

ARGUMENT ESTIMATES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to obtain some argument properties of meromorphically multivalent functions in the punctured open unit disk. We also derive the integral preserving properties in a sector.

1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For f and g which are analytic in $\mathcal{U}$, we say that f is subordinate to g , written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathcal{U}$ such that $f(z) = g(w(z))$.

Let Σ_p denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \dots + a_{k+p-1}z^k + \dots \quad (p \in \mathbb{N} = \{1, 2, \dots\})$$

which are analytic in the annulus $\mathcal{D} = \{z : 0 < |z| < 1\}$. We denote by $\Sigma_p^*(\beta)$ the subclass of Σ_p consisting of all functions which is meromorphically starlike of order β in $\mathcal{U}$.

The Hadamard product or convolution of two functions f and g in Σ_p will be denoted by $f * g$.

Let

$$D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z) \quad (z \in \mathcal{D}) \quad (1.1)$$

or, equivalently,

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$$\begin{aligned}
D^{n+p-1}f(z) &= \frac{1}{z^p} \left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!} \right)^{(n+p-1)} \\
&= \frac{1}{z^p} + (n+p)a_0 \frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!} a_1 \frac{1}{z^{p-2}} + \dots \\
&\quad \dots + \frac{(n+k+2p-1)\dots(n+p)}{(k+p)!} a_{k+p-1} z^k + \dots \quad (z \in \mathcal{D}),
\end{aligned}$$

where n is any integer greater than $-p$.

For various interesting developments involving the operators D^{n+p-1} for functions belonging to Σ_p , the reader may be refereed to the recent works of author[1], Uralegaddi and Path[7], and others[8,9].

Let

$$\Sigma_p^*[n; A, B] = \left\{ f \in \Sigma_p : -\frac{z(D^{n+p-1}f(z))'(z)}{D^{n+p-1}f(z)} \prec_p \frac{1+Az}{1+Bz}, z \in \mathcal{U} \right\}, \quad (1.2)$$

where $-1 \leq B < A \leq 1$. In particular, we note that $\Sigma_p^*[-p+1; 1, -1]$ is the well known class of meromorphically p -valent starlike functions. From (1.2), we observe[6] that a function f is in $\Sigma_p^*[n; A, B]$ if and only if

$$\left| \frac{z(D^{n+p-1}f(z))'(z)}{D^{n+p-1}f(z)} + \frac{p(1-AB)}{1-B^2} \right| < \frac{p(A-B)}{1-B^2} \quad (-1 < B < A \leq 1; z \in \mathcal{U}). \quad (1.3)$$

The object of the present paper is to give some argument estimates of meromorphically multivalent functions belonging to Σ_p and the integral preserving properties in connection with the differential operators D^{n+p-1} defined by (1.1).

2. Main results

To establish our main results, we need the following lemmas.

Lemma 2.1 [2]. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \gamma) > 0$ ($\beta, \gamma \in \mathbb{C}$). If q is analytic in $\mathcal{U}$ with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \quad (z \in \mathcal{U})$$

implies

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$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

Lemma 2.2 [4]. *Let h be convex univalent in $\mathcal{U}$ and $\lambda(z)$ be analytic in $\mathcal{U}$ with $\operatorname{Re} \lambda(z) \geq 0$. If q is analytic in $\mathcal{U}$ and $q(0) = h(0)$, then*

$$q(z) + \lambda(z)zq'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

Lemma 2.3 [5]. *Let q be analytic in $\mathcal{U}$ with $q(0) = 1$ and $q(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\left| \arg q(z) \right| < \frac{\pi}{2}\alpha \text{ for } |z| < |z_0| \quad (2.1)$$

and

$$\left| \arg q(z_0) \right| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1). \quad (2.2)$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha, \quad (2.3)$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg q(z_0) = \frac{\pi}{2}\alpha \quad (2.4)$$

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg q(z_0) = -\frac{\pi}{2}\alpha \quad (2.5)$$

and

$$q(z_0)^{\frac{1}{\alpha}} = \pm ia \quad (a > 0). \quad (2.6)$$

At first, with the help of Lemma 2.1, we obtain the following

Proposition 2.1. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\operatorname{Re} h$ be bounded in $\mathcal{U}$. If $f \in \Sigma_p$ satisfies the condition*

$$-\frac{z(D^{n+p}f(z))'}{pD^{n+p}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z \in \mathcal{U}} \operatorname{Re} h(z) < \frac{n+2p}{p}$ (provided $D^{n+p-1}f(z) \neq 0$ in $\mathcal{U}$).

Proof. Let

$$q(z) = -\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)}.$$

By using the equation

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z), \quad (2.7)$$

we get

$$q(z) - \frac{n+2p}{p} = -\frac{(n+p)D^{n+1}f(z)}{pD^{n+p-1}f(z)}. \quad (2.8)$$

Taking logarithmic derivatives in both sides of (2.8) and multiplying by z , we have

$$\frac{zq'(z)}{-pq(z) + n+2p} + q(z) = -\frac{z(D^{n+p}f(z))'}{pD^{n+p}f(z)} \prec h(z) \quad (z \in \mathcal{U}).$$

From Lemma 2.1, it follows that $q(z) \prec h(z)$ for $\operatorname{Re}(-h(z) + \frac{n+2p}{p}) > 0$ ($z \in \mathcal{U}$), which means

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z \in \mathcal{U}} \operatorname{Re} h(z) < \frac{n+2p}{p}$.

Proposition 2.2. Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\operatorname{Re} h$ be bounded in $\mathcal{U}$. Let F be the integral operator defined by

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (c > 0). \quad (2.9)$$

If $f \in \Sigma_p$ satisfies the condition

$$-\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z \in \mathcal{U}} \operatorname{Re} h(z) < \frac{c+p}{p}$ (provided $D^{n+p-1}F(z) \neq 0$ in $\mathcal{U}$).

Proof. From (2.9), we have

$$z(D^{n+p-1}F(z))' = cD^{n+p-1}f(z) - (c+p)D^{n+p-1}F(z). \quad (2.10)$$

Let

$$p(z) = -\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)}.$$

Then, by using (2.10), we get

$$q(z) - (c+p) = -c\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)}. \quad (2.11)$$

Taking logarithmic derivatives in both sides of (2.11) and multiplying by z , we have

$$\frac{zq'(z)}{-pq(z) + (c+p)} + q(z) = -\frac{z(D^{n+p-1}f(z))'}{pD^{n+p-1}f(z)} \prec h(z) \quad (z \in \mathcal{U}).$$

Therefore, by Lemma 2.1, we have

$$-\frac{z(D^{n+p-1}F(z))'}{pD^{n+p-1}F(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z \in \mathcal{U}} \operatorname{Re} h(z) < \frac{c+p}{p}$ (provided $D^{n+p-1}F(z) \neq 0$ in $\mathcal{U}$).

Remark. Taking $p = 1$ and $h(z) = \frac{1+z}{1-z}$ in Proposition 2.1 and Proposition 2.2, we have the results obtained by Ganigi and Uralegaddi[3].

Applying Lemma 2.2, Lemma 2.3 and Proposition 2.1, we now derive

Theorem 2.1. *Let $f \in \Sigma_p$. Choose an integer n such that*

$$n \geq \frac{p(1+A)}{1+B} - 2p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left| \arg \left(-\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n+1; A, B]$, then

$$\left| \arg \left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} - \gamma \right) \right| < \frac{\pi}{2}\alpha,$$

where α ($0 < \alpha \leq 1$) is the solution of the equation

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$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha \sin \frac{\pi}{2} (1 - t(A, B))}{\frac{(n+2p)(1-B)+A-1}{1-B} + \alpha \cos \frac{\pi}{2} (1 - t(A, B))} \right) \quad (2.12)$$

when

$$t(A, B) = \frac{2}{\pi} \sin^{-1} \left(\frac{p(A - B)}{(n + 2p)(1 - B^2) - p(1 - AB)} \right). \quad (2.13)$$

Proof. Let

$$q(z) = -\frac{1}{p - \gamma} \left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} + \gamma \right).$$

By (2.7), we have

$$\begin{aligned} (p - \gamma)zq'(z)D^{n+p-1}g(z) + (1 - \gamma)q(z)z(D^{n+p-1}g(z))' \\ - (n + 2p)z(D^{n+p-1}f(z))' = -(n + p)z(D^{n+p}f(z))' - \gamma z(D^{n+p-1}g(z))'(z). \end{aligned} \quad (2.14)$$

Dividing (2.14) by $D^{n+p-1}g(z)$ and simplifying, we get

$$q(z) + \frac{zq'(z)}{-r(z) + n + 2p} = -\frac{1}{p - \gamma} \left(\frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)} + \gamma \right), \quad (2.15)$$

where

$$r(z) = -\frac{z(D^{n+p-1}g(z))'}{D^{n+p-1}g(z)}.$$

Since $g \in \Sigma_p^*[n + 1; A, B]$, from Proposition 2.1, we have

$$r(z) \prec p \frac{1 + Az}{1 + Bz}.$$

Using (1.3), we have

$$-r(z) + n + 2p = \rho e^{i\frac{\pi}{2}} \phi,$$

where

$$\begin{cases} \frac{(n+2p)(1+B)-(1+A)}{1+B} < \rho < \frac{(n+2p)(1-B)+A-1}{1-B} \\ -t(A, B) < \phi < t(A, B) \end{cases}$$

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when $t(A, B)$ is given by (2.13). Let h be a function which maps $\mathcal{U}$ onto the angular domain $\{w : |\arg w| < \frac{\pi}{2}\delta\}$ with $h(0) = 1$. Applying Lemma 2.2 for this h with $\lambda(z) = \frac{1}{-r(z)+n+2p}$, we see that $\operatorname{Re} q(z) > 0$ in $\mathcal{U}$ and hence $q(z) \neq 0$ in $\mathcal{U}$.

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}} = ia$ ($a > 0$). Then we obtain

$$\begin{aligned} \arg \left[-\frac{1}{p-\gamma} \left(\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}g(z_0)} + \gamma \right) \right] &= \arg \left(q(z_0) + \frac{z_0 q'(z_0)}{-r(z_0) + n + 2p} \right) \\ &= \frac{\pi}{2}\alpha + \arg \left(1 + i\alpha k(\rho e^{i\frac{\pi}{2}\phi})^{-1} \right) \\ &= \frac{\pi}{2}\alpha + \tan^{-1} \left(\frac{\eta k \sin \frac{\pi}{2}(1-\phi)}{\rho + \alpha k \cos \frac{\pi}{2}(1-\phi)} \right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1} \left(\frac{\alpha \sin \frac{\pi}{2}(1-t(A, B))}{\frac{(n+2p)(1-B)+A-1}{1-B} + \alpha \cos \frac{\pi}{2}(1-t(A, B))} \right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ and $t(A, B)$ are given by (2.12) and (2.13), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia$ ($a > 0$). Applying the same method as the above, we have

$$\begin{aligned} \arg \left[-\frac{1}{p-\gamma} \left(\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}g(z_0)} + \gamma \right) \right] \\ \leq -\frac{\pi}{2}\alpha - \tan^{-1} \left(\frac{\alpha \sin \frac{\pi}{2}(1-t(A, B))}{\frac{(n+2p)(1-B)+A-1}{1-B} + \alpha \cos \frac{\pi}{2}(1-t(A, B))} \right) \\ = -\frac{\pi}{2}\delta, \end{aligned}$$

where δ and $t(A, B)$ are given by (2.12) and (2.13), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem.

Letting $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 2.1, we have

Corollary 2.1. *Let $f \in \Sigma$. If*

$$-\operatorname{Re} \left\{ \frac{z(D^{n+p}f(z))'}{D^{n+p}g(z)} \right\} > \gamma \quad (0 \leq \gamma < p)$$

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for some $g \in \Sigma_p$ satisfying the condition

$$\left| \frac{z(D^{n+p}g(z))'}{D^{n+p}g(z)} + p \right| < p,$$

then

$$-\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} \right\} > \gamma.$$

Taking $A = 1$, $B = 0$ and $g(z) = \frac{1}{z^p}$ in Theorem 2.1, we have

Corollary 2.2. *Let $f \in \Sigma_p$. If*

$$|\arg [-z^{p+1}(D^{n+p}f(z))' - \gamma]| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p; \quad 0 < \delta \leq 1),$$

then

$$|\arg [-z^{p+1}(D^{n+p-1}f(z))' - \gamma]| < \frac{\pi}{2}\delta.$$

Making $n = 0$, $p = 1$ and $\delta = 1$ in Corollary 2.2, we have

Corollary 2.3. *Let $f \in \Sigma_1$. If*

$$-\operatorname{Re} \{z^2(zf''(z) + 3f'(z))\} > \gamma \quad (0 \leq \gamma < 1),$$

then

$$-\operatorname{Re} \{z^2 f'(z)\} > \gamma.$$

By the same techniques as in the proof of Theorem 2.1, we obtain

Theorem 2.2. *Let $f \in \Sigma$. Choose an integer n such that*

$$n \geq \frac{p(1+A)}{1+B} - 2p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left| \arg \left(\frac{z(D^{n+p}f(z))'}{(D^{n+p}g(z))} + \gamma \right) \right| < \frac{\pi}{2}\delta \quad (\gamma > p, 0 < \delta \leq 1)$$

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for some $g \in \Sigma_p^*[n+1; A, B]$, then

$$\left| \arg \left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} + \gamma \right) \right| < \frac{\pi}{2}\alpha,$$

where α ($0 < \alpha \leq 1$) is the solution of the equation given by (2.12).

Next, we prove

Theorem 2.3. Let $f \in \Sigma_p$ and choose a positive number c such that

$$c \geq \frac{1+A}{1+B} - p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left| \arg \left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n; A, B]$, then

$$\left| \arg \left(-\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}G(z)} - \gamma \right) \right| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9),

$$G(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} g(t) dt, \quad (c > 0), \quad (2.16)$$

and α ($0 < \alpha \leq 1$) is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha \sin \frac{\pi}{2}(1 - t(A, B, c))}{\frac{(c+p)(1-B)+A-1}{1-B} + \alpha \cos \frac{\pi}{2}(1 - t(A, B, c))} \right) \quad (2.17)$$

when

$$t(A, B, c) = \frac{2}{\pi} \sin^{-1} \left(\frac{p(A-B)}{(c+p)(1-B^2) - p(1-AB)} \right).$$

Proof. Let

$$q(z) = -\frac{1}{p-\gamma} \left(\frac{z(D^n F(z))'}{D^n G(z)} + \gamma \right).$$

Since $g \in \Sigma_p^*[n; A, B]$, from Proposition 2.2, $g \in \Sigma_p^*[n; A, B]$.

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Using (2.10), we have

$$(p - \gamma)q(z)D^{n+p-1}G(z) - (c + p)D^{n+p-1}F(z) = -cD^{n+p-1}f(z) - \gamma D^{n+p-1}G(z).$$

Then, by a simple calculation, we get

$$(p - \gamma)(zq'(z) + q(z)(-r(z) + c + p)) + \gamma(-r(z) + c + p) = -\frac{cz(D^{n+p-1}f(z))'}{D^{n+p-1}G(z)},$$

where

$$r(z) = -\frac{z(D^{n+p-1}G(z))'}{D^{n+p-1}G(z)}.$$

Hence we have

$$q(z) + \frac{zq'(z)}{-r(z) + c + p} = -\frac{1}{p - \gamma} \left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} + \gamma \right).$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we omit it.

Letting $n = -p + 1$, $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 2.3, we have

Corollary 2.4. *Let $c > 0$ and $f \in \Sigma$. If*

$$-\operatorname{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \gamma \quad (0 \leq \gamma < p)$$

for some $g \in \Sigma_p$ satisfying the condition

$$\left| \frac{zg'(z)}{g(z)} + p \right| < p,$$

then

$$-\operatorname{Re} \left\{ \frac{zF'(z)}{G(z)} \right\} > \gamma,$$

where F and G are given by (2.9) and (2.16), respectively.

Taking $n = 0$, $B \rightarrow A$ and $g(z) = \frac{1}{z^p}$ in Theorem 2.3, we have

Corollary 2.5. *Let $c > 0$ and $f \in \Sigma_p$. If*

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$$|\arg (-z^{p+1}f'(z) - \gamma)| < \frac{\pi}{2}\delta, \quad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

then

$$|\arg (-z^{p+1}F'(z) - \gamma)| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9) and α ($0 < \alpha \leq 1$) is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha}{c + p - 1} \right).$$

By using the same methods as in proving Theorem 2.3, we have

Theorem 2.4. Let $f \in \Sigma_p$ and choose a positive number c such that

$$c \geq \frac{1+A}{1+B} - p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left| \arg \left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} + \gamma \right) \right| < \frac{\pi}{2}\delta \quad (\gamma > p; 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n; A, B]$, then

$$\left| \arg \left(\frac{z(D^{n+p-1}F(z))'}{D^{n+p-1}G(z)} + \gamma \right) \right| < \frac{\pi}{2}\alpha,$$

where F and G are given by (2.9) and (2.16), respectively, and α ($0 < \alpha \leq 1$) is the solution of the equation given by (2.17)

Finally, we derive

Theorem 2.5. Let $f \in \Sigma_p$. Choose an integer n such that

$$n \geq \frac{p(1+A)}{1+B} - 2p,$$

where $-1 < B < A \leq 1$ and $p \in \mathbb{N}$. If

$$\left| \arg \left(-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < p; 0 < \delta \leq 1)$$

for some $g \in \Sigma_p^*[n; A, B]$, then

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$$\left| \arg \left(-\frac{z(D^{n+p}F(z))'}{D^{n+p}G(z)} - \gamma \right) \right| < \frac{\pi}{2}\delta,$$

where F and G are given by (2.9) and (2.16) with $c = n + p$, respectively.

Proof. From (2.7) and (2.8) with $c = n + p$, we have $D^{n+p-1}f(z) = D^{n+p}F(z)$
Therefore

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}g(z)} = \frac{z(D^{n+p}F(z))'}{D^{n+p}G(z)}$$

and the result follows.

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