \{|n - b| + \operatorname{Re}(b) - \alpha\} |a_n| \leq \operatorname{Re}(b) - \alpha - |b - 1|,$$

that is,

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha}.$$

Thus, using the preceding result, we have

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right|$$

$$\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n$$

$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\leq |z| + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|^2.$$

Similarly, we have

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right|$$

$$\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n$$

$$\geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\geq |z| - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|^2.$$

Consequently, if $f(z) \in S_0(\alpha, b)$, then we obtain

$$|z| - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|^2 \leq |f(z)| \leq |z| + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|^2.$$

Finally, taking the function

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} z^2,$$

we have the equalities for $z = |z|$, and for $z = -|z|$, respectively.

This completes the proof of Theorem 2. □

For $f'(z)$ of $f(z)$ belonging to $S_0(\alpha, b)$, we have

Theorem 3. *If $f(z) \in S_0(\alpha, b)$, then*

$$1 - \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z| \leq |f'(z)| \leq 1 + \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z| \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} z^2 \quad (z = \pm |z|).$$

Proof. By the assumption $f(z) \in S_0(\alpha, b)$, we note that

$$\begin{aligned} \sum_{n=2}^{\infty} \{ |n - b| + \operatorname{Re}(b) - \alpha \} |a_n| &= \sum_{n=2}^{\infty} \left\{ n \left| 1 - \frac{b}{n} \right| + \operatorname{Re}(b) - \alpha \right\} |a_n| \\ &= \sum_{n=2}^{\infty} \left\{ n \left| 1 - \frac{b}{n} \right| \right\} |a_n| + \sum_{n=2}^{\infty} (\operatorname{Re}(b) - \alpha) |a_n| \\ &\leq \operatorname{Re}(b) - \alpha - |b - 1|, \end{aligned}$$

which implies that

$$\min_{n \geq 2} \left| 1 - \frac{b}{n} \right| \sum_{n=2}^{\infty} n |a_n| + \sum_{n=2}^{\infty} (\operatorname{Re}(b) - \alpha) |a_n| \leq \sum_{n=2}^{\infty} \left\{ n \left| 1 - \frac{b}{n} \right| \right\} |a_n| + \sum_{n=2}^{\infty} (\operatorname{Re}(b) - \alpha) |a_n|.$$

This gives us that

$$\begin{aligned} \min_{n \geq 2} \left| 1 - \frac{b}{n} \right| \sum_{n=2}^{\infty} n |a_n| &\leq \operatorname{Re}(b) - \alpha - |b - 1| - (\operatorname{Re}(b) - \alpha) \sum_{n=2}^{\infty} |a_n| \\ &\leq \operatorname{Re}(b) - \alpha - |b - 1| - (\operatorname{Re}(b) - \alpha) \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} \\ &\leq \frac{(\operatorname{Re}(b) - \alpha - |b - 1|) \min_{n \geq 2} |n - b|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha}, \end{aligned}$$

that is, that

$$\begin{aligned} \sum_{n=2}^{\infty} n|a_n| &\leq \frac{(\operatorname{Re}(b) - \alpha - |b - 1|) \min_{n \geq 2} |n - b|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} \times \frac{1}{\min_{n \geq 2} \left| 1 - \frac{b}{n} \right|} \\ &= \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha}. \end{aligned}$$

Because, $\min_{n \geq 2} |n - b|$ has the minimum value when $\operatorname{Re}(b)$ is the nearest n , and $\min_{n \geq 2} \left| 1 - \frac{b}{n} \right|$ has the minimum value when $\operatorname{Re} \left(\frac{b}{n} \right)$ is the nearest 1, in other words, when $\operatorname{Re}(b)$ is the nearest n . Consequently, we have $\frac{\min_{n \geq 2} |n - b|}{\min_{n \geq 2} \left| 1 - \frac{b}{n} \right|} = 2$.

In view the above, we see that

$$\begin{aligned} |f'(z)| &= \left| 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\ &\leq 1 + \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \\ &\leq 1 + |z| \sum_{n=2}^{\infty} n |a_n| \\ &\leq 1 + \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |f'(z)| &= \left| 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\ &\geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \\ &\geq 1 - |z| \sum_{n=2}^{\infty} n |a_n| \\ &\geq 1 - \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|. \end{aligned}$$

Consequently, if $f(z) \in S_0(\alpha, b)$, then we obtain

$$1 - \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z| \leq |f'(z)| \leq 1 + \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|.$$

Taking $f(z)$ given by

$$f'(z) = 1 + \frac{2(\operatorname{Re}(b) - \alpha - |b - 1|)}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} z,$$

which is equivalent to

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} z^2,$$

we see that the equalities in Theorem 3 are attained. □

Now, we give an example for Theorem 3.

Example 1. Let $f(z) \in S_0(\alpha, b)$, $b = 3 + \frac{1}{2}i$ and $0 \leq \alpha = \frac{4}{5} < 3 - \frac{\sqrt{17}}{2}$. Then we have

$$f(z) = z + \frac{22 - 5\sqrt{17}}{27} z^2 \text{ and } f'(z) = 1 + \frac{2(22 - 5\sqrt{17})}{27} z.$$

Therefore

$$\frac{z f'(z)}{f(z)} = 2 - \frac{27}{27 + (22 - 5\sqrt{17}) z}.$$

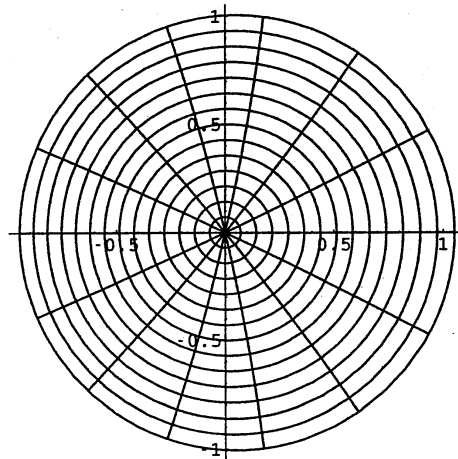


Figure 3: Image of U by $f(z)$ for Example 1.

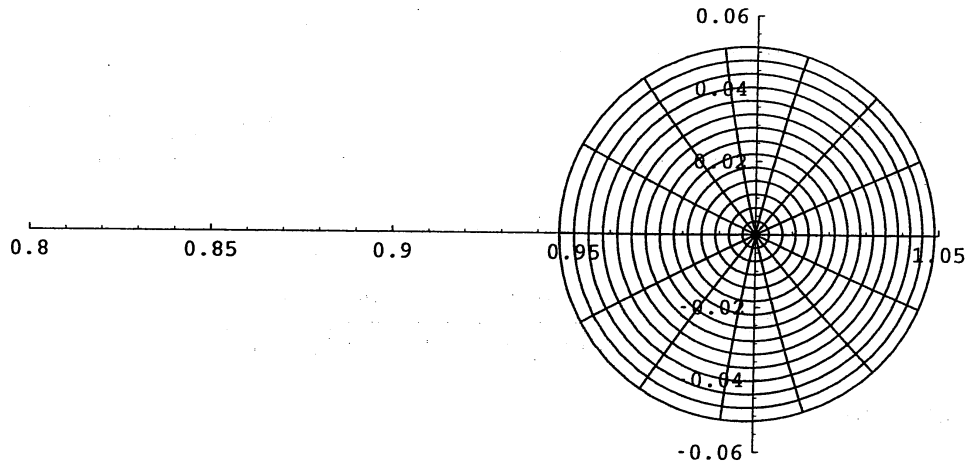


Figure 4: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 1.

Furthermore, for $f(z)$ in the class $C_0(\alpha, b)$, we see

Theorem 4. If $f(z) \in C_0(\alpha, b)$, then

$$|z| - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} |z|^2 \leq |f(z)| \leq |z| + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} |z|^2 \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} z^2 \quad (z = \pm|z|).$$

Proof. For $f(z) \in C_0(\alpha, b)$, we note that

$$2 \left\{ \min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha \right\} \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n \{|n - b| + \operatorname{Re}(b) - \alpha\} |a_n| \leq \operatorname{Re}(b) - \alpha - |b - 1|,$$

that is, that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}}.$$

Applying the above inequality, we have

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right|$$

$$\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n$$

$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\leq |z| + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} |z|^2.$$

Similarly, we have

$$\begin{aligned}
 |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \\
 &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \\
 &\geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\
 &\geq |z| - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} |z|^2.
 \end{aligned}$$

Consequently, if $f(z) \in C_0(\alpha, b)$, then we obtain

$$|z| - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} |z|^2 \leq |f(z)| \leq |z| + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} |z|^2.$$

Finally, letting

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} z^2,$$

we know that the equalities of the theorem are attained. $\square$

For the derivative $f'(z)$ of $f(z)$ in $C_0(\alpha, b)$, we also have

Theorem 5. *If $f(z) \in C_0(\alpha, b)$, then*

$$1 - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z| \leq |f'(z)| \leq 1 + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z| \quad (z \in U).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} z^2 \quad (z = \pm|z|).$$

Proof. Noting that

$$\left\{ \min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha \right\} \sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} n \{ |n - b| + \operatorname{Re}(b) - \alpha \} |a_n| \leq \operatorname{Re}(b) - \alpha - |b - 1|,$$

for $f(z) \in C_0(\alpha, b)$, we have

$$\sum_{n=2}^{\infty} n |a_n| \leq \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha}.$$

Thus, we have

$$\begin{aligned}
 |f'(z)| &= \left| 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\
 &\leq 1 + \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \\
 &\leq 1 + |z| \sum_{n=2}^{\infty} n |a_n| \\
 &\leq 1 + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|,
 \end{aligned}$$

and

$$\begin{aligned}
 |f'(z)| &= \left| 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\
 &\geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \\
 &\geq 1 - |z| \sum_{n=2}^{\infty} n |a_n| \\
 &\geq 1 - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|.
 \end{aligned}$$

Consequently, if $f(z) \in C_0(\alpha, b)$, then we obtain

$$1 - \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z| \leq |f'(z)| \leq 1 + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} |z|.$$

Making

$$f'(z) = 1 + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha} z,$$

or

$$f(z) = z + \frac{\operatorname{Re}(b) - \alpha - |b - 1|}{2\{\min_{n \geq 2} |n - b| + \operatorname{Re}(b) - \alpha\}} z^2,$$

we complete the proof of Theorem 5. □

For Theorem 5, we give the following example.

Example 2. Let $f(z) \in C_0(\alpha, b)$, $b = 3 + \frac{1}{2}i$ and $0 \leq \alpha = \frac{4}{5} < 3 - \frac{\sqrt{17}}{2}$. Then we have

$$f(z) = z + \frac{22 - 5\sqrt{17}}{54} z^2, \quad f'(z) = 1 + \frac{22 - 5\sqrt{17}}{27} z \quad \text{and} \quad f''(z) = \frac{22 - 5\sqrt{17}}{27}.$$

Therefore

$$1 + \frac{z f''(z)}{f'(z)} = 2 - \frac{27}{27 + (22 - 5\sqrt{17}) z}.$$

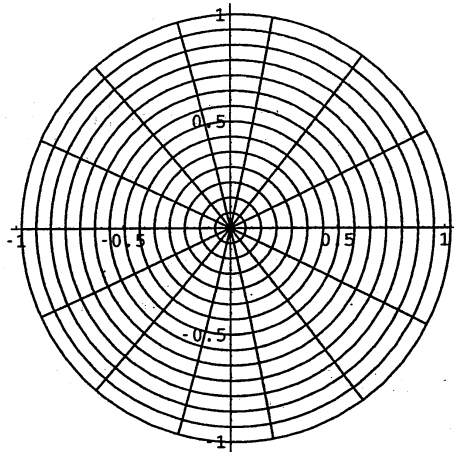


Figure 5: Image of U by $f(z)$ for Example 2.

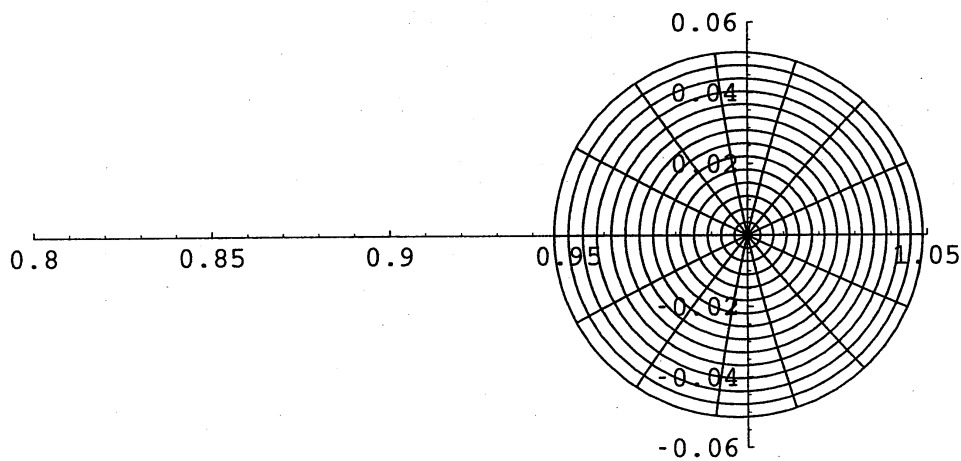


Figure 6: Image of U by $1 + \frac{z f''(z)}{f'(z)}$ for Example 2.

IV. Necessary conditions for the class $S(\alpha, b)$.

In general, we know that the coefficient inequalities which we give in Theorem 1 and Corollary 1 are not necessary conditions for the classes $S(\alpha, b)$ and $C(\alpha, b)$. Therefore, we try to find some necessary conditions for the class $S(\alpha, b)$.

Theorem 6. *Let $f(z)$ be in the class $S(\alpha, b)$ with $a_n = |a_n|e^{in\pi}$ and $b = 1 + ib_2$, then*

$$\begin{aligned} & \left(\sum_{n=2}^{\infty} (n + \alpha) |a_n| + \frac{(1 - b_2 - \alpha)(1 + b_2 - \alpha)}{\alpha - 1} \right) \\ & \times \left(\sum_{n=2}^{\infty} (n - 2 + \alpha) |a_n| - \frac{(1 - b_2 - \alpha)(1 + b_2 - \alpha)}{\alpha - 1} \right) + b_2^2 \left(\sum_{n=2}^{\infty} |a_n| \right)^2 \\ & < \frac{b_2^2(1 - b_2 - \alpha)(1 + b_2 - \alpha)}{(\alpha - 1)^2}. \end{aligned}$$

Proof. Since $f(z) \in S(\alpha, b)$ if and only if $\left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha = 1 - \alpha$, we have, for $z = e^{i\pi}$,

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - b \right| &= \left| \frac{b - 1 - \sum_{n=2}^{\infty} (n - b) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\ &= \frac{|b - 1 + \sum_{n=2}^{\infty} (n - b) |a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\ &= \frac{|b - 1 + \sum_{n=2}^{\infty} n |a_n| - \sum_{n=2}^{\infty} b |a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\ &= \frac{|1 + ib_2 - 1 + \sum_{n=2}^{\infty} n |a_n| - \sum_{n=2}^{\infty} (1 + ib_2) |a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\ &= \frac{|ib_2 + \sum_{n=2}^{\infty} n |a_n| - \sum_{n=2}^{\infty} |a_n| - ib_2 \sum_{n=2}^{\infty} |a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\ &= \frac{|\sum_{n=2}^{\infty} (n - 1) |a_n| + ib_2(1 - \sum_{n=2}^{\infty} |a_n|)|}{1 - \sum_{n=2}^{\infty} |a_n|} \\ &= \frac{\sqrt{(\sum_{n=2}^{\infty} (n - 1) |a_n|)^2 + b_2^2(1 - \sum_{n=2}^{\infty} |a_n|)^2}}{1 - \sum_{n=2}^{\infty} |a_n|} < 1 - \alpha. \end{aligned}$$

It follows from the above that

$$\sqrt{\left(\sum_{n=2}^{\infty} (n - 1) |a_n| \right)^2 + b_2^2 \left(1 - \sum_{n=2}^{\infty} |a_n| \right)^2} < 1 - \alpha - \sum_{n=2}^{\infty} (1 - \alpha) |a_n|,$$

that is, that

$$\begin{aligned} & \left(\sum_{n=2}^{\infty} (n-1)|a_n| \right)^2 + b_2^2 \left(1 - \sum_{n=2}^{\infty} |a_n| \right)^2 \\ & < (1-\alpha)^2 - 2(1-\alpha)^2 \sum_{n=2}^{\infty} |a_n| + \left(\sum_{n=2}^{\infty} (1-\alpha)|a_n| \right)^2. \end{aligned}$$

It is easy to see that

$$\begin{aligned} & \left(\sum_{n=2}^{\infty} (n+\alpha)|a_n| \right) \left(\sum_{n=2}^{\infty} (n-2+\alpha)|a_n| \right) + b_2^2 \left(\sum_{n=2}^{\infty} |a_n| \right)^2 \\ & \quad + (2-4\alpha+2\alpha^2-2b_2^2) \sum_{n=2}^{\infty} |a_n| \\ & < (1-b_2-\alpha)(1+b_2-\alpha). \end{aligned}$$

Thus we obtain

$$\begin{aligned} & \left(\sum_{n=2}^{\infty} (n+\alpha)|a_n| + \frac{(1-b_2-\alpha)(1+b_2-\alpha)}{\alpha-1} \right) \\ & \quad \times \left(\sum_{n=2}^{\infty} (n-2+\alpha)|a_n| - \frac{(1-b_2-\alpha)(1+b_2-\alpha)}{\alpha-1} \right) + b_2^2 \left(\sum_{n=2}^{\infty} |a_n| \right)^2 \\ & < \frac{b_2^2(1-b_2-\alpha)(1+b_2-\alpha)}{(\alpha-1)^2}, \end{aligned}$$

which derives the proof of the theorem. $\square$

Noting that $\frac{1+\alpha}{2} < \operatorname{Re}(b)$, we consider the case of $b = b_1 + ib_2$ with $b_1 > \frac{1+\alpha}{2}$.

Theorem 7. *If $f(z) \in S(\alpha, b)$ with $a_n = |a_n|e^{in\pi}$ and $b = b_1 + ib_2$, then*

$$\begin{aligned} & \left(\sum_{n=2}^{\infty} (n-\alpha)|a_n| + \frac{(\alpha-b_1)^2 - |b|^2 + b_1}{\alpha-b_1} \right) \left(\sum_{n=2}^{\infty} (n-2b_1+\alpha)|a_n| + \frac{(\alpha-b_1)^2 - |b|^2 + b_1}{\alpha-b_1} \right) \\ & \quad + b_2^2 \left(\sum_{n=2}^{\infty} |a_n| \right)^2 + 2(b_1-1) \sum_{n=2}^{\infty} n|a_n| \\ & < |b|^2 - 1 - \frac{|b|^2(|b|^2 - 2b_1) - b_1^2}{(\alpha-b_1)^2}. \end{aligned}$$

Proof. Note that $f(z) \in S(\alpha, b)$ if and only if $\left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha = b_1 - \alpha$. Letting $z = e^{i\pi}$, we have

$$\begin{aligned}
\left| \frac{zf'(z)}{f(z)} - b \right| &= \left| \frac{b - 1 - \sum_{n=2}^{\infty} (n - b)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \\
&= \frac{|b - 1 + \sum_{n=2}^{\infty} (n - b)|a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\
&= \frac{|b - 1 + \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} b|a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\
&= \frac{|b_1 + ib_2 - 1 + \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} (b_1 + ib_2)|a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\
&= \frac{|(b_1 - 1) + ib_2 + \sum_{n=2}^{\infty} n|a_n| - \sum_{n=2}^{\infty} b_1|a_n| - ib_2 \sum_{n=2}^{\infty} |a_n||}{1 - \sum_{n=2}^{\infty} |a_n|} \\
&= \frac{|(b_1 - 1) + \sum_{n=2}^{\infty} (n - b_1)|a_n| + ib_2(1 - \sum_{n=2}^{\infty} |a_n|)|}{1 - \sum_{n=2}^{\infty} |a_n|} \\
&= \frac{\sqrt{(b_1 - 1) + \sum_{n=2}^{\infty} (n - 1)|a_n|)^2 + b_2^2(1 - \sum_{n=2}^{\infty} |a_n|)^2}}{1 - \sum_{n=2}^{\infty} |a_n|} < b_1 - \alpha,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\sqrt{\left((b_1 - 1) + \sum_{n=2}^{\infty} (n - b_1)|a_n| \right)^2 + b_2^2 \left(1 - \sum_{n=2}^{\infty} |a_n| \right)^2} \\
&< b_1 - \alpha - \sum_{n=2}^{\infty} (b_1 - \alpha)|a_n|.
\end{aligned}$$

By using the same manner as in the proof of the previous theorem, we have

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} (n - \alpha)|a_n| \right) \left(\sum_{n=2}^{\infty} (n - 2b_1 + \alpha)|a_n| \right) + b_2^2 \left(\sum_{n=2}^{\infty} |a_n| \right)^2 + 2(b_1 - 1) \sum_{n=2}^{\infty} n|a_n| \\
&\quad + (2b_1 - 4\alpha b_1 + 2\alpha^2 - 2b_2^2) \sum_{n=2}^{\infty} |a_n| \\
&< 2b_1 - 2\alpha b_1 + \alpha^2 - b_2^2 - 1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned} & \left(\sum_{n=2}^{\infty} (n - \alpha) |a_n| + \frac{(\alpha - b_1)^2 - |b|^2 + b_1}{\alpha - b_1} \right) \left(\sum_{n=2}^{\infty} (n - 2b_1 + \alpha) |a_n| + \frac{(\alpha - b_1)^2 - |b|^2 + b_1}{\alpha - b_1} \right) \\ & + b_2^2 \left(\sum_{n=2}^{\infty} |a_n| \right)^2 + 2(b_1 - 1) \sum_{n=2}^{\infty} n |a_n| \\ & < |b|^2 - 1 - \frac{|b|^2(|b|^2 - 2b_1) - b_1^2}{(\alpha - b_1)^2}. \end{aligned}$$

This completes the proof of the theorem. $\square$

V. Some examples of functions belonging to the class $S(\alpha, b)$.

Now, we consider some examples of functions $f(z)$ which belong to the class $S(\alpha, b)$.

Remark 5. Since

$$f(z) \in S(\alpha, b) \iff \left| \frac{zf'(z)}{f(z)} - b \right| < \operatorname{Re}(b) - \alpha,$$

we consider B which satisfies

$$\left| \frac{zf'(z)}{f(z)} - b \right| < |1 - b| + |B| = \operatorname{Re}(b) - \alpha.$$

Thus we get

$$|B| = \operatorname{Re}(b) - \alpha - |1 - b|,$$

and

$$B = (\operatorname{Re}(b) - \alpha - |1 - b|) e^{i\phi}.$$

For such B , we consider

$$\frac{zf'(z)}{f(z)} - b = (1 - b) + Bz,$$

which gives that

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = B.$$

Integrating both sides, we have

$$\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t} \right) dt = B \int_0^z dt,$$

so

$$\frac{f(z)}{z} = e^{Bz}.$$

Thus we obtain

$$f(z) = ze^{\{(\operatorname{Re}(b) - \alpha - |1 - b|)e^{i\phi}\}z}.$$

Taking some b and α in Remark 5, we give

Example 3. Taking $b = 1 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, we have

$$f(z) = ze^{\{(\operatorname{Re}(b)-\alpha-|1-b|)e^{i\phi}\}z} = ze^{\frac{1}{6}z}.$$

Since

$$f'(z) = \left(1 + \frac{1}{6}z\right) e^{\frac{1}{6}z}$$

we see

$$\frac{zf'(z)}{f(z)} = 1 + \frac{1}{6}z.$$

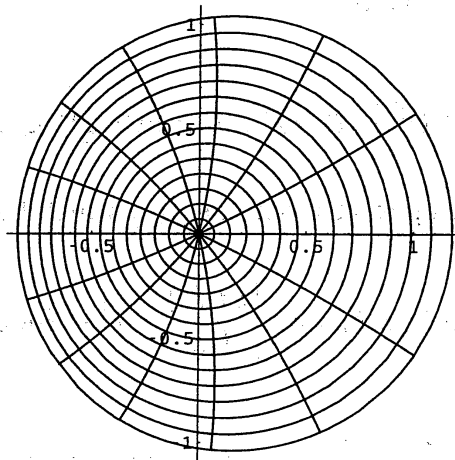


Figure 7: Image of U by $f(z)$ for Example 3.

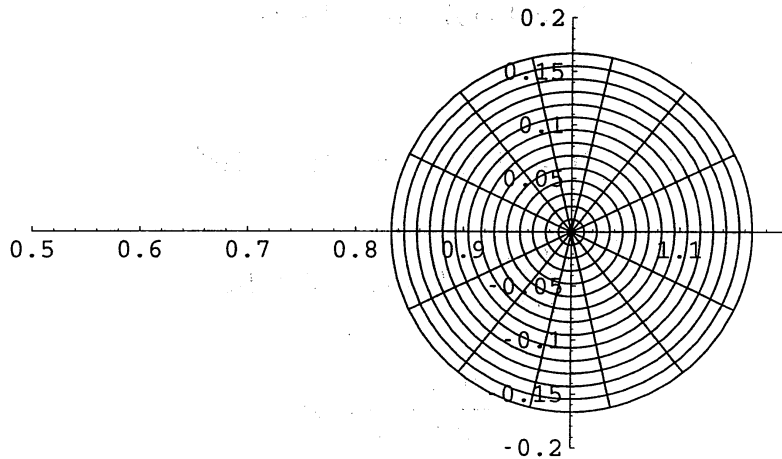


Figure 8: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 3.

Example 4. Taking $b = 2 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, we have

$$f(z) = ze^{\{(\operatorname{Re}(b)-\alpha-|1-b|)e^{i\phi}\}z} = ze^{\left(\frac{3}{2} - \frac{\sqrt{10}}{3}\right)z}.$$

Since

$$f'(z) = \frac{1}{6}(6 + (9 - 2\sqrt{10})z)e^{\left(\frac{3}{2} - \frac{\sqrt{10}}{3}\right)z}$$

we see

$$\frac{zf'(z)}{f(z)} = 1 + \left(\frac{3}{2} - \frac{\sqrt{10}}{3}\right)z.$$

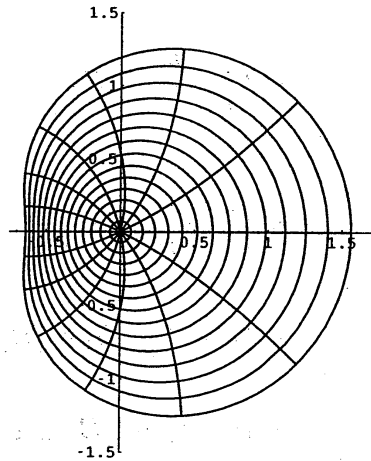


Figure 9: Image of U by $f(z)$ for Example 4.

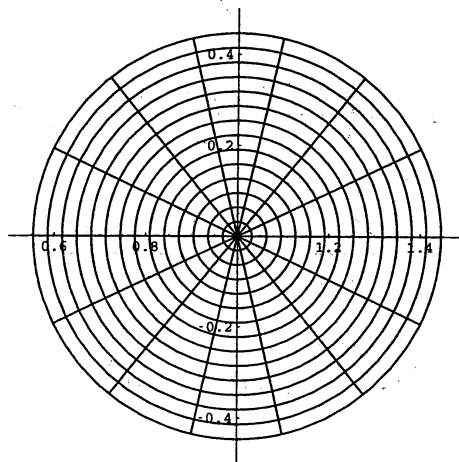


Figure 10: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 4.

Next, we consider

Remark 6. We consider B such that

$$\frac{zf'(z)}{f(z)} - b = \frac{(1-b) + Bz}{1-z}.$$

Noting that

$$\begin{aligned}
 \left| \frac{zf'(z)}{f(z)} - b \right| &= \left| \frac{1-b+Bz}{1-z} \right| \\
 &= \left| \frac{1-b}{1-z} + \frac{Bz}{1-z} \right| \\
 &= \left| \frac{1-b}{1-z} + \frac{B}{1-z} - B \right| \\
 &< \left| \frac{1-b}{1-z} \right| + \left| \frac{B}{1-z} \right| + |B| \\
 &< |1-b| + |B| + |B| \\
 &= |1-b| + 2|B| \leq \operatorname{Re}(b) - \alpha,
 \end{aligned}$$

we find B given by

$$\begin{aligned}
 |B| &= \frac{\operatorname{Re}(b) - \alpha - |1-b|}{2}, \\
 B &= \left(\frac{\operatorname{Re}(b) - \alpha - |1-b|}{2} \right) e^{i\phi}.
 \end{aligned}$$

For such B , we see that

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1-b+B}{1-z}.$$

After integration, we have

$$\log \frac{f(z)}{z} = \log (1-z)^{(b-1-B)},$$

that is

$$f(z) = z(1-z)^{(b-1-B)}.$$

It follows that

$$f(z) = z(1-z)^{(b-1-(\frac{\operatorname{Re}(b)-\alpha-|1-b|}{2})e^{i\phi})}.$$

Taking some b and α , Remark 6 gives us

Example 5. Making $b = 1 + \frac{1}{100}i$, $\alpha = \frac{1}{2}$, we have

$$f(z) = z(1-z)^{(b-1-(\frac{\operatorname{Re}(b)-\alpha-|1-b|}{2})e^{i\phi})} = z(1-z)^{\frac{1}{200}(2i-49(-1)^{\frac{1}{100}})},$$

$$f'(z) = \frac{1}{200}(200 + ((-200-2i) + 49(-1)^{\frac{1}{100}})z)(1-z)^{\frac{1}{200}((-200+2i)-49(-1)^{\frac{1}{100}})}.$$

This gives us that

$$\frac{zf'(z)}{f(z)} = \frac{200 - ((200 + 2i) - 49(-1)^{\frac{1}{100}})z}{200(1 - z)}.$$

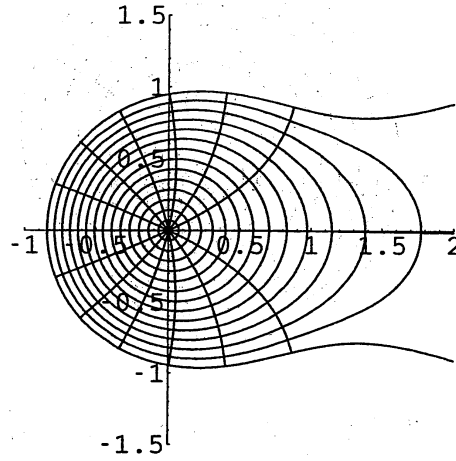


Figure 11: Image of U by $f(z)$ for Example 5.

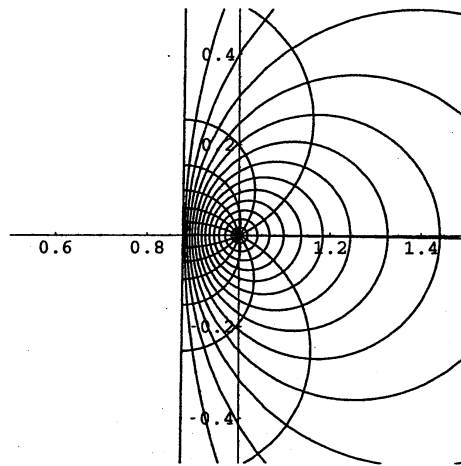


Figure 12: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 5.

Example 6. Making $b = 2 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, we have

$$f(z) = z(1 - z)^{(b-1 - (\frac{\operatorname{Re}(b) - \alpha - |1-b|}{2})e^{i\phi})} = z(1 - z)^{\frac{1}{12}(3+2\sqrt{10}+4i)},$$

$$f'(z) = -\frac{1}{12}(-12 + (15 + 2\sqrt{10} + 4i)z)(1 - z)^{\frac{1}{12}(-9+2\sqrt{10}+4i)}.$$

This gives us that

$$\frac{zf'(z)}{f(z)} = \frac{12 - (15 + 2\sqrt{10} + 4i)z}{12(1 - z)}.$$

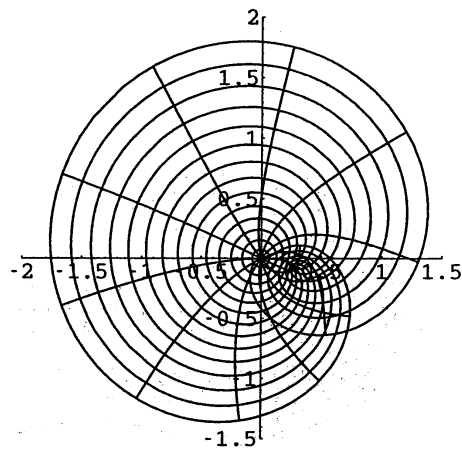


Figure 13: Image of U by $f(z)$ for Example 6.

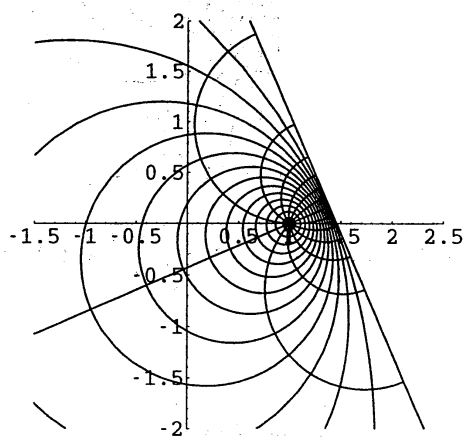


Figure 14: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 6.

Finally, considering $f(z) = z + a_2 z^2$, we derive

Remark 7. Let us consider the function $f(z)$ given by $f(z) = z + a_2 z^2$. For such $f(z)$, we have

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{1 + 2a_2 z}{1 + a_2 z} = 1 + \frac{a_2 z}{1 + a_2 z} \\ &= 1 + \frac{B}{1 + B}, \end{aligned}$$

where $B = a_2 z$. Since

$$\left| \frac{zf'(z)}{f(z)} - b \right| = \left| \frac{B}{1 + B} + 1 - b \right| < \operatorname{Re}(b) - \alpha,$$

we have

$$\left| \frac{B}{1 + B} + 1 - b \right|^2 < (\operatorname{Re}(b) - \alpha)^2.$$

Let $C = (\operatorname{Re}(b) - \alpha)^2$. A simple calculation gives that

$$|B|^2 + \frac{(1 - b)(2 - \bar{b}) - C}{|2 - b|^2 - C} \bar{B} + \frac{(1 - \bar{b})(2 - b) - C}{|2 - b|^2 - C} B < \frac{C - |1 - b|^2}{|2 - b|^2 - C},$$

so

$$\begin{aligned} \left| B + \frac{(1 - b)(2 - \bar{b}) - C}{|2 - b|^2 - C} \right|^2 &= \left| B - \frac{C - (1 - b)(2 - \bar{b})}{|2 - b|^2 - C} \right|^2 \\ &< \frac{1}{(|2 - b|^2 - C)^2} (|2 - b|^2 C - |1 - b|^2 |2 - b|^2 + |1 - b|^2 C \\ &\quad + |(1 - b)(2 - \bar{b})|^2 - (1 - b)(2 - \bar{b})C - (1 - \bar{b})(2 - b)C) \\ &= \frac{C}{(|2 - b|^2 - C)^2}. \end{aligned}$$

Thus we see

$$\left| B - \frac{C - (1 - b)(2 - \bar{b})}{|2 - b|^2 - C} \right| < \frac{\sqrt{C}}{|2 - b|^2 - C}.$$

Consequently, we obtain

$$\begin{aligned} \left| B - \frac{(\operatorname{Re}(b) - \alpha)^2 - (1 - b)(2 - \bar{b})}{|2 - b|^2 - (\operatorname{Re}(b) - \alpha)^2} \right| &< \frac{\sqrt{(\operatorname{Re}(b) - \alpha)^2}}{|2 - b|^2 - (\operatorname{Re}(b) - \alpha)^2} \\ &= \frac{\operatorname{Re}(b) - \alpha}{|2 - b|^2 - (\operatorname{Re}(b) - \alpha)^2}. \end{aligned}$$

This shows that $B = a_2 z$ is inside of the circle with the center at

$$\frac{(\operatorname{Re}(b) - \alpha)^2 - (1 - b)(2 - \bar{b})}{|2 - b|^2 - (\operatorname{Re}(b) - \alpha)^2}$$

and radius

$$\frac{\operatorname{Re}(b) - \alpha}{|2 - b|^2 - (\operatorname{Re}(b) - \alpha)^2}.$$

Making use of $b = 1 + \frac{1}{3}i$ and $\alpha = \frac{1}{2}$ in Remark 7, we see

Example 7. Taking $b = 1 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, we see that a_2 is inside of the circle with the center at $\frac{5}{31} + \frac{12}{31}i$, and radius $\frac{18}{31}$. Thus we have $-\frac{5}{31} = \frac{13}{31} - \frac{18}{31} < 0 < |a_2 z| < \frac{13}{31} + \frac{18}{31} = 1$. If we take $a_2 = \frac{1}{2}$, then

$$f(z) = z + \frac{1}{2}z^2$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{z}{2+z}.$$

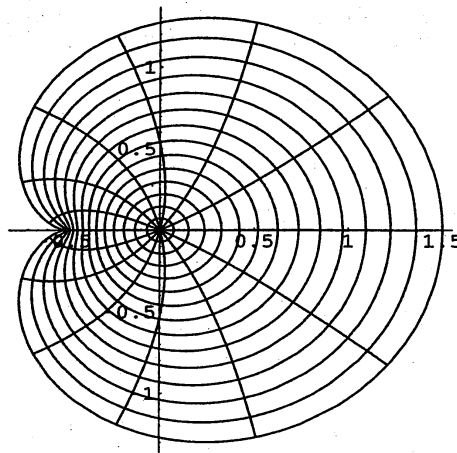


Figure 15: Image of U by $f(z)$ for Example 7.

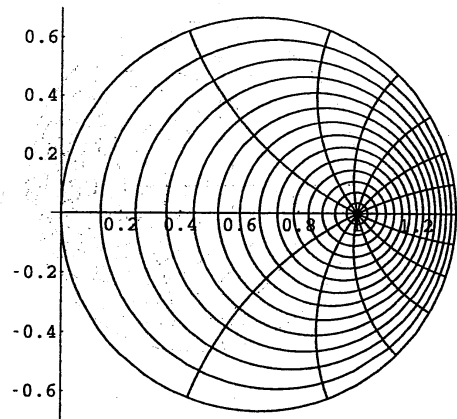


Figure 16: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 7.

Also, letting $b = 2 + \frac{1}{3}i$ and $\alpha = \frac{1}{2}$, Remark 7 leads us

Example 8. Taking $b = 2 + \frac{1}{3}i$, $\alpha = \frac{1}{2}$, we see that a_2 is inside of the circle with the center at $1 + \frac{12}{77}i$, and radius $\frac{54}{77}$. Thus we have $0.310772 = \frac{\sqrt{6073}}{77} - \frac{54}{77} < |a_2z| < 1 < \frac{\sqrt{6073}}{77} + \frac{54}{77} = 1.71337$. If we take $a_2 = \frac{2}{3}$, then

$$f(z) = z + \frac{2}{3}z^2$$

and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2z}{3+2z}.$$

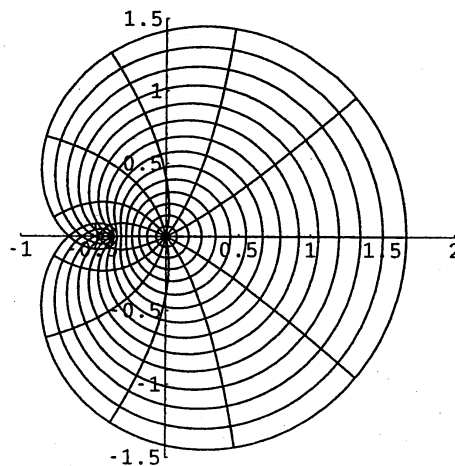


Figure 17: Image of U by $f(z)$ for Example 8.

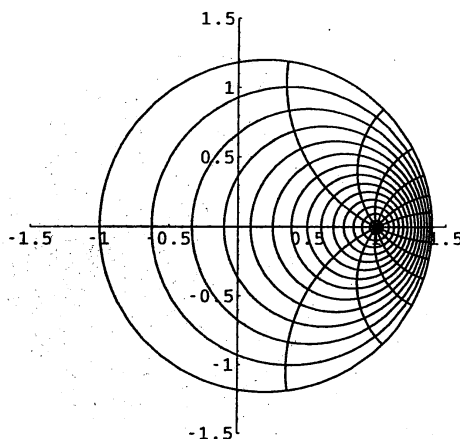


Figure 18: Image of U by $\frac{zf'(z)}{f(z)}$ for Example 8.

References

- [1] T. Sekine and S. Owa, Certain subclasses of starlike functions of order α , *PanAmer. Math. J.* **5**(1995), 95 - 100.
- [2] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51**(1975), 109 - 116.
- [3] H. Silverman, Starlike functions of positive order, *Bull. Inst. Math. Acad. Sinica* **28**(2000), 183 - 187.

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka, 577-8502
Japan